Theory and applications of invariants in Alfred Clebsch's papers

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"One may well pose as the task of algebra in the most general way the problem of searching for the properties of homogeneous functions that are preserved by any univocal algebraic transformation"[1] wrote Alfred Clebsch, then professor of mathematics at the University of Göttingen, in a presentation of his 1872 book *Theorie der binären algebraischen Formen*, [Clebsch 1872a,b]. Although this task was actually too broad to be handled in full generality, Clebsch added, important research had been developed in the past decades in the restricted case where the considered transformations are linear:

> Only such [linear transformations] were to be imposed on the variables; formations which remained unchanged in this process or were only changed by a characteristic factor that could be easily specified became the object of the new theory under the name of *invariants* and *covariants*.[2]

This theory, which was also referred to as "the so-called modern algebra, the imperishable creation of Cayley and Sylvester",[3] is what was called in other places the theory of invariants.

As is alluded to in the preceding lines, in the 19th century invariants and covariants were specific objects that were associated with an algebraic form, that is, a homogeneous polynomial. Given such a form f, an invariant is a

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[1]"Man darf wohl als die Aufgabe der Algebra in allgemeinster Weise das Problem hinstellen, die Eigenschaften der homogenen Functionen zu suchen, welche bei beliebigen eindeutigen algebraischen Umformungen derselben erhalten bleiben." [Clebsch 1872a, p. 323].

[2]"Nur solche [lineare Transformationen] sollten den Veränderlichen auferlegt werden; Bildungen, welche hierbei unverändert blieben oder doch nur um einen leicht anzugebenden characteristischen Factor verändert wurden, werden unter dem Namen der *Invarianten* und *Covarianten* der Gegenstand der neuen Theorie." [Clebsch 1872a, pp. 323–324].

[3]"[D]ie sogenannte neuere Algebra, die unvergängliche Schöpfung Sylvester's und Cayley's [...]" [Clebsch 1872a, p. 323]. I translated "neuere" by "modern" in reference to George Salmon's *Lessons Introductory to Modern Higher Algebra*, [Salmon 1866]. In particular, what is called "neuere Algebra" by Clebsch and "modern algebra" by Salmon is not to be mistaken with the modern algebra of the beginning of the 20th century, marked with the advent of structures. On the latter, see [Corry 2004].

homogeneous polynomial in the coefficients of f that remains essentially the same when f undergoes linear transformations. A covariant has the same property of immutability but is a polynomial in both the coefficients and the variables of the form.[4]

Clebsch's book, which was supposed to synthesize many pieces of the knowledge on forms and invariants and to reach a wide audience [Clebsch 1872a, p. 321], testifies in itself to the commitment of its author to invariant theory. Other clues of the time go in the same direction: the large part that the authors of his scientific obituary devoted to invariant theory, recognised as one of the six mathematical domains that Clebsch had researched [Brill et al. 1873], or his use of Carl Borchardt's alleged despise for invariant theory as one of the reasons for founding, together with Carl Neumann and against the sacred *Journal für die reine und angewandte Mathematik*, the *Mathematische Annalen*, a journal that would be welcoming to this topic.[5]

That Clebsch has been a notable contributor to invariant theory can also be seen in the classical narratives on the theory, where his name occurs recurrently [Fisher 1966; Crilly 1986; Parshall 1989]. In fact, Clebsch appears in such sources mainly for one major achievement: the generalisation in 1861 of the symbolic representation of invariants, of which a first version had been proposed by Siegfried Aronhold in 1858, and which has been used by Paul Gordan later, in 1868, for proving the celebrated finiteness theorem.[6]

The provider of an intermediate, yet important result, Clebsch thus appears as one chain in the sequence of events leading to the finiteness theorem. Little is known, however, on the reasons why Clebsch developed the symbolic representation, and, more generally, on his incentive to work on invariant theory. In particular, if Clebsch's crossed interests in this theory and in projective geometry are mentioned here and there in the historical literature, the effective articulation between the two domains is hardly understood yet.

The present chapter investigates Clebsch's research connected to invariant theory. After an overview of the related corpus and a presentation of some mathematical material on forms and invariants, three selected works are scrutinized to appreciate the role of invariants and covariants in Clebsch's technique. Without entering in the mathematical details to the same extent, other papers are then described because they reveal an intriguing facet of Clebsch's practice of invariants, consisting in linking invariant theory with the question of writing down equations explicitly. All these works showcase different configurations between invariant theory and other mathematical domains that are imple-

[4]More precise definitions will be provided below, together with examples.

[5]See a letter from Clebsch to Wilhelm Fiedler dated 17th October 1868, [Confalioneri, Schmidt, and Volkert 2019, p. 74].

[6]This theorem states that the invariants and covariants associated with a form with two variables can all be expressed algebraically from a finite number of them. David Hilbert extended the theorem to forms with any number of variables two decades later. On Hilbert and this theorem, see [Boniface 2004, ch. 2].

mented around a wide range of applications of invariants. For this reason, and since most of the usual accounts of invariant theory are structured around the finiteness theorem, which concerns aspects that are internal to the theory, the analysis proposed in this chapter constitutes an important complement for our historical understanding of this theory.[7]

The final section expands on what can be seen as traces of a program on forms and invariants that Clebsch began to draw up a few months before his unforeseen death at the end of 1872. This program contains elements moving towards an unification of parts of algebra and geometry by the means of algebraic forms. This general scope and the year 1872 echoing spontaneously Felix Klein's *Erlanger Programm*, a comparison between the two programs is eventually proposed.

1. Invariants and covariants in Clebsch's papers

According to the publication list included in his scientific obituary and the *Jahrbuch über die Fortschritte der Mathematik*, 99 distinct papers written by Clebsch have ever been published.[8] These papers, whose publication years range from 1856 to 1873,[9] include two book reviews and one obituary. The 96 remaining ones are research articles in the narrower sense of the term, six of which have been published jointly with Gordan.

To delimit a sub-corpus on invariant theory can be achieved in several ways. Let me present and confront two of them.

The disciplinary classifications of the time offer a first avenue. In Clebsch's case, it is convenient to use the *Catalogue of scientific papers*, which, at the beginning of the 20th century, inventoried papers of the 19th century and distributed them into disciplinary divisions.[10] Invariant theory does not appear in the names of these divisions but the part on "Algebra and theory of numbers" contains a section on "Linear substitutions" which, in turn, encloses four subsections devoted to forms. These subsections contain 19 papers of Clebsch, of which three have been published in 1861, and the others from 1867 to 1873, with a gap in 1868.

[7] Other recent papers have studied the use of forms and invariants in arithmetic, algebra and geometry. See [Brechenmacher 2011; Goldstein 2023; Parshall 2023] as well as other references given below.

[8] This number counts for one each of the three pairs of duplicates (up to orthographic and typographic conventions) that appeared both in *Mathematische Annalen* and in *Nachrichten von der königlichen Gesellschaften und der Georg-August-Universität*. The *Jahrbuch*, which started in 1868, allows to spot two papers that are forgotten in the publication list [Brill et al. 1873, pp. 51–55].

[9] Four papers have been published in 1873, thus posthumously.

[10] The *Catalogue* thus covers the whole period of publication of Clebsch, unlike the *Jahrbuch*. As for the *Répertoire bibliographique des sciences mathématiques*, it counts only 60 papers of Clebsch (see http://sites.mathdoc.fr/RBSM/).

Strikingly, this chronological break is partially overlaid to a feature related to the content of these papers. The three ones of 1861, indeed, include applications of invariant theory to geometry and elimination theory, whereas it is the case for only the quarter of those published during the period 1867–1873. Papers concerning solely questions related to invariants thus appear quite late in Clebsch's time-line. They count for instance the article [Clebsch and Gordan 1867/1868], devoted to the so-called "typical representation" of forms, or [Clebsch 1870], where Clebsch establishes, thanks to invariants, a condition for two binary forms to be linearly transformable one into the other. Conversely, the paper on the famous symbolic representation of forms and invariants is one of those published in 1861. It contains applications to both elimination theory and geometry, [Clebsch 1861e].

Such links between invariant theory and other mathematical domains naturally suggests to search for invariants in other publications of Clebsch. Starting again from his 96 papers, the systematic search for the words "invariant" and "covariant" isolates 45 of them.[11] This is remarkably more than the 19 preceding ones, which are included in these 45. The publication years range again from 1861 to 1873, but no break is to be observed: even if 1861 is a peak for the number of papers on invariants and 1867 marks an increasing of such publications, articles involving invariants appear in between, as well as in 1868 (see figure 1).

Moreover, the year 1861 is significant from the viewpoint of Clebsch's general chronology: this is exactly the moment when he began to publish massive numbers of papers on algebraic curves and surfaces, and to abandon the topics related to mathematical physics and the calculus of variations in which he had been mainly involved since his 1854 doctoral dissertation.[12] In other words, invariant theory is tied to curves and surfaces from the very beginning of Clebsch's systematic research on the latter.

This observation is reflected in the *Catalogue* indexation, since the half of the papers published in 1861 fall under the scope of geometry. More generally, the 45 papers involving invariants and covariants are classified as follows.[13] First, among the 19 papers contained in a division concerning forms, four possess a double classification: one is also present in the division related to elimination (which belongs to the section on linear substitutions), one appears

[11] Only two papers (which are not counted among these 45) deal with invariants that do not have the same technical meaning as the ones of invariant theory: [Clebsch 1871c, p. 491] defines the invariants of a Cremona transformation and [Clebsch 1872c, p. 18] evokes the "higher invariants" of algebraic surfaces.

[12] Only one paper published before 1861 deals with a problem coming from projective geometry, [Clebsch 1857]. Let me add that 1861 does not correspond to an institutional change in Clebsch's career: he taught mathematics at the *Polytechnische Schule* of Karlsruhe between 1858 and 1863, then moved to the University of Giessen, where he worked in particular with Gordan. In 1868, he was eventually appointed to the University of Göttingen. See [Brill et al. 1873].

[13] I could not find one of these papers in the *Catalogue*.

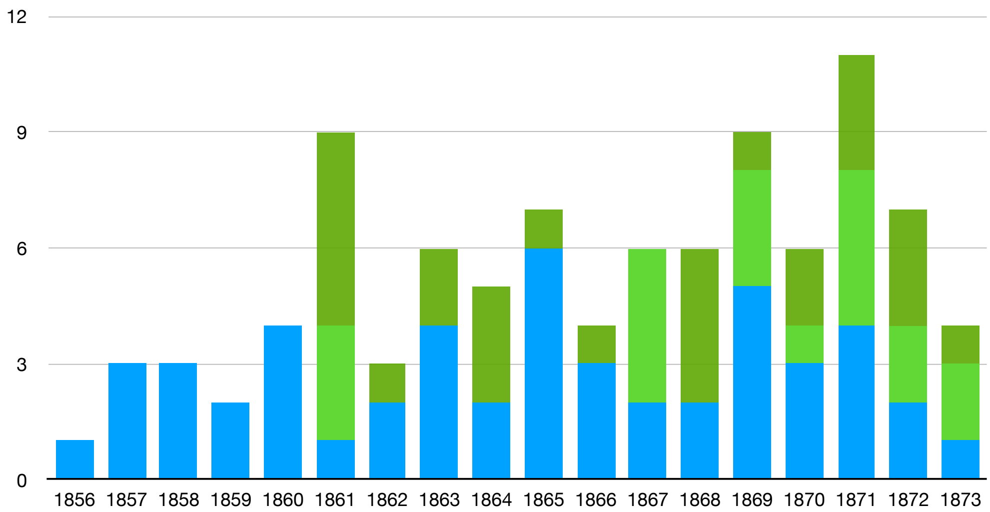

Figure 1: Chronological distribution of Clebsch's 96 published papers. The 45 papers dealing with invariants and covariants are in dark green and light green, the latter standing for those that are classified into the theory of forms in the *Catalogue*. The color blue represents the other papers.

in the part on geometry and two others are in the part on analysis. Among the 26 other articles, five belong to the part on algebra and number theory: two of them fall under the sole scope of elimination, one is devoted to the topic of determinants and two concern the theory of equations. Finally, three papers are located only in the part on analysis, and 16 in that on geometry. A synthetic view on these numbers and more details on the analytic and geometric parts are provided in table 1.

In the image of what has been described above for the corpus made of 19 texts, such a disciplinary distribution covers different phenomena: for instance, while the majority of the papers classified in geometry only make use of known results on invariants and forms, some of them contain more or less long developments pertaining on invariants and forms only, which are immediately applied to prove theorems on curves and surfaces.

