Chapter 3

Analysis of spectral methods for the homogeneous Boltzmann equation

The development of accurate and fast algorithms for the Boltzmann collision integral and their analysis represent a challenging problem in scientific computing and numerical analysis. Recently, several works were devoted to the derivation of spectrally accurate schemes for the Boltzmann equation, but very few of them were concerned with the stability analysis of the method. In particular there was no result of stability except when the method is modified in order to enforce the positivity preservation, which destroys the spectral accuracy. In this paper we propose a new method to study the stability of homogeneous Boltzmann equations perturbed by smoothed balanced operators which do not preserve positivity of the distribution. This method takes advantage of the "spreading" property of the collision, together with estimates on regularity and entropy production. As an application we prove stability and convergence of spectral methods for the Boltzmann equation, when the discretization parameter is large enough (with explicit bound).

3.1 Introduction

This chapter is devoted to the stability and convergence analysis of general spectral algorithms [117, 61, 104, 58]. These methods are based on approximating in the velocity space the distribution function by a periodic function, and on its representation by Fourier series. Recently, fast algorithms were proposed and permit to reduce the computational cost without loosing the spectral accuracy, thus making the method competitive with Monte-Carlo. However an important drawback of the spectral methods up to now had been the lack of proof of stability. Indeed as compared to discrete velocity methods the difficulties are somehow opposite: consistency results are easily obtained, whereas the lack of positivity preservation of the scheme is a major issue when one studies its stability properties. The only paper concerned with the issue of stability for spectral methods applied to the Boltzmann collision operator is [118], but in the latter the author introduce some "filters" on the Fourier modes in order to restore the positivity-preservation of the scheme, which breaks the spectral accuracy.

In this chapter we give the first stability result for the spectral methods applied to the Boltzmann collision operator. Moreover we propose a method which is likely to have other utilizations in collisional kinetic theory:

- we write the Galerkin approximation on the first N Fourier modes of the evolution equation as a *a smooth balanced perturbation* of the original equation, in the sense of a perturbation by some small and mass-preservation (although not positivity-preserving) error term;
- we prove existence and uniqueness of smooth solution for small times, conditionally to a bound on the L^1 norm;
- we use the mixing structure [123, 102] of the collision process to show appearance of positivity after a small time (depending on the size of the box of truncation and the approximation parameter N);
- we use the mass conservation to deduce uniform bounds on the L^1 norm, and therefore regularity bounds growing at most exponentially in time;

Hence our paper introduces a general method on how to exploit fine mixing properties of the collision process in the study of stability of a particular class of perturbed Boltzmann equation, with the application in mind to the stability of spectral methods.

The outline of this chapter is as follows. In Section 3.2 we explain the truncation and periodization associated with spectral methods and fast spectral methods and we formulate the problem of stability of these methods in the general framework of the stability properties of the Boltzmann equation with respect to a smooth balanced perturbation. Section 3.3 is devoted to the proof of the main stability result in the general framework. Finally in Section 3.4 we apply the latter result to the spectral method and establish some stability and convergence results of the numerical solution.

3.2 Formulation of a general stability result

In this section we remind the basic principles leading to the periodized truncations of the Boltzmann collision operator arising in spectral methods. Then, we present the main result of this paper: the stability of the spatially homogeneous Boltzmann equation with respect to a smooth balanced perturbation, preserving mass and smoothness but not non-negativity of the solution. This stability means that we are able to construct global solutions and estimate the error between perturbed and unperturbed solutions.

Any deterministic numerical method requires to work on a *bounded* velocity space. This therefore supposes a non physical truncation (associated with limit conditions) of this velocity space, which we shall discuss below.

3.2.1 General framework

We consider the spatially homogeneous Boltzmann equation written in the following general form

$$\frac{\partial f}{\partial t} = \mathcal{Q}(f), \tag{3.1}$$

where $\mathcal{Q}(f)$ is given by

$$\mathcal{Q}(f) = \int_{\mathcal{C}} \mathcal{B}(y,z)\,\left[f' f'_\star - f_\star f\right] dy\, dz, \quad v \in \mathbb{R}^{d_v} \tag{3.2}$$

with

$$v' = v + \Theta'(y,z), \qquad v'_\star = v + \Theta'_\star(y,z), \qquad v_\star = v + \Theta_\star(y,z).$$

In the equations above, $\mathcal{C}$ is some given (unbounded) domain for y, z, and Θ, Θ', $\Theta'_\star$ are suitable functions, to be defined later. This general framework emphasizes the translation invariance property of the collision operator, which is crucial for the spectral methods.

For the approximation of the velocity space we consider a bounded domain $\mathcal{D}_T = [-T, T]^{d_v}$ with $0 < T < \infty$ and truncate the integration in y and z in (3.2) since periodization would yield infinite result if not: we set y and z to belong to some truncated domain $\mathcal{C}_R \subset \mathcal{C}$ (the parameter R refers to its size and will be defined later). For a compactly supported function with support included in $\mathcal{B}_S$, the ball centered at 0 with radius $S > 0$, one has to prescribe suitable relations (depending on the precise change of variable and truncation chosen) between S, R and T in order to retain all possible collisions and at the same time prevent intersections of the regions where f is different from zero (this is the so-called *dealiasing condition*). Then the *truncated* collision operator reads

$$\mathcal{Q}^R(f) = \int_{\mathcal{C}_R} \mathcal{B}(y,z)\left(f'_\star f' \,-\, f_\star f\right) dy\, dz \tag{3.3}$$

for $v \in \mathcal{D}_T$ (the expression for $v \in \mathbb{R}^{d_v}$ is deduced by periodization).

The *spectral equation* is the projection of the collision equation in $\mathcal{P}_N$, the $(2N+1)^{d_v}$-dimensional vector space of trigonometric polynomials of degree at most N in each direction, *i.e.*,

$$\frac{\partial f_N}{\partial t} = \mathcal{P}_N\, \mathcal{Q}^R(f_N),$$

where $\mathcal{P}_N$ denotes the orthogonal projection on $\mathcal{P}_N$ in $L^2(\mathcal{D}_T)$.

3.2.2 A common abstract formulation for the stability of spectral methods

From now on, $\mathcal{Q}^R$ shall denote a periodized truncated collision operator as in (2.15) or (2.19). As we shall see, using this formulations, both classical and fast spectral methods fall into the following framework:

$$\begin{cases} \dfrac{\partial f}{\partial t} = \mathcal{Q}^R(f) + \mathcal{P}_\varepsilon(f), \quad v \in \mathcal{D}_T, \\[2em] f(0,v) = f_{0,\varepsilon}(v), \quad v \in \mathcal{D}_T, \end{cases} \tag{3.4}$$

where $\mathcal{P}_\varepsilon$ is a "smooth balanced perturbation", which means that it satisfies the following (balanced law)

$$\int_{\mathcal{D}_T} \mathcal{P}_\varepsilon(f)\, dv = 0 \tag{3.5}$$

and preserves the smoothness of the distribution function, *i.e.*, there exist constants $C_0, C_k > 0$ such that

$$\begin{cases} \|\mathcal{P}_\varepsilon(f)\|_{L^2} \;\leq\; C_0\, \|f\|_{L^1}\, \|f\|_{L^2} \\ \\ \|\mathcal{P}_\varepsilon(f)\|_{H^{k+1}_{\text{per}}} \;\leq\; C_{k+1}\, \left(\|f\|^2_{H^k_{\text{per}}} + \|f\|_{L^1}\, \|f\|_{H^{k+1}_{\text{per}}} \right), \quad k \geq 0, \end{cases} \tag{3.6}$$

where $\|\cdot\|_{H^k_{\text{per}}}$ is the usual norm of the Sobolev space of periodic functions $H^k_{\text{per}}(\mathcal{D}_T)$.