For reasons of space, I will not depict more precisely the variety of the questions tackled in the corpus, be they related to geometry, analysis or algebra. Instead, a number of selected papers representing different cases of classification and disciplinary configurations within the technique will be scrutinized. This will enlighten several facets of Clebsch's work on invariants that I find characteristic and that are specifically tied to applications. First I will analyse the 1861 paper on the symbolic representation and its applications to elimination and geometry, a paper classified in form theory only. I will then study an article published in 1863, catalogued in geometry and dealing with

ALGEBRA AND THEORY OF NUMBERS

- Linear substitutions
  - Determinants 2
  - Discriminants and resultants; Elimination . . . . . $2+1$
  - General theory of quantics . . . . . $3+1$
  - Binary forms . . . . . $8+2$
  - Ternary forms . . . . . $3+1$
  - Special developments associated with forms in more than three variables 1
- Theory of equations
  - General resolution of equations; theory of Galois; equations 5th order; equations 6th order . . . . . $1+1$

ANALYSIS

- Algebraic functions and their integrals . . . . . $2+1$
- Differential equations . . . . . $1+1$

GEOMETRY

- Elementary geometry . . . . . $1+1$
- Geometry of conics and quadrics 2
- Algebraic curves and surfaces of degree higher than the second 8
- Transformations and general methods for algebraic configurations $2+1$
- Infinitesimal geometry; applications of differential and integral calculus to geometry 1
- Differential geometry; applications of differential equations to geometry 2

Table 1: Classification in the *Catalogue* of Clebsch's papers dealing with invariants and covariants. The sums $+d$ indicate the double classifications. "Quantics" is a 19th-century English equivalent of "forms".

the so-called problem of normals to a conic or a quadric. The theory of the quintic equation, for which Clebsch used results on the typical representation of forms, constitutes the next investigation; the related paper of 1871 belongs both to equation theory and geometry.

As already evoked, two other sections will be devoted respectively to the link between invariant theory and the explicit writing of given equations, and to Clebsch's last works on invariants. Aimed at accounting for different features, these sections will not go into as much mathematical detail as the preceding ones.

To help the reader understand the technicalities that are part and parcel of Clebsch's papers, I first propose some explanations of mathematical results on invariants, most of which were considered as basic knowledge by Clebsch. These explanations are given from a 19th-century point of view and are augmented with some historical information.

2. Mathematical appetisers

An r-ary algebraic form of order n is a homogeneous polynomial in r variables and of degree n in respect with the variables. Thus a binary form is given by an expression such as:

$$f(x_1, x_2) = A_0 x_1^n + nA_1 x_1^{n-1} x_2 + \binom{n}{2} A_2 x_1^{n-2} x_2^2 + \cdots + nA_{n-1} x_1 x_2^{n-1} + A_n x_2^n.$$

Normalising the different coefficients with binomial numbers is a convention that simplifies computations and that was used by many mathematicians of the 19th century, including Clebsch. Moreover, the latter never specified the nature of the A_i. His different theorems and proofs make clear that these coefficients were never seen as integers. On the other hand, he occasionally drew his attention on the irrationalities that he had to introduce, which suggests that he saw the coefficients as numbers somehow situated between the rational and the complex numbers.

Invertible linear transformations of the variables act on forms: in the case of two variables, such a transformation is defined by equations

$$\begin{cases} x_1 = \alpha_{11} x_1' + \alpha_{12} x_2' \\ x_2 = \alpha_{21} x_1' + \alpha_{22} x_2', \end{cases}$$

where the determinant $r = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$ of the transformation is non zero. Replacing x_1, x_2 by these formulas in $f(x_1, x_2)$ and reorganising the different terms yields

$$\begin{aligned}f(x_1,x_2) &= f(\alpha_{11}x_1' + \alpha_{12}x_2', \alpha_{21}x_1' + \alpha_{22}x_2')\\ &= A_0(\alpha_{11}x_1' + \alpha_{12}x_2')^n + nA_1(\alpha_{11}x_1' + \alpha_{12}x_2')^{n-1}(\alpha_{21}x_1' + \alpha_{22}x_2') +\\ &\qquad + \cdots + A_n(\alpha_{21}x_1' + \alpha_{22}x_2')^n\\ &= A_0'x_1'^n + nA_1'x_1'^{n-1}x_2' + \cdots + A_n'x_2'^n,\end{aligned}$$

the new coefficients A_i' being functions of the A_i and the α_{ij}.

Consider now a homogeneous polynomial $I(A_0,\ldots,A_n)$ in the coefficients of f. Such a polynomial is called an *invariant* of f if there exists an integer λ such that $I(A_0',\ldots,A_n') = r^\lambda I(A_0,\ldots,A_n)$ for any invertible linear transformation, the coefficients A_i' being defined as above. Similarly, a *covariant* of f is a homogeneous polynomial $K(A_0,\ldots,A_n,x_1,x_2)$ for which there exists an integer λ such that $K(A_0',\ldots,A_n',x_1',x_2') = r^\lambda K(A_0,\ldots,A_n,x_1,x_2)$ for any invertible linear transformation.

Let me exemplify these definitions, first in the case of a binary quadratic form

$$f(x_1,x_2) = A_0x_1^2 + 2A_1x_1x_2 + A_2x_2^2.$$

Its *discriminant* is defined as $R = A_1^2 - A_0A_2$. It is an invariant of f since $R(A_0',A_1',A_2') = r^2R(A_0,A_1,A_2)$, as an easy computation shows. Obviously, it coincides with the usual discriminant of the second-degree equation

$$f(x_1,1) = A_0x_1^2 + 2A_1x_1 + A_2 = 0.$$

In particular, the condition $R = 0$ is a necessary and sufficient condition for this equation to have a double root. This amounts to the fact that the binary form $f(x_1,x_2)$ is the square of a linear factor or, as was frequently formulated by Clebsch, that the homogeneous equation $f(x_1,x_2) = 0$ has a double root x_1/x_2.

Similarly, for a binary quartic form

$$f(x_1,x_2) = A_0x_1^4 + 4A_1x_1^3x_2 + 6A_2x_1^2x_2^2 + 4A_3x_1x_2^3 + A_4x_2^4,$$

its discriminant R is an invariant having the property that $R = 0$ if and only if the equation $f(x_1,x_2) = 0$ has a double root x_1/x_2. However, contrary to the quadratic case, the quartic form f has other invariants. In particular, in their very first investigations on invariant theory in the mid-1840s, Arthur Cayley and George Boole had found[14] the two invariants

$$\begin{aligned}i &= A_0A_4 - 4A_1A_3 + 3A_2^2\\ j &= A_0A_2A_4 + 2A_1A_2A_3 - A_2^3 - A_0A_3^2 - A_1^2A_4.\end{aligned}$$

[14] See [Parshall 1989, pp. 160–165] and [Wolfson 2008].

Just like for the discriminant, the vanishing of i or j provides information on the roots of the equation $f(x_1, x_2) = 0$. For instance, Cayley proved in 1858 that this equation has a triple root if $i = j = 0$, [Cayley 1858, p. 454]. Moreover, a remarkable result is that i and j are fundamental invariants of f, which means that every invariant can be expressed as a polynomial in i and j. In particular, Boole had seen in 1845 that the discriminant of the quartic is given by

$$R = i^3 - 27j^2.$$

Another difference between the quadratic and the quartic form is that the latter has non-trivial covariants. Although it will not appear in the rest of this chapter, let me illustrate the notion with

$$\begin{aligned} H = (A_0A_2 - A_1^2)x_1^4 + 2(A_0A_3 - A_1A_2)x_1^3x_2 + (A_0A_4 + 2A_1A_3 - 3A_2^2)x_1^2x_2^2 \\ +2(A_1A_4 - A_2A_3)x_1x_2^3 + (A_2A_4 - A_3^2)x_2^4. \end{aligned}$$

It is a covariant of the fourth order in the variables x_1, x_2 and of the second order in the coefficients A_i. There also exists a covariant of the second order in x_1, x_2 that is associated with the quartic form. However, there does not exist any *linear* covariant, that is, any covariant of the first order in x_1, x_2. Such linear covariants exist for binary forms of odd order greater than 3, in particular for quintic forms. This result will be important for the work on the so-called typical representations of forms.

Increasing by one the number of variables leads to ternary forms, such as the following quadratic one:

$$f(x_1, x_2, x_3) = A_{11}x_1^2 + A_{22}x_2^2 + A_{33}x_3^2 + 2A_{12}x_1x_2 + 2A_{23}x_2x_3 + 2A_{13}x_1x_3.$$

As in the binary case, this form has a discriminant, which is simply the determinant

$$R = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}.$$

The condition $R = 0$ expresses that the form f can be factorised into two linear terms. Because there are now three variables, there is no direct link with algebraic equations in one unknown. However, the same process of setting one unknown, say x_3, to 1 gives the usual equation of a conic section in the plane:

$$f(x_1, x_2, 1) = A_{11}x_1^2 + +2A_{12}x_1x_2 + A_{22}x_2^2 2A_{13}x_1 + 2A_{23}x_2 + A_{33} = 0,$$

and the equality $R = 0$ means that this conic is degenerate. For such a geometric interpretation, it is also to possible to place oneself in the framework of projective geometry, and thus to keep the homogeneous equation $f(x_1, x_2, x_3) = 0$ between the homogeneous coordinates (x_1, x_2, x_3) of a point.

Finally, ternary cubic forms

$$f(x_1, x_2, x_3) = \sum_{i,j,k} A_{ijk} x_i x_j x_k$$

are used to express equations of plane cubic curves. The study of these curves by Otto Hesse in the 1840s is one of the roots of invariant theory: the study of their inflection points is in close relation with that of the Hessian covariant, defined as the determinant made with the coefficients $\frac{\partial^2 f}{\partial x_i \partial x_j}$, [Parshall 1989, pp. 171–172]. When Aronhold tackled the subject in 1850, it was precisely in the continuation of Hesse's works. Among other results, Aronhold put into light two invariants of the cubic, denoted by S and T, and proved that the discriminant of the cubic form is given by $R = S^3 - T^2$. The vanishing of R was interpreted as the existence of a non trivial triple (x_1, x_2, x_3) in which all the first partial derivatives of f vanish, [Aronhold 1850, p. 153]; as will be seen below, this would be interpreted by Clebsch as the condition for the cubic curve $f = 0$ to have a double point.

Before going on, let me eventually remark that all the invariants (and covariants) that have been considered in this section are related to one form. However, there exists also a notion of simultaneous invariant. Several forms being given, a simultaneous invariant is a polynomial in the coefficients of these forms that remains unchanged under the action of any linear transformation.

3. Symbolic representation and tangential equations

The main paper on the symbolic representation of forms, invariants and covariants is dated September 1860 and has been inserted in one of the two 1861 volumes of Crelle's *Journal für die reine und angewandte Mathematik* [Clebsch 1861e]. Its content has also been presented to the Berlin Academy of sciences by Carl Borchardt in October 1860 [Clebsch 1861c].

The first lines of both these papers emphasise three points. First, that Aronhold had already developed a symbolic representation of forms and invariants in the particular case of ternary cubic forms a few years before [Aronhold 1858].[15] Second, that Clebsch had already briefly presented and used the (generalised) symbolic representation in an anterior paper, dated March 1860 [Clebsch 1861d].[16] And third, that the generalisation of Aronhold's work had two main

[15] See [Parshall 1989, pp. 173–175] for a description of Aronhold's version of the symbolic representation. Moreover, Cayley had also developed a kind of symbolic notation, although different in nature than Aronhold's and Clebsch's one, [Crilly 1988, pp. 334–336]. Finally, as is well known, the latter has been re-worked much later to make it more rigorous from a 20th-century point of view, [Kung and Rota 1984].

[16] In my view, this paper does not contain anything that would be decisive for the issue of the symbolic representation. However, I will expound a part of its content in section 6, devoted to the relation between invariant theory and the explicit establishment of given equations.

interests: on one hand, it could be taken as the very definition of forms and invariants in future research and, on the other hand, it presented “benefits [...] for the theory of elimination”.[17] As will be seen, the applications to elimination served in turn to handle problems coming from projective geometry.

3.1 The symbolic representation

The core of Clebsch’s symbolic representation of an r-ary form f of order n is to proceed as if this form were the nth power of a linear form, i.e. $f = (a_1x_1 + \cdots + a_rx_r)^n$. In such an expression, the symbolic coefficients a_i do not have any meaning in themselves; they are related to the actual coefficients of f by equations deduced by identifications after expanding the nth power. For instance, for a binary quartic

$$f = A_0x_1^4 + 4A_1x_1^3x_2 + 6A_2x_1^2x_2^2 + 4A_3x_1x_2^3 + A_4x_2^4,$$

a symbolic representation is $f = (a_1x_1+a_2x_2)^4$. Expanding the latter expression and identifying with the actual expression of f yields the rules

$$a_1^4 = A_0 \quad ; \quad a_1^3a_2 = A_1 \quad ; \quad a_1^2a_2^2 = A_2 \quad ; \quad a_1a_2^3 = A_3 \quad ; \quad a_2^4 = A_4.$$

Depending on the needs, other letters can be used for the symbolic representation: one has also $f = (b_1x_1 + b_2x_2)^4$, with symbolic coefficients b satisfying $b_1^4 = A_0$, $b_1^3b_2 = A_1$, etc. This is the symbolic representation of forms.

That of the associated invariants stems from the consideration of determinants such as

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

Although Clebsch denoted such a determinant by (ab) only in later publications, I will use this convenient abbreviation in the following lines. In the geometric applications that he developed thereafter, Clebsch proved that the symbolic representation of the first fundamental invariant i of the quartic is $i = \frac{1}{2}(ab)^4$. Indeed, expanding this fourth power and using the substitution rules yields

$$\begin{aligned} \frac{1}{2}(ab)^4 &= \frac{1}{2}(a_1^4b_2^4 + a_2^4b_1^4 - 4a_1^3a_2b_1b_2^3 - 4a_1a_2^3b_1^3b_2 + 6a_1^2a_2^2b_1^2b_2^2) \\ &= A_0A_4 - 4A_1A_3 + 3A_2^2, \end{aligned}$$

which is the expression defining i, as announced. Similarly, Clebsch proved that the symbolic representation of the second fundamental invariant j is $j = \frac{1}{6}(ab)^2(bc)^2(ca)^2$, where the letter c refers to yet another series of symbolic

[17]“[E]s ist der Zweck der gegenwärtigen Abhandlung, namentlich auf den Nutzen aufmerksam zu machen, welchen man für die Theorie der Elimination aus dieser Darstellung ziehen kann.” [Clebsch 1861e, p. 1].

coefficients of f. In particular, the invariants i and j are products of constants and powers of symbolic determinants.

More generally, the key theorem proved by Clebsch is that every invariant I of an r-ary form f (of any order) is a linear combination of products of symbolic determinants with r rows and r columns:

$$I = \sum \lambda \prod (ab \dots s).$$

An analogous result was provided for covariants, which are combinations of products of symbolic determinants and symbolic linear factors such as $a_1x_1 + a_2x_2$. The explanation of the symbolic representation and the proof of the latter theorems took about 13 pages out of the 62 pages of Clebsch's paper. I shall skip these technicalities and focus on their applications.

The first one concerned the elimination of the unknown between two algebraic equations, which amounts to finding a polynomial expression of the coefficients of these equations whose vanishing expresses that the latter have a common root. From the homogeneous point of view, the issue consists in eliminating the two homogeneous variables between two binary forms equated to zero. Clebsch remarked that the result of such an elimination is a simultaneous invariant of the forms and managed to find its symbolical representation in the case where the two forms are of the same order [Clebsch 1861e, pp. 18–24].

Another application of the symbolic notation consisted in expressing the invariants of forms with $r + 1$ unknowns linked together by a linear relation. To explain Clebsch's procedure, let me consider the case $r = 2$. Clebsch took a ternary form given by the symbolic representation $f = (a_1x_1 + a_2x_2 + a_3x_3)^n = (b_1x_1 + b_2x_2 + b_3x_3)^n$, and an invertible linear transformation

$$\begin{cases} x_1 = c_0X_1 + c_0'X_2 + c_0''X_3 \\ x_2 = c_1X_1 + c_1'X_2 + c_1''X_3 \\ x_3 = c_2X_1 + c_2'X_2 + c_2''X_3. \end{cases}$$

By acting on the symbolical level, this transformation changes f into $\tilde{f} = (\alpha_1X_1 + \alpha_2X_2 + \alpha_3X_3)^n = (\beta_1X_1 + \beta_2X_2 + \beta_3X_3)^n$, where $\alpha_1 = c_0a_1 + c_0'a_2 + c_0''a_3$, etc. Further, if C denotes the determinant of the linear transformation, one has

$$C \cdot X_3 = u_1x_1 + u_2x_2 + u_3x_3,$$

where $u_1 = c_1c_2' - c_1'c_2$, $u_2 = -c_0c_1' + c_0'c_2$ and $u_3 = c_0c_1' - c_0'c_1$ are cofactors of C.[18] As a result, if one supposes that $u_1x_1 + u_2x_2 + u_3x_3 = 0$, one has $X_3 = 0$ and f becomes $\tilde{f} = (\alpha_1X_1 + \alpha_2X_2)^n = (\beta_1X_1 + \beta_2X_2)^n$.

Now, according to the previous theorem, every invariant of $\tilde{f}$ is a combination $\sum \lambda \prod (\alpha\beta)$, and Clebsch observed that the symbolic determinant $(\alpha\beta)$

[18]From a current point of view, this is just a consequence of the matrix formula $X = \Gamma^{-1}x$, where Γ is the matrix with coefficients $c_i^{(j)}$.

can be expressed with the original coefficients a_i of the form and those of the linear relation $u_1x_1 + u_2x_2 + u_3x_3 = 0$. Specifically, one has

$$(\alpha\beta) = \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = (abu).$$

From this, Clebsch deduced that every invariant of the form $f = (a_1x_1 + a_2x_2 + a_3x_3)^n = (b_1x_1 + b_2x_2 + b_3x_3)^n$, of which the variables satisfy $u_1x_1 + u_2x_2 + u_3x_3 = 0$, is given by a combination

$$\sum \lambda \prod (abu),$$

where $\sum \lambda \prod (\alpha\beta)$ is an invariant of the binary form $\tilde{f}$. This theorem and its extension to the case where several linear relations exist between the variables were the starting point of the geometric applications that followed.

3.2 Geometry enters the picture

As Clebsch expounded, indeed, the question of finding the invariants of a form when the variables are linked by a linear relation "is of the utmost importance for geometry, inasmuch as it concerns the intersection points of a line with a curve, or the properties of a plane intersection of an algebraic surface."[19] This general geometric framework was made more specific through a number of problems to be solved. The first one was the "very important problem, of which the complete solution is contained in the previous considerations, [and which presents itself] in the task of expressing any curve in line coordinates."[20]

This task referred to the dual representation of curves in projective geometry.[21] Usually, an algebraic curve in the projective plane can be defined by an equation $f(x_1, x_2, x_3) = 0$, where (x_1, x_2, x_3) are homogeneous coordinates of a point. Instead of the points, however, one can consider the straight lines to be the basic elements of the plane. From this point of view, instead of being defined a locus of points, a curve is defined as an envelope of lines, that is, as the set of all the lines that are tangent to it. Furthermore, since a line can be defined by an equation $u_1x_1 + u_2x_2 + u_3x_3 = 0$, the coefficients u_i are called its "line coordinates". To describe a curve as an envelope, then, corresponds to

[19] "Diese Frage ist für die Geometrie von höchster Bedeutung, insofern es sich um die Schnittpunkte einer Geraden mit einer Curve handelt, oder um die Eigenschaften eines ebenen Schnitts einer algebraischen Fläche." [Clebsch 1861e, p. 26].

[20] "Ein sehr wichtiges Problem, dessen vollständige Lösung in den vorhergehenden Betrachtungen enthalten ist, bietet sich dar in der Aufgabe, eine beliebige Curve in Liniencoordinaten auszudrücken." [Clebsch 1861e, p. 35].

[21] About duality in projective geometry, see [Lorenat 2015] and the references given on p. 547.

define it by an equation $g(u_1, u_2, u_3) = 0$. This is the tangential equation of the curve, and the task mentioned by Clebsch amounts to writing it down.[22]

Clebsch reformulated this problem as follows. Let $f(x_1, x_2, x_3) = 0$ be the equation of a curve of order n, and consider the binary form obtained from f by supposing that $u_1x_1 + u_2x_2 + u_3x_3 = 0$. One must find the relation between u_1, u_2, u_3 such that this binary form has two equal linear factors, or two equal roots, when considered as an equation: this corresponds to the case of tangency of the line, the intersection of the latter with the curve being counted twice.

This is where invariants came into the picture, since the condition for an equation to have two coinciding roots is provided by the vanishing of its discriminant.

After some general results, Clebsch exemplified his method in the cases of conics, cubics and quartics. Let me explain it for quartics. One starts from an homogeneous equation $f(x_1, x_2, x_3) = 0$, where f is a ternary quartic form, and one considers the binary form given by

$$\tilde{f} = (a_1x_1 + a_2x_2)^4 = (b_1x_1 + b_2x_2)^4 = (c_1x_1 + c_2x_2)^4.$$

As seen above, $\tilde{f}$ has a double root if and only if $i^3 - 27j^2 = 0$. This was attributed to Cayley by Clebsch, who nevertheless cited a memoir of Charles Hermite on this point, [Hermite 1856].

Now, since the symbolic expressions of the two fundamental invariants of $\tilde{f}$ are $i = \frac{1}{2}(ab)^4$ and $j = \frac{1}{6}(ab)^2(bc)^2(ca)^2$, Clebsch's process consists in considering the symbolic invariants $P = \frac{1}{2}(abu)^4$ and $Q = \frac{1}{6}(abu)^2(bcu)^2(cau)^2$. Then, corresponding to $i^3 - 27j^2 = 0$, the condition under which the result of the elimination of one variable between

$$\begin{cases} f(x_1, x_2, x_3) = 0 \\ u_1x_1 + u_2x_2 + u_3x_3 = 0 \end{cases}$$

has a double root is $P^3 - 27Q^2 = 0$. Said differently, this equation, seen as an equation with unknowns u_1, u_2, u_3, is the tangential equation of the quartic curve, which is what was sought.

In this case, Clebsch took the condition $i^3 - 27j^2 = 0$ for granted, thanks to works of Cayley (and Hermite). In other examples, he first devoted some lines to establish himself analogous results. For instance, referring to formulas from Hermite's previously cited paper, he asserted that if $i = j = 0$, the quartic binary form $\tilde{f} = 0$ has three equal roots.[23] After having proved this result, Clebsch used the previous procedure to obtain a geometric result. A

[22] In other words, duality guarantees the existence of an equation in line coordinates, and Clebsch's aim is to calculate its coefficients. This is the kind of mathematical issues that will be discussed in section 6.

[23] Clebsch seemed unaware of the fact that this result had been proved by Cayley in 1858, as mentioned above. Moreover, Hermite's paper does not contain explicitly the property on the triple root, but the latter does ensue directly from some formulas of Hermite (who, in

triple root of the binary form $\tilde{f}$ corresponding to an "inflectional" tangent $u_1x_1 + u_2x_2 + u_3x_3 = 0$ to $f = 0$, i.e. a tangent with a contact of order 3, the line coordinates of such a tangent satisfy $P = Q = 0$. Or, as Clebsch eventually formulated this result, the inflectional tangents to $f = 0$ are the common tangents to the curves $P = 0$ and $Q = 0$.

Many other results on curves were deduced from the same procedure. Clebsch also rapidly tackled the issue of finding the tangential equation of a surface, an issue that he treated in the exact same way as what had been done for curves. In this case, he had to deduce a result on quaternary forms (the variables of which are linked by a linear relation) from a result on ternary forms. Here again, Clebsch relied on the works of some of his predecessors. He recalled that Aronhold [1850] had proved that there are two invariants S, T of the general ternary cubic form such that $T^2 - S^3 = 0$ is the condition for the associated curve to have a double point. Just like before, representing these invariants symbolically and introducing a fourth series of variables $u_1, \ldots, u_4$ in the symbolical determinants allowed to find the tangential equation of a cubic surface under the form $\mathrm{T}^2 - \Sigma^3 = 0$.

3.3 Disciplinary mixes I

The paper on the symbolic representation, which was put exclusively in the theory of forms by the *Catalogue*, contained many applications to elimination theory and, above all, to projective geometry. The question of the writing of the dual equation of a quartic curve, that has been analysed above, is particularly telling on how the different ingredients were successively combined.

Thanks to homogeneous coordinates, the initial geometric problem was translated as a question of elimination between a quartic form and a linear form, which had the effect of lowering the number of variables. The searched properties of the binary form thus obtained was then expressed with the help of invariants, be it by taking already known results of Cayley and Hermite or by proving on the basis of his predecessors' works what he needed. The invariants in question were then expressed symbolically, which allowed Clebsch to apply his procedure of increasing the number of variables. The new invariants (or their adequate combinations) were then directly equated to zero and thus interpreted geometrically as tangential equations of curves.

If the symbolic representation would later be used by Clebsch and others to deal with questions pertaining to invariants only, it was initially entangled in such a disciplinary configuration. Interestingly, this way of writing and conceiving forms and invariants was not systematically adopted by Clebsch himself after 1861, as the next case study shows.

turn, took on some results that he attributed to Cayley). Another paper from which the interpretation of the simultaneous vanishing of i and j could be deduced is [Eisenstein 1844]. Clebsch did not cite this article in any of his published papers, but Aronhold did, [Aronhold 1850, p. 156].