Moreover the perturbation is supposed to be small in the following sense: there exists a function $\varphi(\varepsilon)$ such that for any $p \geq 0$,

$$\|\mathcal{P}_\varepsilon(f)\|_{H^p_{\text{per}}} \;\leq\; \varphi(\varepsilon), \tag{3.7}$$

where $\varphi(\varepsilon)$ depends on $\|f\|_{H^{p+k}_{\text{per}}}$ for some $k > 0$, and goes to zero as ε goes to zero.

Finally in order to prove global existence with uniform regularity bounds, we shall require additional assumptions on the relation between the equilibrium distributions of the perturbed and unperturbed (periodized) Boltzmann equations, and about the stability of the unperturbed equation (see the following statement).

Let us therefore write the unperturbed equation for reference:

$$\begin{cases} \dfrac{\partial f}{\partial t} = \mathcal{Q}^R(f), \quad v \in \mathcal{D}_T, \quad t > 0 \\ \\ f(0, v) = f_0(v), \quad v \in \mathcal{D}_T. \end{cases} \tag{3.8}$$

Let us state the general stability theorem:

Theorem 3.2.1 *Let us consider a perturbed Boltzmann equation (3.4) in the torus $\mathcal{D}_T$, where $\mathcal{Q}^R$ is defined by (2.15) or (2.19), and for a sequence of smooth balanced perturbations $\big(\mathcal{P}_\varepsilon = \mathcal{P}_\varepsilon(f)\big)_{\varepsilon>0}$ which satisfy* (3.5)-(3.6)-(3.7).

Assume that the constant functions are equilibria of the perturbed equation (3.4) *(as for equation* (3.8)*) and that they are nonlinearly locally stable in any $H^k_{\text{per}}(\mathcal{D}_T)$ for equation* (3.8).

We assume that f_0 is a non-negative function, non zero everywhere, belonging to $H^k_{\text{per}}(\mathcal{D}_T)$ with $k \in \mathbb{N}$ and $k > d_v/2$. We consider a sequence of smooth balanced perturbations $f_{0,\varepsilon}$ of the initial datum for the perturbed problem (3.4) *(non necessarily positive) such that*

$$\int_{\mathcal{D}_T} f_{0,\varepsilon} = \int_{\mathcal{D}_T} f_0 \quad \textit{and} \quad \|f_0 - f_{0,\varepsilon}\|_{H^k_{\text{per}}} \;\leq\; \psi(\varepsilon),$$

with $\psi(\varepsilon)$ goes to zero when ε goes to zero.

Then, there exists $\varepsilon_0 > 0$ depending only on the collision kernel B, the truncation R, the constants in (3.6)-(3.7) *for the perturbation, and the $L^1(\mathcal{D}_T)$ and $H^k_{\rm per}(\mathcal{D}_T)$ norms on f_0, such that for any $\varepsilon \in (0, \varepsilon_0)$,*

(i) *there exists a unique global smooth solution f_ε to* (3.4)*;*

(ii) *for any $p < k$, this solution belongs to $H^p_{\rm per}(\mathcal{D}_T)$ for all times with uniform bounds as time goes to infinity;*

(iii) *this solution remains "essentially non-negative" uniformly in time, in the sense that there is $\eta(\varepsilon) > 0$ (with $\eta(\varepsilon) \to 0$ as ε goes to 0) such that the non-positive part is $\eta(\varepsilon)$-small:*

$$\forall\, t \geq 0, \quad \|f^-_\varepsilon(t,\cdot)\|_{L^\infty} \leq \eta(\varepsilon)$$

where f^-_ε denotes $|f_\varepsilon|\,\mathbf{1}_{\{f_\varepsilon \leq 0\}}$;

(iv) *this solution f_ε converges in $H^p_{\rm per}(\mathcal{D}_T)$ for any $p < k$, uniformly on any $[0, \tau]$, $\tau > 0$, to the solution f of the unperturbed equation* (3.8) *when the parameter ε goes to zero;*

(v) *the solution f_ε to* (3.4) *converges in $H^k_{\rm per}(\mathcal{D}_T)$ as time goes to infinity to the constant equilibrium distribution in the torus prescribed by its mass, and it is "asymptotically uniformly positive", that is for t larger than some fixed explicit time.*

We split the proof into two main steps: first in Section 3.3 we prove existence, uniqueness and smoothness of a solution on an arbitrary bounded time interval (as the size of perturbation goes to 0). The main difficulty is to prove that non-negativity of the distribution function is recovered in a certain sense. Finally in Section 3.4 we apply the previous general results to spectral methods, and prove their stability and convergence.

3.3 Proof of stability on an arbitrary bounded time interval

In this section we first give some technical lemmas and next establish a result showing existence and uniqueness of a smooth solution on an arbitrary time interval to the perturbed equation (3.4), depending on an assumption of smallness on the size of the perturbation. Then, in Lemma 3.3.6 we prove the control of negative values of $f(t)$.

3.3.1 Preliminary results

We start this section by a classical result of L^p estimates on the Boltzmann operator $\mathcal{Q}^R(g,h)$ given by

$$\mathcal{Q}^R(g,h) := \int_{\mathcal{C}_R} \mathcal{B}(y,z)\left(g'_\star\, h' \,-\, g_\star\, h\right) dy\, dz.$$

Lemma 3.3.1 *Let the collision kernel B satisfy the assumption* (1.3)-(1.4)-(1.5). *Then, the periodized Boltzmann operator $\mathcal{Q}^R$ (defined by (2.15) or (2.19)) satisfies: for all $p \in [0, \infty]$ there exists a constant $C_p(R, B) > 0$ such that*

$$\|\mathcal{Q}^R(g,h)\|_{L^p},\ \|\mathcal{Q}^R(h,g)\|_{L^p} \ \leq\ C_p(R,B)\,\|g\|_{L^1}\,\|h\|_{L^p}. \tag{3.9}$$

Proof : The proof is exactly similar to the case of the usual Boltzmann collision operator for a collision kernel bounded with compact support, see for instance [105] for a recent proof. ■

Now, we prove smoothness of the solution to the perturbed problem (3.4) on a fixed time interval under the assumption of an *a priori* bound on the L^1 norm of the solution.

Lemma 3.3.2 *Let us consider a collision kernel B which satisfies the assumptions* (1.3)-(1.4)-(1.5) *and a sequence of smooth balanced perturbations $\left(\mathcal{P}_\varepsilon = \mathcal{P}_\varepsilon(f^\varepsilon)\right)_{\varepsilon>0}$ which satisfy* (3.5)-(3.6)-(3.7), *and let $\tau > 0$ be the length of the time interval. Assume that $f_0 \in H^k(\mathcal{D}_T)$ for $k \in \mathbb{N}$ and that $f(t)$ is a (non necessarily positive) solution to* (3.4) *with initial datum f_0, which satisfies the L^1-estimate*

$$\forall\, t \in [0,\tau], \quad \|f(t)\|_{L^1} \leq M. \tag{3.10}$$

Then, there exists a constant $C_k(M) > 0$, only depending on M, R, T and $\|f_0\|_{H^k_{\mathrm{per}}}$ such that

$$\forall\, t \in [0,\tau], \quad \|f(t)\|_{H^k_{\mathrm{per}}} \ \leq\ C_k(M). \tag{3.11}$$

Proof :[Proof of Lemma 3.3.2] We proceed by induction on $k \geq 0$. For the first stage $k = 0$, we apply Lemma 3.3.1 with $p = 2$ and $g = h = f$ and we use assumption (3.6) on the perturbation:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f(t)\|^2_{L^2} \ &\leq\ \|\mathcal{Q}^R(f) \,+\, \mathcal{P}_\varepsilon(f)\|_{L^2}\,\|f(t)\|_{L^2} \\ &\leq\ (C_2(R,B) \,+\, C_0)\ \|f(t)\|_{L^1}\,\|f(t)\|^2_{L^2}.\end{aligned}$$

Hypothesis (3.10) provides a control on $\|f(t)\|_{L^1}$ and we can apply Gronwall's lemma to get the result (3.11) at stage $k = 0$.