4. The problem of normals

Clebsch tackled what he called the problem of the normals of curves and surfaces of the second order in a paper dated 31st January 1862 and published in Crelle's journal one year after, [Clebsch 1863]. Some of the results were also exposed in a shorter article in *Annali di matematica pura ed applicata*, dated 2nd February 1862, [Clebsch 1861/1862].

As Clebsch recalled, the problem of drawing the normals to a conic passing through a given point had been generalised by Cayley [1859] and by Wilhelm Fiedler in his translation of Salmon's book on conic sections [Salmon and Fiedler 1860]. Clebsch's aim was to "attach an analytic treatment to a discussion of the problem which [...] leads to some new results."[24]

4.1 The problem and its equation

The generalised problem of normals involves the notion of *polar* of a point with respect to a conic, which designates the line joining the points of contact of the two tangents to the conic drawn from the given point. A point x and two conics of equations $u = 0$ and $v = 0$ being given, the question is to determine the points X belonging to $u = 0$, such that the tangent to $u = 0$ at X, the polar of X with respect to $v = 0$ and the polar of x with respect to $v = 0$ are concurrent (see figure 2).[25] Clebsch also provided a dual version of the problem, as well as its extension to the case of quadric surfaces: a point x and two quadrics of equations $u = 0$ and $v = 0$ being given, the problem is to find the points X such that the polar of x with respect to the tangent cone of X to $v = 0$ coincides with the tangent plane of X at $u = 0$.

Without any justification, Clebsch asserted that these problems were "contained" in the the system of equations

$$\begin{cases} v_1 = \lambda U_1 + \mu V_1 \\ v_2 = \lambda U_2 + \mu V_2 \\ \dots\dots\dots\dots\dots\dots \\ v_n = \lambda U_n + \mu V_n \end{cases} \quad \text{and} \quad U = 0, \tag{1}$$

"where the λ, μ are indeterminate factors" and where the values $n = 3$ and $n = 4$ correspond to the cases of conics and quadrics, respectively.[26] The number n, indeed, represents the number of variables of the quadratic forms u and v,

[24] "Es sei mir erlaubt an eine analytische Behandlung eine Discussion des Problems zu knüpfen, welche auch für das Normalenproblem einige neue Resultate mit sich führt." [Clebsch 1863, p. 64].

[25] If the two conics are confocal, these points X are such that the line xX is a normal to $u = 0$. This explains the name of the problem.

[26] "[...] so sind die beiden angeführten Probleme in dem folgenden System von Gleichungen enthalten, wenn darin $n = 3$ oder $n = 4$ gesetzt wird: [...], wobei die λ, μ unbestimmte Factoren sind." [Clebsch 1863, p. 65]

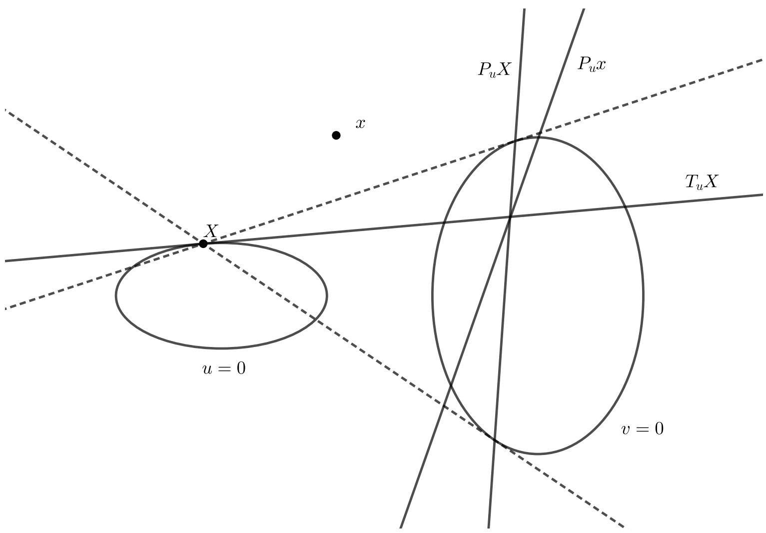

Figure 2: The problem of the normals, in the case of conics. The line T_uX is the tangent to $u = 0$ at the point X. The lines P_vX and P_vx are the polars of X and x with respect to $v = 0$. In this figure, the point X is such that these three lines are concurrent.

so that $u = 0$ and $v = 0$ are the homogeneous equations of conics or quadrics according to the value of n. Moreover the lower-case, resp. upper-case, letters correspond to functions evaluated at $x = (x_1, \dots, x_n)$, resp. $X = (X_1, \dots, X_n)$, and a subscript i symbolises half of the derivative with respect to the ith variable. For instance, if $v = v_{11}x_1^2 + 2v_{12}x_1x_2 + \ldots + v_{nn}x_n^2$, then v_1 and V_1 are the linear expressions $v_1 = v_{11}x_1 + v_{12}x_2 + \cdots + v_{1n}x_n$ and $V_1 = v_{11}X_1 + v_{12}X_2 + \cdots + v_{1n}X_n$.

The link between the equations (1) and the problem of normals comes directly from the equations of the different objects involved in the problem. In the case $n = 3$, for instance, the equation of the tangent to $u = 0$ at X is $U_1z_1 + U_2z_2 + U_3z_3 = 0$, where z_1, z_2, z_3 are the current coordinates of the plane. The equation of the polar of X in respect with $v = 0$ is $V_1z_1 + V_2z_2 + V_3z_3 = 0$, and that of the polar of x with respect with $v = 0$ is $v_1z_1 + v_2z_2 + v_3z_3 = 0$.[27] That the latter line passes through the intersection of the two former is reflected by the fact that its equation is a linear combination of the two others, whence the existence of λ, μ such that $v_i = \lambda U_i + \mu V_i$ for each i, as in (1).

Clebsch was first and foremost interested in the equation obtained by eliminating $X_1, \dots, X_n$ from the equations (1). To find it, he denoted by Δ

[27]By definition, an equation of P_vX is $\frac{\partial v}{\partial z_1}(z)X_1 + \frac{\partial v}{\partial z_2}(z)X_2 + \frac{\partial v}{\partial z_2}(z)X_2 = 0$. Computing the derivatives and reorganising the terms yields $V_1z_1 + V_2z_2 + V_3z_3 = 0$. For a nice modern exposition on tangents and polars, see [Fischer 2001].

the determinant made of the elements $\lambda u_{ik} + \mu v_{ik}$, where the u_{ik} and v_{ik} are the coefficients of the quadratic forms u and v. The sub-determinant of Δ obtained by erasing the row i and the column k being noted Δ_{ik}, the system in (1) is equivalent to

$$\begin{cases} \Delta X_1 = v_1\Delta_{11} + v_2\Delta_{12} + \cdots + v_n\Delta_{1n} \\ \Delta X_2 = v_2\Delta_{21} + v_2\Delta_{22} + \cdots + v_n\Delta_{2n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta X_n = v_n\Delta_{n1} + v_2\Delta_{n2} + \cdots + v_n\Delta_{nn}. \end{cases}$$

The sought elimination equation is then obtained by using these expressions in $U = 0$, which yields[28]

$$\sum_{i,k,p,q} u_{ik} v_p \Delta_{pi} v_q \Delta_{qk} = 0.$$

Some basic operations involving transformations of determinants eventually allowed Clebsch to re-write this equation as:

$$\Omega \frac{\partial \Delta}{\partial \lambda} - \Delta \frac{\partial \Omega}{\partial \lambda} = 0, \tag{2}$$

with $\Omega = \sum v_p v_q \Delta_{pq}$. A mere inspection of the coefficients of Ω and Δ shows that the equation (2) is homogeneous and of degree $2(n-1)$ in λ, μ, and of degree 2 in x.

Clebsch indicated that this elimination equation had already been studied by Ferdinand Joachimstahl [1857] in the case of two confocal conics, and that he wanted to investigate it in his generalised frame, especially when $n = 4$. As will be seen, Clebsch aimed at studying the cases where this equation has particular properties, such that of having two or three coinciding roots, and at interpreting these cases geometrically. In the image of what we saw in the preceding section, the properties of the equation (2) would be expressed with the help of invariants.

4.2 The case of two coinciding roots

Investigating first the case of two coinciding roots λ/μ, Clebsch formed the discriminant R of the equation (2). He proved that, because of the special form of this equation, R is of the form

$$R = bFG,$$

[28] From a current point of view, let $\xi = (X_1, \ldots, X_n)^T$, $\eta = (v_1, \ldots, v_n)^T$ and $D = \lambda A + \mu B$, where A, B are the matrices of u, v in the canonical basis of k^n, where k is the ground field. The first equations in (1) can be written $\eta = D\xi$, which amounts to $\xi = D^{-1}\eta$. This yields the formulas for the ΔX_i thanks to the expression of D^{-1} with the cofactor matrix. Moreover, the existence of ξ such that $\eta = D\xi$ and $u(\xi) = 0$ is equivalent to the sole equation $u(D^{-1}\eta) = 0$, which is exactly that given by Clebsch.

where b is a constant, G is a factor of degree $6(n-2)$ in x and F is the resultant of Ω and Δ:

$$F = \prod \Omega(\lambda_i/\mu_i),$$

where the λ_i/μ_i are the roots of $\Delta = 0$.

For the geometric interpretation of the equation $F = 0$, Clebsch endeavoured to show that F is the product of square factors up to a multiplicative constant. To do so, he remarked that the condition $\Delta = 0$ implies the existence of numbers α_i such that $\Delta_{pq} = \alpha_p\alpha_q$ for each pair of indices p, q.[29] Thus the function Ω becomes the square of a function that is linear in x:

$$\Omega = \sum_{p,q} \Delta_{pq} v_p v_q = \sum_{p,q} \alpha_p\alpha_q v_p v_q = \left(\sum_p \alpha_p v_p\right)^2.$$

Because of the expression of F given above, this immediately implies that F is the product of squares of linear factors.

On the other hand, Clebsch introduced linear functions $\xi_1, \xi_2, \ldots, \xi_n$ of the x_i such that

$$u = p_1\xi_1^2 + p_2\xi_2^2 + \cdots + p_n\xi_n^2$$
$$v = q_1\xi_1^2 + q_2\xi_2^2 + \cdots + q_n\xi_n^2,$$

with constant coefficients p_i and q_i.[30] The determinant Δ then becomes

$$\Delta = (\lambda p_1 + \mu q_1)(\lambda p_2 + \mu q_2) \ldots (\lambda p_n + \mu q_n),$$

which shows that its roots λ_i/μ_i are the $-q_i/p_i$. From this, it can be inferred via a short computation that $\Omega(\lambda_i/\mu_i)$ is proportional to ξ_i^2, which implies that

$$F = c \cdot \xi_1^2\xi_2^2 \ldots \xi_n^2$$

for a certain constant c.

Clebsch then wrote:

> Since, according to invariant theory, the expression thus obtained can differ from the initial expression only by a constant factor, the ξ manifestly differ from the expressions found above only by constant factors.[31] [Clebsch 1863, p. 70]

[29]This is to be linked to the fact that the rank of the cofactor matrix associated with $\lambda U + \mu V$ is equal to 1 if $\Delta = 0$.

[30]This result, which is not justified by Clebsch, can be seen anachronistically as the simultaneous orthogonalisation of u and v. Here it is implicitly supposed that either u or v is nondegenerate.

[31]"Da nun nach der Theorie der Invarianten der so erhaltene Ausdruck von dem ursprünglichen sich nur durch einen constanten Factor unterscheiden kann, so sind offenbar die ξ von den oben gefundenen linearen Ausdrücken nur um constante Factoren verschieden."

No further explanations were given. The invocation of invariant theory probably alludes to the fact that F is a simultaneous covariant of u and v, since Δ is a simultaneous invariant and Ω a simultaneous covariant of them.[32] Then, having expressed the quadratic forms with the ξ_i can be seen as the effect of a linear transformation, so that the expression of F in terms of the new variables ξ_i must be the same as the original one, up to a constant. In other words, the linear factors (in the original variables x_i) whose squares compose F are nothing else than the ξ_i.

This result allowed Clebsch to conclude that, for $n = 3$, the points x "for which two solutions of the problem coincide constitute the doubly counted polar triangle common to the conics $u = 0$ and $v = 0$, and a curve of the sixth order $G = 0$ besides."[33]

These curves, indeed, correspond to the interpretation of $R = bFG = 0$ as the equation of the locus of the mentioned points x. In particular, since F is essentially the product $\xi_1^2\xi_2^2\xi_3^2$, the equation $F = 0$ represents the triangle made of the three lines $\xi_i = 0$, each of them being counted twice. That this triangle is the common polar triangle of the two conics follows directly from the expressions $u = p_1\xi_1^2 + p_2\xi_2^2 + p_3\xi_3^2$ and $v = q_1\xi_1^2 + q_2\xi_2^2 + q_3\xi_3^2$. Indeed, as is proved by a plain computation, each side $\xi_i = 0$ of the triangle is the polar of the opposite vertex with respect both to $u = 0$ and $v = 0$. This is the definition of the common polar triangle.

The theory of invariants and covariants thus intervened to help Clebsch recognise the geometrical locus defined by $F = 0$. Because this locus is more easily understood in the case where the quadratic forms are sums of squares, the approach consisted in operating a linear change of variables to reduce the given forms to such sums. The crucial fact that F is a simultaneous covariant of u and v then permitted to identify its factors as representing the sides of the common polar triangle.

Invariant theory appeared again in the rest of the paper, but in different disguises. Instead of its link with the linear transformations of the variables x, it is their role as conveying information on algebraic equations that was used.

To take a first example, in the case $n = 3$, the equation (2) is of degree 4. Clebsch noted i, j its two fundamental invariants. After having introduced explicit writings of Δ and Ω as polynomials, he computed the value of i and remarked that it is equal to a complete square, up to a multiplicative constant: $i = 3h^2$. This implies that the discriminant $R = i^3 - 27j^2 = 27(h^3 - j)(h^3 + j)$ can be factorised into two factors. After having compared this expression of

[32] A simultaneous linear transformation of variables being represented by an invertible matrix P, the new matrices A' and B' of u and v are given by $A' = P^TAP$ and $B' = P^TBP$. Thus one has plainly $\Delta' = (\det P)^2\Delta$, which proves that Δ is a simultaneous invariant. A similar argument can be developed for Ω.

[33] "Diejenigen Punkte, für die zwei Lösungen des Problems zusammenfallen, bilden das doppelt gerechnete gemeinsame Polardreieck der Kegelschnittte $u = 0$, $v = 0$, und ausserdem eine Curve sechster Ordnung $G = 0$." [Clebsch 1863, p. 70].

R with the previous one $R = bFG$, Clebsch concluded that $G = 8h^3 - b^3H^2$, where H is defined by $F = b^2H^2$. This expression of G yielded new properties of the curve defined by $G = 0$. For instance, it proved that the six points of intersection of the conic $h = 0$ and the polar triangle $H = 0$ are cusps of this curve, and that the sides of the polar triangle are the associated cuspidal tangents.

4.3 Other special cases

In other cases, Clebsch first replaced the initial equation (2) by something else. Specifically he remarked that this equation $\Delta\frac{\partial\Omega}{\partial\lambda} - \Omega\frac{\partial\Delta}{\partial\lambda} = 0$ is the necessary and sufficient condition for the existence of a number m such that[34]

$$\begin{cases} \Delta + m \cdot \mu\Omega = 0 \\ \frac{\partial\Delta}{\partial\lambda} + m \cdot \frac{\partial\mu\Omega}{\partial\lambda} = 0. \end{cases}$$

Since the second of these equations is the derivative of the first one with respect to λ, it follows that $\Delta\frac{\partial\Omega}{\partial\lambda} - \Omega\frac{\partial\Delta}{\partial\lambda} = 0$ if and only if there exists m such that the equation $\Delta + m \cdot \mu\Omega = 0$ has a double root λ/μ. In a similar, yet more sophisticated way, Clebsch proved that $G = 0$ if and only if the same equation has a triple root.

If $n = 4$, this equation $\Delta + m \cdot \mu\Omega = 0$ is of degree 4 in λ/μ and Clebsch used again the fundamental invariants i and j. By introducing the explicit expansions

$$\begin{cases} \Delta = b\lambda^4 + 4b'\lambda^3\mu + 6b''\lambda^2\mu^2 + 4b'''\lambda\mu^3 + b^{(4)}\mu^4 \\ \Omega = a\lambda^3 + 3a'\lambda^2\mu + 3a''\lambda\mu^2 + a'''\mu^3, \end{cases}$$

he computed i and j, and wrote the results in the form

$$\begin{cases} i = A + 2mA_1 + m^2A_2, \\ j = B + 3mB_1 + 3m^2B_2 + m^3B_3, \end{cases}$$

where the coefficients A_i and B_i are functions of the $a^{(i)}$ and the $b^{(i)}$. At this point it is worth recalling that Δ and Ω depend on x, and thus that the A_i and B_i depend on x as well. More precisely, observing the preceding equations shows that A_i and B_i are both of degree $2i$ in x.

Now, as Clebsch recalled, the equation $\Delta + m \cdot \mu\Omega = 0$ has three equal roots if and only if its fundamental invariants vanish simultaneously. Thus the existence of m such that $\Delta + m \cdot \mu\Omega = 0$ has three equal roots is equivalent to the vanishing of the result of the elimination of m between the equations $i = 0$

[34] This comes from the fact that $\Delta\frac{\partial\Omega}{\partial\lambda} - \Omega\frac{\partial\Delta}{\partial\lambda}$ is the determinant of this system (with unknown m).

and $j = 0$. In other words, G is given by the resultant of i and j with respect to m:

$$G = \begin{vmatrix} A & 2A_1 & A_2 & 0 & 0 \\ 0 & A & 2A_1 & A_2 & 0 \\ 0 & 0 & A & 2A_1 & A_2 \\ B & 3B_1 & 3B_2 & B_3 & 0 \\ 0 & B & 3B_1 & 3B_2 & B_3 \end{vmatrix}.$$

From the knowledge of the degrees in x of the different coefficients, Clebsch found that $G = 0$ represents a surface of degree 12, as expected from the first considerations – indeed, it had been showed that for any n, the polynomial G is of degree $6(n-2)$ in x.

As for the initial equation, we saw that it expresses the existence of m such that $\Delta + m \cdot \mu\Omega = 0$ has a double root. In terms of invariants, this amounts to the existence of m such that $i^3 - 27j^2 = 0$. In Clebsch's terms, "to the primitive equation of the sixth degree $\Delta\frac{\partial\Omega}{\partial\lambda} - \Omega\frac{\partial\Delta}{\partial\lambda} = 0$, one can substitute another one, of particular form, in which m is the unknown":[35]

$$(A + 2mA_1 + m^2A_2)^3 - 27(B + 3mB_1 + 3m^2B_2 + m^3B_3)^2 = 0.$$

This equation was of special interest to Clebsch because it would "lead to some cases for which the problem is algebraically solvable."[36]

For instance, Clebsch remarked that if $AA_2 - A_1^2 = 0$, the previous equation can be transformed into

$$(A + A_1m)^3 = \pm\sqrt{27A} \cdot (B + 3mB_1 + 3m^2B_2 + m^3B_3),$$

which represents "two equations of the third degree instead of one equation of the sixth degree",[37] the number two corresponding to the two possibilities of the sign. Although Clebsch did not make it explicit, this is the reason why the problem is algebraically solvable: this solvability is to be understood as that of the associated algebraic equation in m, which was factorised into two equations of the third degree in m in this case.

Moreover, still under the assumption that $AA_2 - A_1^2 = 0$, Clebsch transformed the equation of $G = 0$ into

$$(BA_1^3 - 3B_1AA_1^2 + 3B_2A^2A_1 - B_3A^3)^2 = 0.$$

Re-introducing a geometric vocabulary, he concluded that the problem is algebraically solvable for each point x of the surface of the fourth order

[35] "[Man kann] an Stelle der ursprünglichen Gleichung sechsten Grades $\Delta\frac{\partial\Omega}{\partial\lambda} - \Omega\frac{\partial\Delta}{\partial\lambda} = 0$ eine andere setzen [...] von eigenthümlicher Form, in welcher m die Unbekannte ist." [Clebsch 1863, p. 73].

[36] "Diese Form ist unter Anderem dadurch merkwürdig, dass sie auf einige Fälle führt, in denen das Problem algebraisch lösbar ist." [Clebsch 1863, p. 73].

[37] "zwei Gleichungen dritten Grades an Stelle einer Gleichung vom sechsten." [Clebsch 1863, p. 73].

$AA_2 - A_1^2 = 0$, which touches the surface $G = 0$ along a curve contained in the sixth-order surface $BA_1^3 - 3B_1AA_1^2 + 3B_2A^2A_1 - B_3A^3 = 0$.

In the other sections of the paper, Clebsch pursued his investigation of the initial equation $\Omega\frac{\partial\Delta}{\partial\lambda} - \Delta\frac{\partial\Omega}{\partial\lambda} = 0$. For instance, he researched the situations where this equation has three equal roots, or two pairs of double roots, or two pairs of triple roots, etc. Many of these conditions were expressed with the help of invariants, and the final results were of the same vein as those that have been presented above. In particular, the properties of the locus defined by $G = 0$ were investigated thoroughly.

4.4 Disciplinary mixes II

Some pieces of Clebsch's approach of the problem of normals recall what we saw in the case of the search for tangential equations, although they were arranged and handled a bit differently.

The initial translation of a geometric question into equations, the elimination process, the use of invariants to express particular conditions on the roots of equations (especially of degree lesser than 4) are indeed common to both works. However, the relations between invariants and geometry were not quite the same. In the case of the tangential equations, invariants of binary forms were turned into invariants of ternary forms which, equated to zero, represented equations of curves directly. On the contrary, in the problem of normals, invariants of binary forms remained closer to the core of the technical machinery, since they only served to transform the equations that would be eventually interpreted geometrically.

The only invariant that was interpreted as defining a geometrical locus was the discriminant R of the equation under scrutiny. As we saw, results on invariants and covariants of quite a different nature as those pertaining to binary forms were mobilised by Clebsch. In particular, the linear change of variables that allowed to understand the geometric significance of the factor F was not interpreted as a geometric transformation *per se*. Indeed, while Clebsch did not even use the terms "transformation" or "change of coordinates" at all in this context, neither did he describe the action that it could have had on points or lines, in the cases $n = 3$ and $n = 4$.

Contrary to the case of invariants expressing conditions on the roots of equations, I have found very few occurrences of such geometrical uses of linear transformations in Clebsch's corpus. In the next section, linear transformations are much more visible and play a critical role within invariant theory and equation theory. Other, "higher" transformations intervene as well, and are put in relation with geometry in an original way.

5. Typical representations and the quintic equation

Clebsch published a paper (dated June 1871) on the geometric interpretation of the theory of the quintic equation in 1871, [Clebsch 1871a]. It was of course well known, at that time, that Niels Abel had proved at the end of the 1820s that the general quintic cannot be solved by radicals. Still, other research had been done on the topic later, and Hermite and Kronecker had both shown in the 1850s how to solve the quintic with the help of the theory of elliptic functions.[38]

Hermite's approach crucially used the result according to which the quintic can be transformed into the so-called Jerrard form $x^5 - x - a = 0$ thanks to a polynomial transformation of the variable. It is this form of the quintic that would allow its identification with an equation associated with elliptic functions and thus the expression of its roots by those of the latter.[39] Accordingly, Clebsch considered the quintic as solved as soon as it has been put into the Jerrard form. One of the aims of his 1871 paper was to interpret geometrically the possibility of finding such a transformation.

This was split into two main steps, both of them making invariants intervene. The first one did not involve any geometric object. It consisted in proving that if a certain invariant C of the quintic vanishes, the latter can be linearly transformed into a Jerrard equation. The second step dealt with the question of the possibility of making the invariant C equal to zero, a question that Clebsch formulated in his geometrical framework.

In what follows I present the outlines of this research by focussing on the first step, which makes a certain use of what was called the typical representation of forms.[40] Although it is not compulsory to understand Clebsch's work on the quintic, I begin with some lines on this topic, which is essentially absent from the recent historiography.

5.1 Typical representations

According to several testimonies of the 19th century,[41] the notion of typical representation has been introduced by Hermite in a paper published in 1854, [Hermite 1854]. In the case of a binary form of odd degree greater than 3, Hermite proved that if two independent linear covariants are taken as new variables, the coefficients of the transformed form are invariants. This new expression of the form, where the unknowns are covariants and the coefficients are

[38] See [Goldstein 2011; Petri and Schappacher 2004]. Francesco Brioschi had also worked on the topic, [Houzel 2002, pp. 77–79].

[39] As Catherine Goldstein pointed out to me, Hermite all the more praised the Jerrard form because he saw it as the way to access the true nature of the roots of the quintic. See [Hermite 1858a, p. 508].

[40] The geometrical interpretation and, more generally, how geometry intervenes in Clebsch's paper, are studied in [Lê 2017].

[41] See for instance [Clebsch 1872a, p. 331] and [Meyer 1892, p. 156].

invariants, is a *typical representation* of the form (or a *forme-type*, in Hermite's French words). Hermite then computed explicitly a typical representation of a binary quintic form. Assuming that this form has real coefficients, he used this result to determine conditions on the invariants of the form that correspond to different cases of reality of the roots of the associated quintic equation.