Let us now assume that (3.11) holds at stage $k \geq 0$ and let us prove that it also holds at stage $(k+1)$.

Let us first recall a formula on the derivatives of $\mathcal{Q}^R$: from the bilinearity of $\mathcal{Q}^R$ and the translation invariance property of the periodized Boltzmann collision operator, one has

$$\nabla_v \mathcal{Q}^R(f) = \mathcal{Q}^R(\nabla_v f, f) + \mathcal{Q}^R(f, \nabla_v f),$$

which yields a Leibniz formula at any order $s \in \mathbb{N}$:

$$\begin{aligned} \left\|\mathcal{Q}^R(f)\right\|^2_{H^s_{\text{per}}} &= \sum_{|\nu|\le s} \left\|\partial^\nu \mathcal{Q}^R(f)\right\|^2_{L^2} \\ &\le C \sum_{|\nu|\le s} \sum_{|\mu|\le|\nu|} \begin{pmatrix} \nu \\ \mu \end{pmatrix} \left\|\mathcal{Q}^R(\partial^\mu f, \partial^{\nu-\mu} f)\right\|^2_{L^2}. \end{aligned} \tag{3.12}$$

Now, using (3.12) with $s = k+1$ we have

$$\left\|\mathcal{Q}^R(f)\right\|^2_{H^{k+1}_{\text{per}}} \le C \left\|\mathcal{Q}^R(f)\right\|^2_{H^k_{\text{per}}} + C \sum_{|\nu|=k+1} \sum_{|\mu|\le k+1} \begin{pmatrix} \nu \\ \mu \end{pmatrix} \left\|\mathcal{Q}^R(\partial^\mu f, \partial^{\nu-\mu} f)\right\|^2_{L^2}. \tag{3.13}$$

From Lemma 3.3.1 with $p = 2$ and $g = \partial^\mu f$, $h = \partial^{\nu-\mu} f$ together with the hypothesis (3.11), we get

$$\left\|\mathcal{Q}^R(f)\right\|_{H^k_{\text{per}}} \le C(k,R) \left\|f\right\|_{H^k_{\text{per}}} \left\|f\right\|_{H^k_{\text{per}}} \le C_2(k,R,B)\, C_k(M)^2. \tag{3.14}$$

Then we split the last term of (3.13) in two parts for $\mu \neq 0$ and $\mu = 0$. For $\mu \neq 0$, we again apply Lemma 3.3.1 with $p = 2$ and $g = \partial^\mu f$, $h = \partial^{\nu-\mu} f$ and use the fact that both derivatives $|\mu| = \sum_{i=1}^{d_v} |\mu_i| \le k$ and $|\nu - \mu| \le k$:

$$\sum_{|\nu|=k+1} \sum_{\substack{|\mu|\le k+1 \\ \mu\neq 0}} \begin{pmatrix} \nu \\ \mu \end{pmatrix} \left\|\mathcal{Q}^R(\partial^\mu f, \partial^{\nu-\mu} f)\right\|^2_{L^2} \le C_3(k,R,B)\, C_k(M)^2. \tag{3.15}$$

Finally, for $\mu = 0$ we apply Lemma 3.3.1 with $p = 2$ and $g = f$, $h = \partial^\nu f$:

$$\sum_{|\nu|=k+1} \left\|\mathcal{Q}^R(f, \partial^\nu f)\right\|^2_{L^2} \le C_4(k,R,B) \left\|f\right\|^2_{L^1} \left\|f\right\|^2_{H^{k+1}_{\text{per}}}. \tag{3.16}$$

Then, gathering inequalities from (3.14) to (3.16) and using the assumption (3.6) on the smooth balanced perturbation $\mathcal{P}_\varepsilon$, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\|f\right\|^2_{H^{k+1}_{\text{per}}} &\le \left\|\mathcal{Q}^R(f) + \mathcal{P}_\varepsilon(f)\right\|_{H^{k+1}_{\text{per}}} \left\|f\right\|_{H^{k+1}_{\text{per}}} \\ &\le C\,(C_2 + C_3)\, C_k(M)^2 \left\|f\right\|_{H^{k+1}_{\text{per}}} + C\,(C_4 + C) \left\|f\right\|_{L^1} \left\|f\right\|^2_{H^{k+1}_{\text{per}}}. \end{aligned}$$

Finally using the control (3.10) on $\|f(t)\|_{L^1}$ we apply Gronwall's lemma to get (3.11) at stage $k+1$: there exists a constant $C_{k+1}(M)$, only depending on M, R, T and $\|f_0\|_{H^{k+1}_{\text{per}}}$ such that

$$\forall\, t \in [0,\tau], \quad \left\|f(t)\right\|_{H^{k+1}_{\text{per}}} \le C_{k+1}(M).$$

■

Then, we establish existence and uniqueness of a smooth solution for the perturbed problem (3.4) on a small time interval $[0, \bar{\tau}]$, $\bar{\tau} > 0$.

Proposition 3.3.3 *Let us consider a collision kernel B which satisfies the assumptions* (1.3)-(1.4)-(1.5) *and a sequence of smooth balanced perturbations* $\left(\mathcal{P}_\varepsilon = \mathcal{P}_\varepsilon(f)\right)_{\varepsilon>0}$ *which satisfy* (3.5)-(3.6)-(3.7). *We assume that $f_0 \in H^k_{\text{per}}(\mathcal{D}_T)$, for $k \in \mathbb{N}$ and set*

$$M = 2\,\|f_0\|_{L^1}. \tag{3.17}$$

Then, there exists $\bar{\tau} = \bar{\tau}(M) > 0$ such that for all $\varepsilon > 0$ the perturbed Boltzmann equation (3.4) *admits a unique solution (non necessarily positive) on the time interval $[0, \bar{\tau}]$, where the solution f satisfies*

$$\forall\, t \in [0, \bar{\tau}], \quad \|f(t)\|_{L^1} \;\leq\; M. \tag{3.18}$$

Moreover, there exists a constant $C_k(M) > 0$, only depending on M, R, and $\|f_0\|_{H^k_{\text{per}}}$ such that

$$\forall\, t \in [0, \bar{\tau}], \quad \|f(t)\|_{H^k_{\text{per}}} \;\leq\; C_k(M). \tag{3.19}$$

Proof :[Proof of Proposition 3.3.3] First, we apply Lemma 3.3.1 with $p = 2$ and $g = h = f$

$$\left\|\mathcal{Q}^R(f)\right\|_{L^2} \leq C_2(R, B)\, \|f\|_{L^1}\, \|f\|_{L^2}.$$

Moreover, using assumption (3.6) on the perturbation, there exists $C > 0$ such that for all $\varepsilon > 0$

$$\|\mathcal{P}_\varepsilon(f)\|_{L^2} \leq C\, \|f\|_{L^1}\, \|f\|_{L^2}.$$

Therefore, we obtain a constant $C > 0$, only depending on R and the collision kernel B such that

$$\frac{d}{dt}\|f(t)\|_{L^2} \leq C\, \|f\|^2_{L^2}.$$

This implies that

$$\|f(t)\|_{L^2} \;\leq\; \frac{\|f_0\|_{L^2}}{1 \,-\, C\, \|f_0\|_{L^2}\, t}.$$

In particular, we get an estimate for the L^1 norm

$$\|f(t)\|_{L^1} \;\leq\; (2T)^{d_v/2}\, \|f(t)\|_{L^2} \;\leq\; (2T)^{d_v/2}\, \frac{\|f_0\|_{L^2}}{1 \,-\, C\, \|f_0\|_{L^2}\, t}.$$

Now, setting $M = 2\,(2T)^{d_v/2}\, \|f_0\|_{L^2}$ and from the latter inequality, we show that there exists $\bar{\tau} < 1/(2\,C\, \|f_0\|_{L^2})$ such that

$$\forall\, t \in [0, \bar{\tau}], \quad \|f(t)\|_{L^1} \leq M,$$

which gives (3.18).