The topic of the quintic equation thus appears to be closely linked to typical representations, but Hermite's results on the reality of the roots were not taken over by Clebsch. In the latter's publications, the topic of typical representations first occurred in a paper entirely devoted to it, and written together with Gordan, [Clebsch and Gordan 1867/1868]. The two authors began by briefly recalling the existence of typical representations of binary forms of any degree. Their intention, however, was to determine such representations for forms of degree 5 and 6. Contrary to the sextic case, Clebsch and Gordan asserted, their contribution for what was related to the quintic consisted mainly in a "systematic exposition", since almost everything had been proved in the past by Hermite, Cayley, Sylvester and Salmon [Clebsch and Gordan 1867/1868, p. 24]. If the final formulas were already known, a notable difference of Clebsch and Gordan's research with that of the cited mathematicians was the massive use of the symbolic notation of forms, invariants and covariants.

The computations being quite involved, I will content myself to show one formula that would be important for Clebsch's 1871 work on the quintic equation. As already said, the aim of Clebsch and Gordan was to find a typical representation of a binary quintic form f. To do so, they were led to find that of some of the covariants of f. However, to provide a complete picture of the situation and show the efficiency of their method, the two mathematicians also tackled the case a cubic covariant j, the typical representation of which was not useful to establish that of f. Specifically, assuming that the discriminant R of f is non zero, they exhibited two linear covariants α, δ such that[42]

$$R^2 j = -\delta^3 - \frac{3}{2} N \delta \alpha^2 + \frac{1}{2}(CM - BN)\alpha^3,$$

where the capital letters designate invariants of f. In particular, the invariant C is the one which has been evoked earlier and whose vanishing would be studied by Clebsch later. An important property that would be used is that C can be defined as the discriminant of the cubic covariant j.

As for the typical representation of f, it looked like that of j, although being more complex. The use of both the representations in 1871 to tackle the quintic equation was somewhat incidental, and linked to the mathematical question of transforming a binary quintic form into a sum of three fifth powers.[43]

[42] Clebsch and Gordan use other numerical conventions. Here I adopt the normalisations and notations that fit with Clebsch's later paper on the quintic equation.

[43] For the sake of brevity, I will not expand on the applications that Clebsch made of typical representations in his other papers. The most spectacular, perhaps, was the determination of

5.2 Towards the Jerrard form of the quintic

In the 1871 paper, Clebsch supposed that the discriminant R of the quintic f does not vanish. He recalled that the problem of representing f as a sum of three fifth powers could be done by using the two linear covariants α, δ as new variables: the problem amounted to determining six numbers κ, κ', κ'', m, m', m'' such that

$$f = \kappa(\delta + m\alpha)^5 + \kappa'(\delta + m'\alpha)^5 + \kappa''(\delta + m''\alpha)^5.$$

Citing Salmon's *Lessons Introductory to Modern Higher Algebra* [Salmon 1866], he stated that the three factors $\delta + m\alpha$, $\delta + m'\alpha$, $\delta + m''\alpha$ are necessarily the factors of the cubic covariant j, that is:

$$j = k(\delta + m\alpha)(\delta + m'\alpha)(\delta + m''\alpha),$$

the number k being just a multiplicative constant. This is where the typical representation of j,

$$R^2 j = -\delta^3 - \frac{3}{2}N\delta\alpha^2 + \frac{1}{2}(CM - BN)\alpha^3,$$

was used. Indeed, setting $\delta = -m\alpha$ in this expression proved that m, m' and m'' are the three roots of

$$m^3 + \frac{3}{2}Nm + \frac{1}{2}(CM - BN) = 0.$$

The determination of the numbers κ was based, on its part, on the typical representation of f, of which Clebsch made explicit its first terms:

$$R^4 f = \left(\frac{2A^2}{3} - B\right)\delta^5 + 5\left(\frac{N}{2} - \frac{AM}{3}\right)\delta^4\alpha + 10\frac{M^2}{6}\delta^3\alpha^2 + \cdots$$

Identifying between these coefficients and those given by the expression of f as a sum of fifth powers then yielded a system of three linear equations for κ, κ', κ'': for instance, considering the coefficient of δ^5 gives $\kappa + \kappa' + \kappa'' = \frac{1}{R^4}\left(\frac{2A^2}{3} - B\right)$.

However, the computation of the numbers κ, that is, the inversion of this linear system, could be carried out only for $C \neq 0$. For this reason, Clebsch supposed that C converged to 0. Using what we can see as first-order approximations of the different functions, he obtained formulas for all the coefficients $\kappa, \ldots, m''$, so that the expression of f as a sum of fifth powers then became

$$R^4 f = \frac{2A^2}{3}(\delta - B\alpha)^5 - B\left(\delta + \frac{B}{2}\alpha\right)^5 + \frac{5B^2}{4}\alpha\left(\delta + \frac{B}{2}\alpha\right)^4.$$

a necessary and sufficient condition for two binary forms of the same order to be linearly transformable one into the other, [Clebsch 1870].

Finally, he introduced the two linear covariants $\xi = \delta - B\alpha$ and $\eta = \delta + \frac{B}{2}\alpha$, thanks to which the previous equation is transformed into

$$R^4 f = \frac{2A^2}{3}\xi^5 - \frac{5B}{6}\xi\eta^4 - \frac{B}{6}\eta^5.$$

Setting $x = \xi/\eta$, this proves that the quintic equation $f = 0$ is equivalent to

$$x^5 - \frac{B}{4A^2}(5x + 1) = 0,$$

which, since it contains no term in x^4, x^3 and x^2, is a Jerrard form.[44] Because the new homogeneous variables ξ, η are independent linear covariants of f, and thus linear expressions of the initial variables, this proved that a quintic equation for which $C = 0$ can be linearly transformed into a Jerrard equation.

5.3 The art of making invariants vary

The question, then, was to search a way to pass from the general quintic equation $f(x) = 0$ to one for which the invariant C vanishes. Since invariants remain the same under the action of linear transformations, this can only be done by using what Clebsch called a higher transformation

$$x' = \frac{\varphi(x)}{\psi(x)},$$

where φ and ψ are polynomials. Using such transformations to investigate algebraic equations was not new: Clebsch referred to the works of Hermite and Gordan, who had both used this technique to tackle the quartic equation, [Hermite 1858b; Gordan 1870]. In particular, Hermite had proved that it is possible to find a polynomial transformation making the fundamental invariant i of the quartic vanish; just like what has been seen above with C in the case of the quintic, the condition $i = 0$ implied that the quartic can be transformed into a special form related to the theory of elliptic functions, [Goldstein 2011, p. 249].

In his 1871 paper, Clebsch focussed on quadratic transformations, which correspond to the case where the above polynomials φ and ψ are of degree 2. Each of them is thus determined by three coefficients, defined up to a non zero multiplicative constant. Clebsch interpreted these coefficients as homogeneous coordinates of points P, Q in the plane: a quadratic substitution being defined by two such points, it corresponds to a straight line PQ.

The key point in this interpretation is that a quadratic substitution having the effect of making an invariant J vanish is associated with a line PQ that is tangent to a certain curve defined by J. More precisely, Clebsch relied on the

[44] Clebsch did not took the time to transform this equation into $x^5 - x - a = 0$, which is the Jerrard form in the strict sense of the term. This can be done easily by replacing x by ax, where $a = \sqrt[4]{-\frac{5B}{4A^2}}$.

process he had proposed in the 1861 paper on the symbolic representation of invariants: if $J = \sum K \prod(ab)$, the curve in question is defined by the tangential equation

$$\sum K \prod(abu) = 0,$$

an equation obtained by adding line coordinates in the symbolic determinants. The curve associated with the invariant C was studied very thoroughly by Clebsch, with techniques combining projective geometry and invariant theory. This allowed him to prove the existence of tangents to the curve having good properties. The aim being to solve the quintic, indeed, the coefficients of the equation of the tangent must not contain any inadequate irrationality, such as fifth roots of numbers.[45]

5.4 Disciplinary mixes III

Contrary to the two previous cases, Clebsch treated here a problem centred around an algebraic equation. Instead of resting on already-known properties, Clebsch had to prove himself that a quintic equation can be transformed into a Jerrard equation if its invariant C vanishes. His demonstration displays a wide range of techniques of the theory of forms and invariants, such as the typical representations and the issue of expressing a quintic form as a sum of three fifth powers. Works on invariant theory that Clebsch had previously developed hence found an interesting application to equation theory.

That said, Clebsch's principal objective was to present a "geometrical image" of the elements dealing with the solution of the quintic.[46] Geometry occurred first to interpret the parameters of a quadratic substitution as the coordinates of two points in the plane – the transformation itself was not interpreted geometrically. Finding a transformation making $C = 0$ was then tantamount to determining a tangent to the curve defined from C by method developed in the 1861 memoir.

The three case-studies that I have depicted thus display different disciplinary configurations within the mathematical technique. They exemplify that the interactions of invariant theory with geometry and equation theory took different shapes according to the initial situations and the results that needed to be developed. I hope to have thus helped to enlighten the kind of practice that could be made of invariants, especially form the point of view of their applications.

[45]See [Lê 2017] for more details on this and other features of Clebsch's paper that have not been dealt with here.

[46]"So finden sich denn wirklich alle Elemente der Auflösung der Gleichungen 5ten Grades hier in einem geometrischen Bilde zusammengefasst und verbunden." [Clebsch 1871a, p. 354].

6. Making equations explicit

There is yet another facet of invariant theory that can be seen in Clebsch's papers. More vague and more discrete, it has to do with the issue of "really" establishing given equations. Such allusions are scattered throughout the publications of Clebsch. Let me make a quick tour on a selection of them.

The first one appears in the very first paper of Clebsch where invariants intervene, where Clebsch proposed a quick, not detailed version of the symbolic representation of invariants, [Clebsch 1861d]. Clebsch tackled the problem of transforming a quaternary cubic form into the sum of five cubes, a problem linked with the theory of cubic surfaces. Given a cubic form $u = \sum a_{ikh}x_i x_h x_k$ with variables x_1, x_2, x_3, x_4, the problem was to determine linear functions $A_i = \alpha_{1i}x_1 + \alpha_{2i}x_2 + \alpha_{3i}x_3 + \alpha_{4i}x_4$ such that

$$u = A_1^3 + A_2^3 + A_3^3 + A_4^3 + A_5^3.$$

Clebsch simply asserted that such a determination was possible because the constants α were in an adequate number, and he explained that the A_i are linked together by a linear relation

$$k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4 + k_5A_5 = 0,$$

the k_i being constants.

Invariants were first mentioned a few pages later, when Clebsch asserted that "with the help of invariant theory, it is now possible to really set the equation of the fifth degree on which depends the [above] transformation".[47] As the following lines of the paper make clear, this equation is the one of which the numbers k_i^6 are the roots. Such an equation obviously exists, and Clebsch denoted it by

$$k^{30} - C_1k^{24} + C_2k^{18} - C_3k^{12} + C_4k^6 - C_5 = 0.$$

Clebsch then computed the coefficients C_i explicitly by providing their expression as polynomials in five invariants $J_1, \ldots, J_5$ of the cubic form u. The proof made use of the symbolic notation and a the property according to which every invariant of u is a polynomial in the J_i.[48]

Invariant theory thus appeared to compute explicitly an object of which the existence was known *a priori*, and Clebsch emphasised this specific role. Other situations of the same vein can be observed in his papers, although the

[47] "Mit Hülfe der Invariantentheorie ist es nun möglich, die Gleichung fünften Grades wirklich aufzustellen, von welcher die Transformation (5.) abhängt."

[48] As Clebsch realised shortly after, [Clebsch 1861a], a paper of Salmon mentioned an invariant that could not be deduced from the J_i, which evidenced that his proof was erroneous. Clebsch also admitted that his argument for the existence of the transformation, based on a constant counting, was dubious. His correction did not involve invariant theory any more.

vocabulary was not exactly the same and the link with forms and invariants was not always introduced explicitly.

For instance, let me briefly return to the main paper devoted to the symbolical representation [Clebsch 1861e]. As accounted for above, one of the applications of this representation and the related techniques was to provide the "complete solution" of the problem of "expressing any curve in line coordinates". The existence of a tangential equation of a curve of a given order, indeed, ensues directly from general consideration pertaining to projective duality, and Clebsch's "complete solution" consisted in writing down this equation, which was achieved by the addition of a series of line coordinates u in the symbolic expression of the discriminant of a binary form.

Still in 1861, another example is contained in a paper devoted to the inflectional tangents to a curve of the third order, that is, the tangents at each of the nine inflection points of the curve, [Clebsch 1861b]. Clebsch's main theorem was that if $u = 0$ is the equation of a cubic curve, the inflectional tangents of the latter are also tangents to the curve $\Delta(u) = 0$, where $\Delta(u)$ designates the Hessian determinant of u. This theorem, Clebsch explained, could be deduced from "the establishment of the expression of the ninth order which [...] represents the product of the equations of the nine inflectional tangents". He immediately added:

> That such an expression, with rational coefficients, exists is clear from the outset; the representation of it can be undertaken in the following way.[49] [Clebsch 1861b, p. 323]

Here again, the role of invariants in the process of searching for the establishment of the ninth-degree equation was not made explicit in Clebsch's comments but it appears clearly when one looks at the rest of the paper. Many results on binary and ternary forms were developed, in particular with the help of the symbolic notation, and Clebsch ended up with the equation

$$\Delta(v)^3 + 72Sv^2\Delta(v) - v \cdot \frac{1}{6} \sum \sum V_{ik}\Delta_i\Delta_k = 0,$$

where v is a quartic form associated with the cubic one, S is one of its invariants and the Δ_i and V_{ik} are other quantities linked to v.

Because this equation represents the geometrical locus made of the nine inflectional tangents, it is different in nature than the equations that would be called "geometrical equations" from the end of the 1860s on. A geometrical equation was an algebraic equation in one unknown, each root of which corresponds to an object of a given geometric configuration. A famous example was

[49] "Dieser Satz [...] folgt [...] aus der Aufstellung des Ausdruckes neunter Ordnung, welcher [...] das Product der Gleichungen der neun Wendepunktstangenten darstellt. Dass ein solcher Ausdruck, mit rationalen Coefficienten, existirt, ist von vorn herein klar; die Darstellung desselben kann etwa in folgender Weise unternommen werden."

the nine-points equation, an equation of degree 9 associated with the nine inflection points of a cubic curve. Especially around 1870, geometrical equations gave rise to a series of activities by mathematicians, including Clebsch, who used them to better understand Galois' ideas on equations. The search for resolvents of a geometrical equation, for instance, was replaced by the search of specific objects that could be made from those of the geometric configuration.[50]

In particular, in this system of activities centred around questions of resolution, the issue of the explicit writing of the equations and their resolvents was never posed: their existence stemmed directly from that of the associated geometric configurations. However, commenting on such works in his obituary of Julius Plücker, Clebsch wrote:

> It was reserved for the advances of the modern algebra created by Sylvester, Cayley and Salmon, and in particular for the beautiful discoveries of Aronhold, to really form all the equations [associated with the nine-points equation] to be solved, and thus to settle the problem.[51]

The question of "really" forming geometrical equations and resolvents thus belonged to invariant theory, and was solved by techniques pertaining to it.[52]

A few other examples of this association between the invariant theory and the issue of making explicit objects whose existence is known *a priori* can be found in Clebsch's papers, in relation with similar mathematical situations or not [Clebsch 1868, 1869].

On the whole, such occurrences do not proliferate in these papers, nor are they commented on more than through the quotes given above. In particular, Clebsch did not elaborate on the mathematical reasons explaining why invariant theory would be especially suited to make equations explicit. Moreover, the issue of the explicit was never thematised as such in the few passages where Clebsch presented invariant theory as a mathematical discipline, with its privileged objects, theorems and problems.[53] Nevertheless the traces that

[50] The activities linked with geometrical equations are analysed in [Lê 2015a, 2016]. The link with invariants theory is mentioned in [Lê 2015b, p. 234].

[51] "Den Fortschritten der von Sylvester, Cayley und Salmon geschaffenen neuern Algebra, und zwar insbesondere den schönen Entdeckungen Aronholds, war es vorbehalten, alle zu lösenden Gleichungen wirklich zu bilden, und damit das Problem zu erledigen." [Clebsch 1872e, p. 22].

[52] Clebsch's formulation is too vague to understand if he saw one of the mentioned equations in Aronhold's work or if he only meant that the results developed by the latter could be used to establish them. For instance, one resolvent of the nine-points equation is an equation associated with four special triangles. Although it is not presented as such, it coincides with the equation (27) in [Aronhold 1850, p. 154], which is used for instance in [Clebsch 1861b, p. 231]. The same equation is more explicitly associated with the four triangles in [Clebsch and Lindemann 1876, p. 563]. In all these references, the equation is written down thanks to invariant theory.

[53] See [Clebsch 1872a], as well as the introductions of [Clebsch 1871b, 1872d]. Clebsch did not use the word "discipline". Following [Goldstein and Schappacher 2007, p. 54], I employ this term to refer to an "object-oriented system of scholarly activities".

have been presented in this section seem strong enough to indicate a sort of coherence which is important to account for to get a better view on Clebsch's work on invariants.

7. Aborted projects

During the last months of his life, Clebsch produced a number of works whose nature and content seem to set up fragments of a certain program on forms and invariants: in addition to syntheses on these subjects, several articles on new notions were published, in which research questions were posed and placed in broad descriptions where geometry had a major role.

Indeed, inspired by past works of Julius Plücker and Hermann Grassmann, Clebsch defined forms with several series of non independent variables,[54] and he launched their study and that of their invariants, formulating the "fundamental task of invariant theory" of establishing a finiteness theorem in this extended framework [Clebsch 1872d]. While he did not reach this ultimate goal in this publication, he still immediately applied some of the obtained results to geometric issues that he had never tackled before [Clebsch 1873b]. Further, in the special case of two series of variables, he was led to turn his attention to "intermediate forms" (*Zwischenformen*), which had already been considered by some of his predecessors but which he took as the basis to define "connexes", a new kind of geometric objects whose investigation was supposed to "include in itself the whole analytic geometry of the plane", [Clebsch 1873a, p. 203]. As Clebsch explained,

> For the analytic geometry of the plane (algebra of ternary forms), the necessity arose to investigate a structure [i.e. connexes] which includes the algebraic curve as a very special case, and which is at the same time the most comprehensive one whose study can be demanded by the theory of invariants for ternary forms.[55] [Clebsch 1873a, p. 203]

Connexes and intermediate forms were thus objects at the core of a renewed connection between plane geometry and the theory of ternary forms, which Clebsch's words almost assimilated as one and the same domain.

Applications to geometry were also one of the main motivations behind the vast summary on cubic ternary forms that Clebsch and Gordan wrote at about the same time, [Clebsch and Gordan 1873]. Another publication meant to gather knowledge on forms was Clebsch's book *Theorie der binären*

[54] From a current point of view, these variables appear as Grassmannian coordinates.

[55] "Für die analytische Geometrie der Ebene (Algebra der ternären Formen) ergab sich hiebei die Nothwendigkeit, ein Gebilde zu untersuchen, welches die algebraische Curve als sehr besondern Fall einschliesst und welches zugleich das umfassendste ist, dessen Studium durch die Invariantentheorie bei ternären Formen gefordet werden kann. Dieses Gebilde, dessen Untersuchung so zu sagen die ganze analytische Geometrie der Ebene in sich schliesst [...]."

algebraischen Formen, which, as we saw in our introduction, was accompanied by a separate presentation aimed at explaining to a large circle of readers the ins and outs of invariant theory [Clebsch 1872a,b].

In this presentation, Clebsch explained that algebra was historically rooted in both equation theory and analytic geometry. Although the two domains seemed completely disconnected at first sight, the mathematicians who had been engaged in them had recognized that "[i]t was the study of what remains fixed by the change of manifold reorganisations that soon appeared as being the most important and the most useful."[56] Such hints referred to resolvents of equations on one hand, and to the properties of curves and surfaces that remain unchanged by projections on the other hand. The "common viewpoint" on these phenomena, Clebsch continued, had been yielded by the notion of algebraic form:

> All these investigations could be summarised in an elegant form under a common point of view by introducing the concept of homogeneous function. It turned out that the theory of homogeneous functions led to equations, curves or surfaces, depending on the number of homogeneous variables being 2, 3 or 4.[57] [Clebsch 1872a, p. 323]

The understanding of forms, which went through the study of their invariants, was thus crucial for anyone who wanted to research geometry.

Together with the year 1872, all these traces which evoke a unifying view on (parts of) geometry and algebra are irresistibly reminiscent of Felix Klein's *Erlanger Programm* [Klein 1872], whose author was in close scientific relation with Clebsch at that time.[58] Even if Klein's unifying perspective is not exactly the same as Clebsch's, the *Programm* contains many ideas that we have seen in the latter's publications: the insistence on invariants, the assimilation of form theory with some pieces of geometry, the parallel made between geometry and equation theory, and even the mention of connexes. However, the crucial objects for Klein were groups of transformations (of space), and when he stated what he considered as a generalisation of geometry, he associated invariants with groups, not with forms:

> Given a manifold and a group of transformations of the same; to develop the theory of invariants relating to that group.[59]

[56]"Es war das Studium des Festen im Wechsel mannigfacher Umgestaltung, was [...] bald als das Wichtigste und Förderlichste erschien." [Clebsch 1872a, p. 323].

[57]"Alle diese Untersuchungen liessen sich durch Einführung des Begriffes homogener Function unter gemeinsamen Gesichtpunct und in eleganter Form zusammenfassen. Es zeigte sich, dass die Theorie der homogenen Functionen auf die Gleichungen, auf Curven oder Oberflächen führte, jenachdem die Anzahl der homogenen Veränderlichen 2, 3 oder 4 war."

[58]On this program, see [Rowe 1983; Hawkins 1984; Gray 2005; Lê 2015a]. On Klein and his links with Clebsch, see [Tobies 2019, pp. 37–51].

[59]"Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben. Man entwickele die auf die Gruppe bezügliche Invariantentheorie." [Klein 1872, p. 7]. The translation comes from [Klein 1893, p. 219].

In this global framework, forms, which were the overarching objects for Clebsch, were thus relegated to a place of a lesser rank, to the profit of groups.[60] The disciplinary coherence constructed in the *Erlanger Programm* thus came in part from the dislocation and recomposition of elements which formed an other kind of coherence in Clebsch's research.[61]

We know the formidable influence that the *Erlanger Programm* eventually had on mathematicians. But only eventually: as Thomas Hawkins [1984] evidenced, Klein's viewpoints did not become widely known before the very end of the 19th century, after other works on groups and geometry had been published by other mathematicians. And Clebsch's proposal reminds us that in the early 1870s, objects other than groups could be put forward to make algebra and geometry coalesce.

As seen in this chapter, such a proposal was fuelled by a practice of algebraic forms that Clebsch had shaped over more than a decade by knitting together invariants, geometry and algebraic equations, in particular for studying curves and surfaces. These numerous and variegated applications constitute an important part of Clebsch's contribution to invariant theory, a part complemented by two others, which somewhat correspond to other levels in the development of invariant theory. On one hand, the investigations on forms and invariants without immediate applications participated to the setting-up of a certain body of research on invariants with its key issues, such as the systematic listing of invariants and covariants associated with a given situation.[62] On the other hand, the elaboration of a program of unification of geometry and algebra by the means of algebraic forms and their invariants constitutes yet another kind of commitment in the theory.

Clebsch's role in the history of invariant theory cannot therefore be reduced to the systematisation of the symbolic notation. Moreover, his case – and, potentially, that of the numerous other contributors[63] – shows that this history is far from being limited to a linear sequence of events associated with the finiteness theorem and involving only a handful of mathematicians. Contrary

[60] This does not mean that forms were not important in other papers of Klein. His research on the icosahedron and the quintic equation, for instance, involved groups dramatically all the while making considerable use of binary forms together with their invariants and covariants, [Gray 2000, pp. 81–87, 126–135].

[61] On this point, see also [Lê 2015a]. The same kind of mechanism can be seen in the case of the icosahedron, see [Goldstein 2011, pp. 259–260; Lê 2017, pp. 67–68]. The latter reference recalls that groups (be they of transformations or substitutions) are absent from Clebsch's entire work.

[62] This facet of Clebsch's research is a bit less visible in the present chapter, precisely because I chose not to focus on them. Nevertheless it has been encountered in his work on typical forms, as well as in his summaries on binary forms and on ternary cubic forms.

[63] For instance, the sections of the *Catalogue of scientific papers* devoted to forms count 246 different authors. In *Mathematische Annalen*, where invariant theory has been recognized as a specific topic, 39 mathematicians published papers on it between 1868 and 1898. See [Lê 2022, pp. 17–18].

to what this theorem states for invariants, there can be no question of writing a history of invariant theory by expressing all the works relating to it on the mere basis of a too small number of them.

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References

Aronhold Siegfried (1850), "Zur Theorie der homogenen Functionen dritten Grades von drei Variabeln", *Journal für die reine und angewandte Mathematik* **39**, 140–159 (↑ 10, 15, 31).

— (1858), "Theorie der homogenen Functionen dritten Grades von drei Veränderlichen.", *Journal für die reine und angewandte Mathematik* **55**, 97–191 (↑ 10).

Boniface Jacqueline (2004), *Hilbert et la notion d'existence en mathématiques*, Paris: Vrin (↑ 2).