From the estimate in $L^1(\mathcal{D}_T)$ on the function $f(t)$ on the time interval $[0,\bar{\tau}]$, we prove existence and uniqueness of a solution by Cauchy-Lipschitz theorem in $L^1(\mathcal{D}_T)$ (because of the truncation on $\mathcal{D}_T$ the collision kernel is a bounded bilinear function from $L^1(\mathcal{D}_T) \times L^1(\mathcal{D}_T)$ to $L^1(\mathcal{D}_T)$). Finally, from the bound (3.18) and the smoothness assumption $f_0 \in H^k_{\rm per}(\mathcal{D}_T)$ on the initial datum, we are able to apply Lemma 3.3.2, which proves that there exists a constant $C_k(M) > 0$, only depending on M, R, T and $\|f_0\|_{H^k_{\rm per}}$ such that

$$\|f(t)\|_{H^k_{\rm per}} \leq C_k(M).$$

This concludes the proof. ■

By iterating Proposition 3.3.3 and Lemma 3.3.2, we observe that uniform control on the $L^1(\mathcal{D}_T)$ norm on an arbitrarily large time interval $[0,\tau]$ together with smoothness on the initial datum will ensure existence and uniqueness of a smooth solution on this time interval $[0,\tau]$. Furthermore we observe that the control on the $L^1(\mathcal{D}_T)$ norm is obvious for the classical Boltzmann equation thanks to the positivity and mass preservations. Therefore, we shall now focus on the control of positivity of the solution, showing that the solution remains "almost positive" for arbitrarily large time interval, hence allowing to produce uniform control on the $L^1(\mathcal{D}_T)$ norm for arbitrarily large time interval.

We first state a technical lemma which takes advantage of the mixing property of $\mathcal{Q}^{R,+}$ in order to show spreading of the support of a characteristic function of a ball.

Lemma 3.3.4 *Let us consider a collision kernel B which satisfies the assumptions* (1.3)-(1.4)-(1.5) *and a truncated operator $\mathcal{Q}^R$ defined by (2.15) or (2.19). Then for all $0 < r < \sqrt{2}\,T$, we have*

$$\mathcal{Q}^{R,+}(\mathbf{1}_{\mathcal{B}(v,r)}) \geq C_0\, \mathbf{1}_{\mathcal{B}(v,\mu\, r)}$$

for some explicit $\mu = \mu(R,L) > 1$ and $C_0 > 0$.

Remark 3.3.5 *Note that for $r \geq \sqrt{2}\,T$, one has $\mathbf{1}_{\mathcal{B}(v,r)} = 1$ on the torus $[-T,T]^{d_v}$ and there is nothing to prove.*

Proof :[Proof of Lemma 3.3.4] The invariance by translation allows to reduce the proof to the case $v = 0$. The invariance by rotations implies that $I_r := \mathcal{Q}^{R,+}(\mathbf{1}_{\mathcal{B}(0,r)})$ is radially symmetric. More precisely, taking a $\mathcal{C}^\infty$ radially symmetric function ϕ such that $\phi > 0$ on $\mathcal{B} = \mathcal{B}(0,r)$ and $\phi \leq \mathbf{1}_\mathcal{B}$ on $\mathbb{R}^{d_v}$, we have

- the function $v \longrightarrow \mathcal{Q}^{R,+}(\phi)(v)$ is continuous,
- for all $v \in \mathbb{R}^{d_v}$, $I_r(v) \geq \mathcal{Q}^+(\phi)(v)$,
- for all $v \in \mathcal{B}$, $\mathcal{Q}^{R,+}(\phi)(v) > 0$.

As a consequence, for any ball $\mathcal{B}' = \mathcal{B}(0,r')$ strictly included in $\mathcal{B}$, there exists $\kappa_{r'} > 0$ such that $I_r \geq \kappa_{r'}\, \mathbf{1}_{\mathcal{B}'}$.

In order to conclude, we just need to estimate the support of I_r close to the ball $\mathcal{B}$.

Let us fix $r' \in (0, r)$ and choose $v', v'_\star \in \mathcal{B}'$ such that $|v'| = |v'_\star| = r'$ and $|v' - v'_\star| = \min\{R; \sqrt{2}r'\}$. Then taking σ in $\mathcal{B}'$ such that $v, v_\star, v', v'_\star$ is a square, we find $|v| = \sqrt{2}\,r'$ if $\sqrt{2}\,r' \leq R$ and

$$|v| = \left[(r')^2 - \frac{R^2}{4}\right]^{1/2} + \frac{R}{2}$$

else. We define

$$\mu_0 = \min_{R/\sqrt{2} \leq r' \leq \sqrt{2}\,T} \left[1 - \frac{R^2}{4(r')^2}\right]^{1/2} + \frac{R}{2r'} = \min_{R/(2\sqrt{2}T) \leq y \leq 1/\sqrt{2}} \left(\sqrt{1-y^2} + y\right) \in (1, \sqrt{2}).$$

This concludes the proof: we deduce that for any v such that $|v| \in (r', \mu_0\, r')$, we have

$$I_r(v) \geq \mathcal{Q}^+(\phi)(v) > 0$$

since ϕ is strictly positive in the neighborhood of the $v', v'_\star$ associated to v constructed above. Hence we deduce by taking $r' < r$ close to r that for any $0 < \mu < \mu_0$, we have

$$I_r \geq C(\mu, r)\, \mathbf{1}_{\mathcal{B}(0, \mu\, r)}$$

for some constant $C(\mu, r) > 0$ depending continuously on r. We can choose $\mu = (1 + \mu_0)/2$ for instance, and, for this choice of μ, we take

$$C_0 \;=\; \min_{0 \leq r \leq \sqrt{2}} C(\mu, r) \;>\; 0.$$

∎

Finally we establish the following positivity result on the solution to the perturbed problem (3.4):

Lemma 3.3.6 *Let us consider a collision kernel B which satisfies the assumptions* (1.3)-(1.4)-(1.5)*, a truncated operator $\mathcal{Q}^R$ defined by (2.15) or (2.19), and a sequence of smooth balanced perturbations $\left(\mathcal{P}_\varepsilon = \mathcal{P}_\varepsilon(f)\right)_{\varepsilon>0}$ which satisfy* (3.5)-(3.6)-(3.7).

We assume that f_0 is a non-negative function such that $f_0 \in H^k_{\mathrm{per}}(\mathcal{D}_T)$ with $k \in \mathbb{N}$ and $k > d_v/2$. Moreover, we define $M = 2\,\|f_0\|_{L^1}$ and $f_{0,\varepsilon}$ a smooth balanced perturbation of f_0, which is non necessarily positive and such that

$$\int_{\mathcal{D}_T} f_{0,\varepsilon} = \int_{\mathcal{D}_T} f_0 \quad \text{and} \quad \|f_0 - f_{0,\varepsilon}\|_{H^k_{\mathrm{per}}} \;\leq\; \psi(\varepsilon)$$

where $\psi(\varepsilon)$ goes to zero when ε goes to zero. We also set $\bar{\tau}, C_k(M) > 0$ the constant defined in Lemma 3.3.2 such that from Proposition 3.3.3 we have

$$\forall\, t \in [0, \bar{\tau}], \quad \|f(t)\|_{H^k_{\mathrm{per}}} \;\leq\; C_k(M).$$