Brechenmacher Frédéric (2011), "Autour de pratiques algébriques de Poincaré". Preprint, https://hal.science/hal-00630959v3 (↑ 3).

Brill Alexander, Gordan Paul, Klein Felix, Lüroth Jacob, Mayer Adolph, Noether Max, and Mühll Karl von der (1873), "Rudolf Friedrich Alfred Clebsch – Versuch einer Darlegung und Würdigung seiner wissenschaftlichen Leistungen", *Mathematische Annalen* **7**, 1–55 (↑ 2–4).

Cayley Arthur (1858), "A Fifth Memoir upon Quantics", *Philosophical Transactions of the Royal Society of London* **148**, 429–460 (↑ 9).

— (1859), "Note sur les normales d'une conique", *Journal für die reine und angewandte Mathematik* **56**, 182–185 (↑ 16).

Clebsch Alfred (1857), "Anwendung der elliptischen Funktionen auf ein Problem der Geometrie des Raumes", *Journal für die reine und angewandte Mathematik* **53**, 292–308 (↑ 4).

— (1861a), "Ueber die Knotenpunkte der Hesseschen Fläche, insbesondere bei Oberflächen dritter Ordnung", *Journal für die reine und angewandte Mathematik* **59**, 193–228 (↑ 29).

— (1861b), "Ueber die Wendetangente der Curven dritter Ordnung", *Journal für die reine und angewandte Mathematik* **58**, 229–239 (↑ 30, 31).

— (1861c), "Ueber eine symbolische Darstellungsweise algebraischer Formen, und über die davon zu machende Anwendung auf Probleme der Elimination", *Monatsberichte der Königlichen Preußischen Akademie der Wissenschaften zu Berlin* **1860**, 536–540 (↑ 10).

Clebsch Alfred (1861d), “Ueber eine Transformation der homogenen Functionen dritter Ordnung mit vier Veränderlichen”, *Journal für die reine und angewandte Mathematik* **58**, 109–126 (↑ 10, 29).

— (1861e), “Ueber symbolische Darstellung algebraischer Formen”, *Journal für die reine und angewandte Mathematik* **59**, 1–62 (↑ 4, 10–13, 30).

— (1863), “Ueber das Problem der Normalen bei Curven und Oberflächen der zweiten Ordnung”, *Journal für die reine und angewandte Mathematik* **62**, 64–109 (↑ 16, 19, 20, 22).

— (1868), “Ueber die Flächen vierter Ordnung, welche eine Doppelcurve zweiten Grades besitzen”, *Journal für die reine und angewandte Mathematik* **69**, 142–184 (↑ 31).

— (1869), “Note zu dem Aufsatze ‘über eine Eigenschaft von Functionaldeterminanten’ ”, *Journal für die reine und angewandte Mathematik* **70**, 175–181 (↑ 31).

— (1870), “Ueber die Möglichkeit, zwei gegebene binäre Formen linear in einander zu transformiren”, *Mathematische Annalen* **2**, 373–381 (↑ 4, 26).

— (1871a), “Ueber die Anwendung der quadratischen Substitution auf die Gleichungen 5$^{\text{ten}}$ Grades und die geometrische Theorie des ebenen Fünfseits”, *Mathematische Annalen* **4**, 284–345 (↑ 24, 28).

— (1871b), “Zur Theorie der binären algebraischen Formen”, *Mathematische Annalen* **3**, 265–267 (↑ 31).

— (1871c), “Zur Theorie der Cremona’schen Transformationen”, *Mathematische Annalen* **4** (490-496) (↑ 4).

— (1872a), “Clebsch, *Theorie der binären algebraischen Formen*”, *Göttingische gelehrte Anzeigen* **1872** (1), 321–334 (↑ 1, 2, 24, 31, 33).

— (1872b), *Theorie der binären algebraischen Formen*, Leipzig: Teubner (↑ 1, 33).

— (1872c), “Ueber die geradlinigen Flächen vom Geschlechte $p = 0$”, *Mathematische Annalen* **5**, 1–26 (↑ 4).

— (1872d), “Ueber eine Fundamentalaufgabe der Invariantentheorie”, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen* **17**, 3–62 (↑ 31, 32).

— (1872e), “Zum Gedächtniss an Julius Plücker”, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen* **16**, 1–40 (↑ 31).

— (1873a), “Ueber ein neues Grundgebilde der analytischen Geometrie der Ebene”, *Mathematische Annalen* **6**, 203–215 (↑ 32).

— (1873b), “Zur Theorie der Charakteristiken”, *Mathematische Annalen* **6**, 1–15 (↑ 32).

— (1861/1862), “Sur un problème concernant la théorie des surfaces du 2$^{\text{e}}$ ordre”, *Annali di Matematica Pura ed Applicata*, 1st ser. **4**, 195–198 (↑ 16).

Clebsch Alfred and Gordan Paul (1873), “Ueber cubische ternäre Formen”, *Mathematische Annalen* **6**, 436–512 (↑ 32).

Clebsch Alfred and Gordan Paul (1867/1868), “Sulla rappresentazione tipica delle forme binarie”, *Annali di Matematica Pura ed Applicata*, 2nd ser. **1**, 23–79 (↑ 4, 25).

Clebsch Alfred and Lindemann Ferdinand (1876), *Vorlesungen über Geometrie*, vol. 1, Leipzig: Teubner (↑ 31).

Confalioneri Sara, Schmidt Peter-Maximilian, and Volkert Klaus (eds.) (2019), *Der Briefwechsel von Wilhelm Fiedler mit Alfred Clebsch, Felix Klein und italienischen Mathematikern*, Siegen: Universitätsverlag Siegen (↑ 2).

Corry Leo (2004), *Modern Algebra and the Rise of Mathematical Structures*, 2nd ed., Basel, Boston, Berlin: Birkhäuser (↑ 1).

Crilly Tony (1986), “The Rise of Cayley’s Invariant Theory (1841–1862)”, *Historia Mathematica* **13**, 241–254 (↑ 2).

— (1988), “The Decline of Cayley’s Invariant Theory (1863-1895)”, *Historia Mathematica* **15** (4), 332–347 (↑ 10).

Eisenstein Gotthold (1844), “Allgemeine Auflösung der Gleichungen von den ersten vier Graden”, *Journal für die reine und angewandte Mathematik* **27**, 81–83 (↑ 15).

Fischer Gerd (2001), *Plane Algebraic Curves*, Providence: American Mathematical Society. English translation by Leslie Kay (↑ 17).

Fisher Charles S. (1966), “The Death of a Mathematical Theory: A Study in the Sociology of Knowledge”, *Archive for History of Exact Sciences* **3** (2), 137–159 (↑ 2).

Goldstein Catherine (2011), “Charles Hermite’s Stroll through the Galois Field”, *Revue d’histoire des mathématiques* **17**, 211–270 (↑ 24, 27, 34).

— (2023), “Poincaré and Arithmetic Revisited”. To appear (↑ 3).

Goldstein Catherine and Schappacher Norbert (2007), “A Book in Search of a Discipline”, in Catherine Goldstein, Norbert Schappacher, and Joachim Schwermer (eds.), *The Shaping of Arithmetic after C. F. Gauss’s Disquisitiones Arithmeticae*, Berlin: Springer, 3–65 (↑ 31).

Gordan Paul (1870), “Ueber die Invarianten binären Formen bei höheren Transformationen”, *Journal für die reine und angewandte Mathematik* **71**, 164–194 (↑ 27).

Gray Jeremy (2000), *Linear Differential Equations and Group Theory from Riemann to Poincaré*, 2nd ed., Boston: Birkhäuser (↑ 34).

— (2005), “Felix Klein’s Erlangen Program, ‘Comparative Considerations of Recent Geometrical Researches’ (1872)”, in Ivor Grattan-Guinness (ed.), *Landmark Writings in Western Mathematics, 1640-1940*, Amsterdam: Elsevier Press, 544–552 (↑ 33).

Hawkins Thomas (1984), “The *Erlanger Programm* of Felix Klein: Reflexions on Its Place in the History of Mathematics”, *Historia Mathematica* **11**, 442–470 (↑ 33, 34).

Hermite Charles (1854), "Sur la théorie des fonctions homogènes à deux indéterminées", *The Cambridge and Dublin Mathematical Journal* **9**, 172–271 (↑ 24).

— (1856), "Sur la théorie des fonctions homogènes à deux indéterminées. Premier mémoire", *Journal für die reine und angewandte Mathematik* **52**, 1–17. Œuvres I, 350–371 (↑ 14).

— (1858a), "Sur la résolution de l'équation du cinquième degré", *Comptes rendus hebdomadaires des séances de l'Académie des sciences* **46**, 508–515 (↑ 24).

— (1858b), "Sur quelques théorèmes d'algèbre et la résolution de l'équation du quatrième degré", *Comptes rendus des séances de l'Académie des sciences* **46**, 961–967 (↑ 27).

Houzel Christian (2002), *La géométrie algébrique : recherches historiques*, Paris: Albert Blanchard (↑ 24).

Joachimstahl Ferdinand (1857), "De aequationibus quarti et sexti gradus quae in theoria linearum et superficierum secundi gradus occurunt", *Journal für die reine und angewandte Mathematik* **53**, 149–172 (↑ 18).

Klein Felix (1872), *Vergleichende Betrachtungen über neuere geometrische Forschungen*, Erlangen: Andreas Deichert (↑ 33).

— (1893), "A Comparative Review of Recent Researches in Geometry", trans. by M. W. Haskell, *Bulletin of the New York Mathematical Society* **2**, 215–249 (↑ 33).

Kung Joseph P. S. and Rota Gian-Carlo (1984), "The Invariant Theory of Binary Forms", *Bulletin of the American Mathematical Society* **10** (1), 27–85 (↑ 10).

Lê François (2016), "Reflections on the Notion of Culture in the History of Mathematics: The Example of 'Geometrical Equations'", *Science in Context* **29** (3), 273–304 (↑ 31).

— (2017), "Alfred Clebsch's 'Geometrical Clothing' of the Theory of the Quintic Equation", *Archive for History of Exact Sciences* **71** (1), 39–71 (↑ 24, 28, 34).

— (2022), "La théorie des surfaces algébriques dans les *Mathematische Annalen* à l'épreuve de la textométrie (1869-1898)", *Revue d'histoire des mathématiques* **28** (1), 1–45 (↑ 34).

— (2015a), "'Geometrical Equations': Forgotten Premises of Felix Klein's *Erlanger Programm*", *Historia Mathematica* **42** (3), 315–342 (↑ 31, 33, 34).

— (2015b), "Vingt-sept droites sur une surface cubique : rencontres entre groupes, équations et géométrie dans la deuxième moitié du XIX^e siècle", PhD thesis, Université Pierre et Marie Curie (↑ 31).

Lorenat Jemma (2015), "Polemics in Public: Poncelet, Gergonne, Plücker, and the Duality Controversy", *Science in Context* **28** (4), 545–585 (↑ 13).

Meyer Wilhelm Franz (1892), “Bericht über den gegenwärtigen Stand der Invariantentheorie”, *Jahresbericht der Deutschen Mathematiker-Vereinigung* **1**, 79–292 (↑ 24).

Parshall Karen (1989), “Toward a History of Nineteenth-Century Invariant Theory”, in John McCleary and David E. Rowe (eds.), *The History of Modern Mathematics*, vol. 1. Ideas and their Reception, Boston, San Diego, New York: Academic Press, 157–206 (↑ 2, 8, 10).

— (2023), “A Convergence of Paths: Arthur Cayley, Charles Hermite, James Joseph Sylvester, and the Early Development of a Theory of Invariants”, *Revue d’histoire des mathématiques*. To appear (↑ 3).

Petri Birgit and Schappacher Norbert (2004), “From Abel to Kronecker. Episodes from 19th Century Algebra”, in Olav Arnfinn Laudal and Ragni Piene (eds.), *The Legacy of Niels Henrick Abel*, Berlin, Heidelberg, New York: Springer, 227–266 (↑ 24).

Rowe David E. (1983), “A Forgotten Chapter in the History of Felix Klein’s *Erlanger Programm*”, *Historia Mathematica* **10**, 448–457 (↑ 33).

Salmon George (1866), *Lessons Introductory to the Modern Higher Algebra*, 2nd ed., Dublin: Hodges, Smith, and Co. (↑ 1, 26).

Salmon George and Fiedler Wilhelm (1860), *Analytische Geometrie der Kegelschnitte, mit besonderer Berücksichtigung der neueren Methoden*, Leipzig: Teubner. Translated and revised by Wilhelm Fiedler (↑ 16).

Tobies Renate (2019), *Felix Klein: Visionen für Mathematik, Anwendungen und Unterricht*, Berlin, Heidelberg: Springer (↑ 33).

Wolfson Paul R. (2008), “George Boole and the Origins of Invariant Theory”, *Historia Mathematica* **35**, 37–46 (↑ 8).