Then, there exists $\hat{\tau} \in (0, \bar{\tau})$ which only depends on M, R and the collision kernel B, and there exists $\hat{\varepsilon} > 0$ which only depends on $\hat{\tau}$, $C_k(M)$, such that for all ε such that $0 < \varepsilon < \hat{\varepsilon}$ and for any smooth solution $H^k_{\mathrm{per}}(\mathcal{D}_T)$ to the perturbed Boltzmann equation (3.4) *with perturbed initial datum $f_{0,\varepsilon}$, we have*

$$\forall\, v \in \mathcal{D}_T, \quad f_\varepsilon(\hat{\tau}, v) > 0.$$

Moreover there exists $\eta(\varepsilon)$, which goes to 0 as ε goes to zero, such that the non-positive part of f satisfies

$$\|f^-(t)\|_{L^\infty} \;\leq\; \eta(\varepsilon), \quad t \in [0, \hat{\tau}], \tag{3.20}$$

Proof :[Proof of Lemma 3.3.6] Let $\bar{\tau} > 0$ be the length of the time interval for which there exists a smooth solution to the perturbed Boltzmann equation (3.4) with perturbed initial datum $f_{0,\varepsilon}$ such that (in the following we omit the subscript ε for the solution)

$$\|f(t)\|_{H^k_{\mathrm{per}}} \leq C_k(M), \quad t \,\in\, [0, \bar{\tau}].$$

We split the proof into three steps: first, we give a classical estimate on the loss term $\mathcal{L}^R(f)$, second we establish an estimate of f^- with respect to the amplitude of the perturbation $\mathcal{P}_\varepsilon(f)$, and third we use the spreading properties of $\mathcal{Q}^{R,+}(f)$ to prove that there exists $\hat{\tau} \in (0, \bar{\tau})$ such that

$$f(\hat{\tau}, v) > 0, \qquad v \in \mathcal{D}_T.$$

Step 1. Applying Proposition 3.3.3 for the control of $\|f(t)\|_{L^1}$ on the time interval $[0, \bar{\tau}]$, we get

$$\|\mathcal{L}^R(f)\|_{L^\infty} \;\leq\; C(R,B)\,\|f\|_{L^1} \;\leq\; C(R,B)\,M, \tag{3.21}$$

which gives for all $t \,\in\, [0, \tau_0]$, with $\tau_0 = \min\{\bar{\tau}, \ln 2/(M\,C(R,B))\}$

$$2 \,\geq\, e^{M\,C(R,B)\,t} \,\geq\, e^{-\int_0^t \mathcal{L}^R(f(s))\,ds} \,\geq\, e^{-\,M\,C(R,B)\,t} \,\geq\, \frac{1}{2}. \tag{3.22}$$

Step 2. Let us split f as $f = f^+ - f^-$, with $f^\pm = \max\{0, \pm f\}$ and use the monotonicity of $\mathcal{Q}^{R,+}$ for nonnegative distribution functions; it follows that

$$\begin{aligned} \mathcal{Q}^{R,+}(f) \;&=\; \mathcal{Q}^{R,+}\left(f^+ - f^-\right) \\ &\geq\; -\left[\mathcal{Q}^{R,+}\left(f^+, f^-\right) + \mathcal{Q}^{R,+}\left(f^-, f^+\right)\right]. \end{aligned} \tag{3.23}$$

On the time interval $[0, \tau_0]$, we apply Proposition 3.3.3 to estimate $\|f(t)\|_{L^1}$ and since $\|f^+(t)\|_{L^1} \leq \|f(t)\|_{L^1}$; we get from Lemma 3.3.1

$$\|\mathcal{Q}^{R,+}\left(f^+, f^-\right)\|_{L^\infty},\; \|\mathcal{Q}^{R,+}\left(f^-, f^+\right)\|_{L^\infty} \;\leq\; C_\infty(R,B)\,M\,\|f^-(t)\|_{L^\infty}, \tag{3.24}$$

which yields using (3.23)

$$\mathcal{Q}^{R,+}(f) \;\geq\; -2\,C_\infty(R,B)\,M\,\|f^-(t)\|_{L^\infty}. \tag{3.25}$$

Thus from the Duhamel representation of the solution f, we have for $v \in \mathcal{D}_T$,

$$\begin{aligned} f(t,v) &= f_{0,\varepsilon}(v)\, e^{-\int_0^t \mathcal{L}^R(f(s))(v)\,ds} \\ &\quad + \int_0^t \left[\mathcal{Q}^{R,+}(f(s)) + \mathcal{P}_\varepsilon(f(s))\right](v)\, e^{-\int_s^t \mathcal{L}^R(f(u))(v)\,du}\,ds, \\ &\geq -2\,\psi(\varepsilon) - \int_0^t \left[\mathcal{Q}^{R,+}(f(s)) + \mathcal{P}_\varepsilon(f(s))\right](v)\, e^{-\int_s^t \mathcal{L}^R(f(u))(v)\,du}\,ds. \end{aligned}$$

Hence, we get from the lower estimate (3.25) of $\mathcal{Q}^{R,+}(f)$ and the smallness assumption (3.7) of the perturbation $\mathcal{P}_\varepsilon(f)$, for all $v \in \mathcal{D}_T$

$$f^-(t,v) \;=\; \max\{0, -f(t,v)\} \;\leq\; 2\,\psi(\varepsilon) + 2\int_0^t \left(2\,C_\infty(R,B)\,M\,\|f^-(s)\|_{L^\infty} + \varphi(\varepsilon)\right)\,ds.$$

Finally, we take the supremum in $v \in \mathcal{D}_T$ and apply Gronwall's lemma to get for any $0 \leq \tau \leq \tau_0$

$$\|f^-(t)\|_{L^\infty} \;\leq\; \left(2\,\psi(\varepsilon) + 2\,\tau_0\,\varphi(\varepsilon)\right)\, e^{4\,C_\infty(R,B)\,M\,t} := \eta(\varepsilon), \quad t \in [0,\tau_0],$$

which proves (3.20) on the time interval $[0,\tau_0]$.

Step 3. Let us prove that there exists $\hat{\tau} \in (0,\tau_0)$ such that

$$f(\hat{\tau}, v) > 0, \quad v \in \mathcal{D}_T.$$

We start again with the Duhamel representation of the solution

$$\begin{aligned} f(t,v) &= f_{0,\varepsilon}(v)\, e^{-\int_0^t \mathcal{L}^R(f(s))(v)\,ds} \\ &\quad + \int_0^t \left[\mathcal{Q}^{R,+}(f(s)) + \mathcal{P}_\varepsilon(f(s))\right](v)\, e^{-\int_s^t \mathcal{L}^R(f(u))(v)\,du}\,ds, \end{aligned}$$

but we now take into account the fact that the first term is essentially positive and we use the spreading property of the operator $\mathcal{Q}^{R,+}$ (Lemma 3.3.4).

On the one hand since the initial datum is smooth enough ($k > d_v/2$ is large enough such that $f_{0,\varepsilon}$ is Hölder), there exists an explicit $\delta > 0$ depending on $C_k(M)$ such that for ε small enough, there exists $v_0 \in \mathcal{D}_T$ such that

$$f_{0,\varepsilon}(v) \;\geq\; \frac{\eta}{2}\,\mathbf{1}_{\mathcal{B}(v_0,\delta)}(v) - \psi(\varepsilon), \quad \text{with } \eta \;=\; \frac{\|f_0\|_{L^1}}{(2L)^{d_v}}.$$

On the other hand, using the lower bound on the gain operator (3.25) and the estimate (3.20) of $f^-(t)$, it gives

$$\mathcal{Q}^{R,+}(f) \;\geq\; -2\,C_\infty(R,B)\,M\,\eta(\varepsilon)$$

Finally, using (3.22) and the smallness assumption on $\mathcal{P}_\varepsilon(f)$, it first yields for any $0 < \tau < \tau_0$

$$f(t,v) \;\geq\; A_0\,\mathbf{1}_{\mathcal{B}(v_0,\delta)}(v) \;-\; \varphi_1(\varepsilon),$$

with

$$A_0 := \frac{\eta}{4}, \qquad \varphi_0(\varepsilon) := 2\,\tau_0\,\big(2\,C_\infty(R,B)\,M\,\eta(\varepsilon) + \varphi(\varepsilon)\big) \;+\; \psi(\varepsilon). \tag{3.26}$$

Now we choose $\varepsilon_1 > 0$ small enough such that

$$0 \;<\; \frac{A_0}{2} \;\le\; A_0 - \varphi_0(\varepsilon_1) \quad\Longleftrightarrow\quad \varphi_0(\varepsilon_1) \;\le\; \frac{A_0}{2}.$$

Thus, we get on any time interval $[0,\tau] \subset [0,\tau_0]$

$$f^+(t,v) \;\ge\; \frac{A_0}{2}\,\mathbf{1}_{\mathcal{B}(v_0,\delta)}(v).$$

Hence, using the spreading properties of the operator $\mathcal{Q}^{R,+}(f)$ of Lemma 3.3.4 and the monotonicity of $\mathcal{Q}^{R,+}$ for nonnegative distribution functions, it follows that

$$\begin{aligned}\mathcal{Q}^{R,+}(f^+) &\;\ge\; \frac{A_0^2}{4}\,\mathcal{Q}^{R,+}\big(\mathbf{1}_{\mathcal{B}(v_0,\delta)}\big)\\ &\;\ge\; \frac{A_0^2}{4}\,C_0\,\mathbf{1}_{\mathcal{B}(v_0,\mu\,\delta)}.\end{aligned}$$

Next, we again use the uniform bounds previously established in (3.24) on $\mathcal{Q}^{R,+}(f^+,f^-)$ and $\mathcal{Q}^{R,+}(f^-,f^+)$:

$$\begin{aligned}\mathcal{Q}^{R,+}(f) &\;\ge\; \mathcal{Q}^{R,+}(f^+) - \mathcal{Q}^{R,+}(f^-,f^+) - \mathcal{Q}^{R,+}(f^+,f^-)\\ &\;\ge\; \frac{A_0^2}{4}\,C_0\,\mathbf{1}_{\mathcal{B}(v_0,\mu\,\delta)} - 2\,C_\infty(R,B)\,M\,\big(\eta(\varepsilon)+\varphi_0(\varepsilon)\big).\end{aligned}$$

Finally, from the smallness assumption (3.7) of the perturbation $\mathcal{P}_\varepsilon(f)$, it yields for $t \in [\tau/2,\tau]$ and with (3.22)

$$\begin{aligned}f(t,v) &\;\ge\; -\psi(\varepsilon) + \int_0^t \big[\mathcal{Q}^{R,+}(f(s)) + \mathcal{P}_\varepsilon(f(s))\big]\,(v)\,e^{-\int_s^t \mathcal{L}^R(f(u))(v)\,du}\,ds,\\ &\;\ge\; \frac{\tau}{2}\,\frac{A_0^2}{4}\,C_0\,\mathbf{1}_{\mathcal{B}(v_0,\mu\,\delta)} -\, 2\,\tau\,\big[2\,C_\infty(R,B)\,M\,\big(\eta(\varepsilon)+\varphi_0(\varepsilon)\big) \;+\; \varphi(\varepsilon)\big] - \psi(\varepsilon)\\ &\;=\; A_1\,\mathbf{1}_{\mathcal{B}(v_0,\mu\,\delta)} \;-\; \varphi_1(\varepsilon)\end{aligned}$$

with

$$A_1 = \frac{\tau}{8}\,A_0^2\,C_0, \qquad \varphi_1(\varepsilon) := 2\,\tau_0\,\big[2\,C_\infty(R,B)\,M\,\big(\eta(\varepsilon)+\varphi_0(\varepsilon)\big) \;+\; \varphi(\varepsilon)\big] - \psi(\varepsilon).$$

Now, we proceed by induction: assume that there exists $(A_j,\varepsilon_j,\varphi_j)$ such that on the time interval $[\tau - \tau/2^j, \tau] \subset [0,\tau_0]$ and for $\varepsilon \in (0,\varepsilon_j)$, we have

$$f(t,v) \;\ge\; A_j\;\mathbf{1}_{\mathcal{B}(v_0,\mu^j\,\delta)}(v) \;-\; \varphi_j(\varepsilon),$$

where

$$A_j := \left(\frac{\tau}{8}\right)^{2^j-1} A_0^{2^j}\, C_0^{2^{j-1}-1}$$

and $\varphi_j(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Using the same method as before, we first set ε_{j+1} such that

$$f^+(t,v) \;\geq\; \frac{A_j}{2}\mathbf{1}_{\mathcal{B}(v_0,\mu^j\,\delta)}$$

and prove that for $t \in [\tau - \frac{\tau}{2^{j+1}}, \tau]$

$$\mathcal{Q}^{R,+}(f) \;\;\geq\;\; \left(\frac{A_j}{2}\right)^2 C_0\, \mathbf{1}_{\mathcal{B}(v_0,\mu^{j+1}\,\delta)} - \, 2\, C_\infty(R,B)\, M\, \big(\eta(\varepsilon) + \varphi_j(\varepsilon)\big).$$

Then, from the Duhamel formula and the smallness assumption of $\mathcal{P}_\varepsilon(f)$, we finally get the following lower bound

$$f(t,v) \;\geq\; A_{j+1}\, \mathbf{1}_{\mathcal{B}(v_0,\mu^{j+1}\,\delta)} \;-\; \varphi_{j+1}(\varepsilon),$$

with

$$A_{j+1} = \frac{\tau}{8}\, A_j^2\, C_0, \qquad \varphi_{j+1}(\varepsilon) := 2\,\tau_0\, \left[2\, C_\infty(R,B)\, M\, \big(\eta(\varepsilon) + \varphi_j(\varepsilon)\big) \;+\; \varphi(\varepsilon)\right] - \psi(\varepsilon).$$

Since $\mu > 1$, the ball $\mathcal{B}(v_0, \mu^j\,\delta)$ eventually recovers the periodic box $[-L, L]^{d_v}$ *i.e.*, for some J large enough: $[-L, L]^{d_v} \subset \mathcal{B}(v_0, \mu^J\,\delta)$, and for all $t \in [\tau - \frac{\tau}{2^J}, \tau]$, by applying J's times the previous induction we get for $\varepsilon \in (0, \varepsilon_J)$:

$$f(t,v) \;\geq\; A_J\, \mathbf{1}_{\mathcal{B}(v_0,\mu^J\,\delta)}(v) \;-\; \varphi_J(\varepsilon).$$

Finally, up to reducing ε further, we have proved that there exists $(\hat{\tau}, \hat{\varepsilon})$ which only depend on the collision kernel B, the initial datum f_0, L and the perturbation function $\varphi = \varphi(\varepsilon)$ such that for all $0 < \varepsilon < \hat{\varepsilon}$,

$$\forall\, v \in \mathcal{D}_T, \quad f(\hat{\tau}, v) > 0.$$

■

3.3.2 Existence and regularity on a bounded time interval

Proposition 3.3.7 *Let us consider a fixed time $\tau > 0$, a collision kernel B which satisfies the assumptions* (1.3)-(1.4)-(1.5)*, a truncation $\mathcal{Q}^R$ defined by (2.15) or (2.19), and a sequence of smooth balanced perturbations $\big(\mathcal{P}_\varepsilon = \mathcal{P}_\varepsilon(f)\big)_{\varepsilon>0}$ which satisfy* (3.5)-(3.6)-(3.7).

We assume that f_0 is a non-negative function, not zero everywhere, and such that $f_0 \in H^k_{\rm per}(\mathcal{D}_T)$ with $k \in \mathbb{N}$ and $k > d_v/2$. We define $M = 2\,\|f_0\|_{L^1}$ and $(f_{0,\varepsilon})_{\varepsilon>0}$ a sequence of smooth perturbations of f_0 (which is non necessarily positive) such that

$$\int_{\mathcal{D}_T} f_{0,\varepsilon} = \int_{\mathcal{D}_T} f_0 \quad \textit{and} \quad \|f_0 - f_{0,\varepsilon}\|_{H^k_{\rm per}} \;\leq\; \psi(\varepsilon)$$

where $\psi(\varepsilon)$ goes to zero when ε goes to zero.

Then, there exists $\hat{\varepsilon} > 0$, which only depends on the $H^k_{\mathrm{per}}(\mathcal{D}_T)$ and $L^1(\mathcal{D}_T)$ norms of f_0, such that for all $\varepsilon \in (0, \hat{\varepsilon})$,

(i) *there is a unique smooth solution $f_\varepsilon = f_\varepsilon(t, \cdot)$ on $[0, \tau]$ to the perturbed equation* (3.4) *with initial datum $f_{0,\varepsilon}$;*

(ii) *this belongs to $H^k_{\mathrm{per}}(\mathcal{D}_T)$ (with bound growing at most exponentially);*

(iii) *there is some explicit $\eta(\varepsilon) > 0$ (with $\eta(\varepsilon) \to 0$ as ε goes to 0) such that the non-positive part is $\eta(\varepsilon)$-small:*

$$\forall\, t \in [0, \tau], \quad \|f_\varepsilon^-(t, \cdot)\|_{L^\infty} \le \eta(\varepsilon)$$

where f_ε^- denotes $\max\{0, -f_\varepsilon\}$;

(iv) *this solution satisfies for any $p < k$*

$$\forall\, t \in [0, \tau], \quad \|f(t, \cdot) - f_\varepsilon(t, \cdot)\|_{H^p_{\mathrm{per}}} \;\le\; \bar{\varphi}(\varepsilon)$$

where $f(t, \cdot)$ is the solution of unperturbed periodized Boltzmann equation (3.8), and $\bar{\varphi}(\varepsilon)$ is another explicit function which goes to zero as ε goes to zero. Hence up to reducing ε, the perturbed solution remains close to the unperturbed solution on the finite time interval on which we have constructed it.

Proof : We set

$$M_k(\tau) := \|f_0\|_{H^k_{\mathrm{per}}}\, e^{C\,M\,\tau}. \tag{3.27}$$

First, applying Proposition 3.3.3 we have proven that there exists a small $\bar{\tau} > 0$ such that the perturbed Boltzmann equation (3.4) admits an unique smooth solution on the time interval $[0, \bar{\tau}]$ with

$$\|f_\varepsilon(t)\|_{L^1} \;\le\; M$$

and

$$\|f_\varepsilon(t)\|_{H^k_{\mathrm{per}}} \;\le\; M_k(\bar{\tau}) \le M_k(\tau).$$

Moreover from Lemma 3.3.6, there exist $\hat{\tau} \le \bar{\tau}$ and $\hat{\varepsilon} > 0$, only depending on M, $M_k(\bar{\tau}) < M_k(\tau)$, R and the collision kernel B such that for all $0 < \varepsilon < \hat{\varepsilon}$,

$$\forall\, v \in \mathcal{D}_T, \quad f_\varepsilon(\hat{\tau}, v) > 0,$$

$$\|f^-(t)\|_{L^\infty} \;\le\; \eta(\varepsilon), \quad t \in [0, \hat{\tau}].$$

and

$$\forall\, t \in [0, \hat{\tau}], \quad \|f(t)\|_{H^k_{\mathrm{per}}} \le \|f_0\|_{H^k_{\mathrm{per}}}\, e^{C\,M\,2\,\hat{\tau}}.$$

Then, from the preservation of mass under the action of $\mathcal{Q}^R$ and $\mathcal{P}_\varepsilon$:

$$\int_{\mathcal{D}_T} \mathcal{Q}^R(f_\varepsilon)(v)\, dv \;=\; \int_{\mathcal{D}_T} \mathcal{P}_\varepsilon(f_\varepsilon)(v)\, dv \;=\; 0,$$

we have that

$$\int_{\mathcal{D}_T} f_\varepsilon(\hat\tau, v)\, dv \;=\; \int_{\mathcal{D}_T} f_0(v)\, dv.$$

Since f_0 is a nonnegative function, it gives that at time $\hat\tau$

$$\|f_\varepsilon(\hat\tau)\|_{L^1} \;=\; \|f_0\|_{L^1},$$

and on the time interval $t \,\in\, [0, \hat\tau]$ we have

$$\|f_\varepsilon(t)\|_{L^1} \;\leq\; M \quad \text{and} \quad \|f_\varepsilon(t)\|_{H^k_{\text{per}}} \;\leq\; M_k(\tau).$$

Therefore, we consider the perturbed Boltzmann equation (3.4) starting from $f_\varepsilon(\hat\tau)$ as initial data. On the time interval $[\hat\tau, 2\,\hat\tau]$, we apply Proposition 3.3.3 and get that

$$\|f_\varepsilon(t)\|_{L^1} \;\leq\; 2\,\|f_\varepsilon(\hat\tau)\|_{L^1} \;=\; 2\,\|f_0\|_{L^1} \;=\; M, \quad \forall\, t \,\in\, [\hat\tau, 2\,\hat\tau]$$

and

$$\begin{aligned}
\|f_\varepsilon(t)\|_{H^k_{\text{per}}} &\leq \|f_\varepsilon(\hat\tau)\|_{H^k_{\text{per}}}\, e^{C\,M\,(2\,\hat\tau\,-\,\hat\tau)} \\
&\leq \|f_0\|_{H^k_{\text{per}}}\, e^{C\,M\,2\,\hat\tau} \\
&\leq \|f_0\|_{H^k_{\text{per}}}\, e^{C\,M\,\tau} \;=\; M_k(\tau), \quad \forall\, t \,\in\, [\hat\tau, 2\,\hat\tau].
\end{aligned}$$

Moreover, since $\hat\tau$ only depends on M, $M_k(\tau)$, B and R, we can again apply Lemma 3.3.6 on the time interval $[\hat\tau, 2\,\hat\tau]$, which yields that

$$f_\varepsilon(2\,\hat\tau) > 0.$$

We finally proceed by induction to prove existence and uniqueness of a smooth solution f_ε of the perturbed Boltzmann equation (3.4) on the time interval $[0, \tau]$, which proves assertions (i), (ii) and (iii).

To prove (iv), we compute the difference between the solution $f(t)$ to the unperturbed problem (3.8) and the solution $f_\varepsilon(t)$ to (3.4):

$$\frac{\partial (f - f_\varepsilon)}{\partial t} = \frac{1}{2}\left(\mathcal{Q}^R(f - f_\varepsilon, f + f_\varepsilon) + \mathcal{Q}^R(f + f_\varepsilon, f - f_\varepsilon)\right) + \mathcal{P}_\varepsilon(f_\varepsilon).$$

Then, using the smoothness of f and f_ε, we have from Lemma 3.3.2 for any $p < k$

$$\|\mathcal{Q}^R(f - f_\varepsilon, f + f_\varepsilon)\|_{H^p_{\text{per}}},\; \|\mathcal{Q}^R(f + f_\varepsilon, f - f_\varepsilon)\|_{H^p_{\text{per}}} \;\leq\; C_p(M)\, \|f + f_\varepsilon\|_{H^p_{\text{per}}}\, \|f - f_\varepsilon\|_{H^p_{\text{per}}}$$

and since the perturbation is small (assumption (3.7))

$$\|\mathcal{P}_\varepsilon(f)\|_{H^p_{\text{per}}} \;\leq\; \varphi(\varepsilon),$$

it yields that for all $t \,\in\, [0, \tau]$

$$\|f(t) - f_\varepsilon(t)\|_{H^p_{\text{per}}} \;\leq \bar\varphi(\varepsilon),$$

for some function $\bar\varphi(\varepsilon)$ going to zero as ε goes to zero. ∎

3.4 Application: stability and convergence of spectral methods

In this section we consider the following spectral approximation of (3.8)

$$\frac{\partial f_N}{\partial t} = \mathcal{P}_N \, \mathcal{Q}^R(f_N),$$

where $\mathcal{P}_N$ denotes the orthogonal projection on $\mathcal{P}_N$ in $L^2(\mathcal{D}_T)$ (the space of trigonometric polynomials with degree less at most N in each direction).

The goal of this section is to prove the following theorem:

Theorem 3.4.1 *Consider any nonnegative initial datum $f_0 \in H^k_{\rm per}(\mathcal{D}_T)$, with $k > d_v/2$, which is not zero everywhere. Then there exists $N_0 \in \mathbb{N}$ (depending on the mass and $H^k_{\rm per}(\mathcal{D}_T)$ norm of f) such that for all $N \geq N_0$:*

(i) *there is a unique global solution $f_N = f_N(t, \cdot)$ to the following problem*

$$\begin{cases} \dfrac{\partial f_N}{\partial t} = \mathcal{P}_N \, \mathcal{Q}^R(f_N), \quad \forall t \in [0, \tau], \\ \\ f_N(t = 0) = \mathcal{P}_N f_0; \end{cases} \tag{3.28}$$

(ii) *for any $p < k$, there exists $C > 0$ such that*

$$\forall\, t \geq 0, \quad \|f_N(t, \cdot)\|_{H^k_{\rm per}} \;\leq\; C; \quad \forall t \in [0, \tau];$$

(iii) *this solution is nor positive everywhere but the mass of its negative values can be made uniformly small as $N \to \infty$ for all time $t \in [0, \tau]$;*

(iv) *this solution f_N converges to $f(t)$ the solution to (3.8) with the spectral accuracy, globally in time;*

To prove Theorem 3.4.1, we want to apply Theorem 3.2.1 with the perturbation

$$\mathcal{P}^R_N(f_N) := \mathcal{P}_N \, \mathcal{Q}^R(f_N) \, - \, Q^R(f_N),$$

which preserves the mass:

$$\int_{\mathcal{D}_T} \mathcal{P}^R_N(f_N) \, dv \;=\; \int_{\mathcal{D}_T} \big(\mathcal{P}_N \, \mathcal{Q}^R(f_N) \, - \, \mathcal{Q}^R(f_N)\big) \, dv \;=\; 0,$$

In the next Lemma, we prove a consistency and smoothness result for this approximation.

Lemma 3.4.2 *Consider a nonnegative function $f \in H^k_{\rm per}(\mathcal{D}_T)$, with $k > d_v/2$, which is not zero everywhere. Then, there exists $C > 0$ depending only on the collision kernel B and the truncation such that for all $l \geq 0$ we have*

$$\begin{cases} \|\mathcal{P}^R_N(f)\|_{L^2} \;\leq\; C\, \|f\|_{L^1}\, \|f\|_{L^2}, \\ \\ \|\mathcal{P}^R_N(f)\|_{H^{l+1}_{\rm per}} \;\leq\; C\, \left(\|f\|^2_{H^l_{\rm per}} \,+\, \|f\|_{L^1}\, \|f\|_{H^{l+1}_{\rm per}} \right). \end{cases} \tag{3.29}$$

Moreover, for all $l \,\in\, [0, k]$

$$\|\mathcal{P}^R_N(f)\|_{H^l_{\rm per}} \;\leq\; C\, \frac{\|f\|^2_{H^k_{\rm per}}}{N^{k-l}}. \tag{3.30}$$

Proof : First, we split the operator $\mathcal{P}_N$ as

$$\|\mathcal{P}^R_N(f)\|_{H^l_{\rm per}} \leq \|\mathcal{Q}^R(f)\|_{H^l_{\rm per}} + \|\mathcal{P}_N\, \mathcal{Q}^R(f)\|_{H^l_{\rm per}}.$$

From the Parseval identity for a smooth function , $\mathcal{Q}^R(f)$ and apply Lemma 3.3.1 with $p = 2$, we have

$$\|\mathcal{P}_N\, \mathcal{Q}^R(f)\|_{L^2} \;\leq\; \|\mathcal{Q}^R(f)\|_{L^2} \;\leq\; C_2(R,B)\, \|f\|_{L^1}\, \|f\|_{L^2}.$$

Then we have

$$\|\mathcal{P}^R_N(f)\|_{L^2} \;\leq\; C\, \|f\|_{L^1}\, \|f\|_{L^2}.$$

On the other hand, we prove the second estimate in (3.29). As in the proof of Lemma 3.3.2 we get that for all $p \,\in\, [0, k]$

$$\|\mathcal{Q}^R(f)\|_{H^{l+1}_{\rm per}} \;\leq\; C(R,B)\, \left(\|f\|^2_{H^l_{\rm per}} \,+\, \|f\|_{L^1}\| \, \|f\|_{H^{l+1}_{\rm per}} \right) \tag{3.31}$$

Concerning the interpolation error estimate, the following result holds. If $u \in H^l_{\rm per}(\mathcal{D}_T)$ for some $l \geq 1$, then

$$\|u - \mathcal{P}_N u\|_{H^{l+1}_{\rm per}} \leq \frac{C}{N^{k-(l+1)}}\, \|u\|_{H^k_{\rm per}}. \tag{3.32}$$

Then, taking $k = l + 1$ in the latter inequality and from (3.31) we obtain

$$\|\mathcal{P}_N \mathcal{Q}^R(f)\|_{H^{l+1}_{\rm per}} \;\leq\; \|\mathcal{Q}^R(f)\|_{H^{l+1}_{\rm per}} \;\leq\; C_l(R,B)\, \left(\|f\|^2_{H^l_{\rm per}} \,+\, \|f\|_{L^1}\, \|f\|_{H^{l+1}_{\rm per}} \right). \tag{3.33}$$

Gathering (3.31) and (3.33), we finally get

$$\|\mathcal{P}^R_N(f)\|_{H^{l+1}_{\rm per}} \;\leq\; C_l(R,B)\, \left(\|f\|^2_{H^l_{\rm per}} \,+\, \|f\|_{L^1}\, \|f\|_{H^l_{\rm per}} \right).$$

Moreover, using again the error estimate (3.32), it yields

$$\begin{aligned}\|\mathcal{P}_N^R(f)\|_{H^l_{\rm per}} &\leq C\,\frac{\|\mathcal{Q}^R(f)\|_{H^k_{\rm per}}}{N^{k-l}}\\ &\leq C(R,B)\,\frac{\|f\|^2_{H^k_{\rm per}}}{N^{k-l}}.\end{aligned}$$

■

3.4.1 Proof of Theorem 3.4.1

Consider the numerical solution f_N given by solving (3.28). We can formulate the problem as a perturbation of the truncated Boltzmann equation. Indeed setting

$$\mathcal{P}_N(f_N) = -(Id - \mathcal{P}_N)\mathcal{Q}^R(f_N),$$

the problem (3.28) can be written as

$$\frac{\partial f_N}{\partial t} = \mathcal{Q}^R(f_N) + \mathcal{P}_N(f_N).$$

Then, applying Lemma 3.4.2, the perturbation $\mathcal{P}_N$ satisfies the assumptions of Theorem 3.2.1. Moreover since $f_0 \in H^k_{\rm per}(\mathcal{D}_T)$, we have straightforwardly

$$\|f_N(0)\|_{H^p_{\rm per}} \leq \|f_0\|_{H^p_{\rm per}}, \quad \|f_N(0) - f_0\|_{H^p_{\rm per}} \to 0.$$

Hence, we can directly apply Theorem 3.2.1 to the perturbation $\mathcal{P}_N$, which proves that there exists N_0 large enough and only depending on f_0, the kernel B and the truncation, such that for all $N \geq N_0$, the perturbed system admits a unique uniformly smooth solution, which converges to a constant, and satisfies all the points in Theorem 3.2.1.