Alfred Clebsch's "Geometrical Clothing" of the Theory of the Quintic Equation

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December 2015

Abstract

This paper describes a 1871 article of the mathematician Alfred Clebsch presenting a geometrical interpretation of elements of the theory of the general algebraic equation of degree 5. Clebsch's approach, which has been completely forgotten by the historiography, is here used to discuss the relations between geometry, intuition, figures, and visualization. To be more precise, although the article of Clebsch does not contain any figure, we try to clearly delineate what he perceived as geometric in his approach. In particular, we show that Clebsch's use of geometrical objects and techniques does not point to visualization matters, but rather to a way of guidance in algebraic calculations.

1 Introduction

In 1871, the mathematician Alfred Clebsch (1833-1872) published an article in which he aimed at presenting a geometrical interpretation of elements of the theory of the general algebraic equation of degree 5, (Clebsch 1871b). More specifically, Clebsch intended to expose "a complete geometrical overall view on the connections existing between the equations of degree 5 and their resolvents, and especially on the connection with the Jerrard form and the modular equation,"[1] alluding to the researches that Charles Hermite and Leopold Kronecker had carried out on this topic in the 1850s.

This effort to make geometry intervene in another domain of mathematics is far from being an isolated case in the 19th century. Several examples of such geometrical interventions, occurring whether in algebra, in analysis, or in number theory, have already been observed and discussed by historians of mathematics. For instance, let us only mention

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[1]"So erhält man als eine erste Anwendung der im Eingange der Abhandlung entwickelten allgemeinen Principien eine vollständige geometrische Uebersicht über die Zusammenhänge, welche zwischen den Gleichungen 5^{ten} Grades und ihren Resolventen bestehen, insbesondere über den Zusammenhang mit der Jerrard'schen Form und der Modulargleichung." (Clebsch 1871b, p. 285).

the problem of the geometrical representation of complex numbers, (Flament 2003), the role of geometrical diagrams in the theory of differential equations, (Tournès 2012), or the use of geometry in arithmetic as advocated in the geometry of numbers, (Gauthier 2009). Beyond the peculiarities of the cases they described, these studies have in common to have insisted on the fact that some past mathematicians grasped geometry because it is intuitive and allows visualization, in particular through figures and drawings.[2]

It would be misleading, however, to think of geometry as a systematic and exclusive affair of figures, diagrams, visualization, and spatial intuition. For example, when some of the "friends and old students" of Clebsch wrote an obituary of him in 1873,[3] they depicted another conception of geometry to be found in some of his works:

> Clebsch was mainly an algebraist, and the totality of his works is a complete mastery of the algebraic machinery. Alongside this machinery, a clear geometrical perception, by virtue of which each step composing the calculation is brought to an intuitive understanding, sets up in his later research. But this is not a concrete way consisting in seeing spatial circumstances as it can be for other geometers; rather, geometrical intuition is for him symbols and means of orientation for the algebraic problems he deals with.[4] (Brill et al. 1873, pp. 3-4)

So this quotation describes, in the context of algebraic calculations, a geometric intuition as a way of guidance in these calculations explicitly disconnected from visualization matters.[5]

My aim here is to confront this description with Clebsch's mathematical practice in order to ascertain how the "clear geometrical perception" supposed to enlightened "each step of the calculation" expresses itself in the technique. Three main reasons suggest Clebsch's article on the geometrical interpretation of the quintic equation to be a fruitful field for such a confrontation. First, the paper is dated June 1871 while Clebsch died in November 1872, making it part of the later research referred to in the preceding quotation.

[2]The issue of visualization has been extensively studied by philosophers of mathematics: see (Giaquinto 2007) and the references given in this book. See also the works of Valeria Giardino about cognitive aspects of the use of diagrams in specific parts of mathematics, for example in knot theory, (Giardino & De Toffoli 2013).

[3]The authors of the obituary are the mathematicians Alexander Brill, Paul Gordan, Felix Klein, Jacob Lüroth, Adolf Mayer, Max Noether, and Karl von der Mühll. They present themselves as Clebsch's "Freunde und ehemaligen Schüler" at the beginning of the text, (Brill et al. 1873, p. 1). These mathematicians closely worked with Clebsch indeed, so they surely knew how the latter used to do mathematics.

[4]"Clebsch war in erster Linie Algebraiker, und allen seinen Arbeiten gemeinsam ist die vollendete Beherrschung des algebraischen Apparates. Ihr zur Seite stellt sich in den späteren Untersuchungen die klare geometrische Auffassung, vermöge deren jeder Schritt, den die Rechnung vollführt, zu einem anschaulichen Verständnissse gebracht wird. Aber es ist nicht die concrete Art, die räumlichen Verhältnisse zu sehen, wie wir sie bei manchen anderen Geometern finden; die geometrische Anschauung ist ihm mehr Symbol und Orientirungsmittel für die algebraischen Probleme, mit denen er sich beschäftigt."

[5]That being said, I would like to stress that it does *not* mean that Clebsch did not use or praise visualization of geometrical objects in other contexts than the mentioned algebraic calculations. Moreover, let us note that Jemma Lorenat recently analyzed relations between geometry, algebraic computations, figures, and visualization in the first half of the 19th century, (Lorenat 2015). Her study notably brings to light the role of equations as representatives of geometrical objects in Julius Plücker's practice.

Second, the absence of figures in the paper[6] indicates its adequacy with the frame depicted in this quotation, and will encourage us to closely seek geometry in the calculation. Third, because its very topic consists in geometrically interpreting the theory of the quintic, the paper seems fit to clearly delineate what was perceived as geometric in the mathematical technique[7] and so to evaluate how Clebsch coped with an algebraic problem thanks to geometry.

Before entering into the technique, I would like to expose complementary information about Clebsch and his article, which have both been largely forgotten by the historiography. This will contribute to put them in context and understand the ins and outs of Clebsch's approach.

1.1 Biographical elements about Clebsch

Rudolf Friedrich Alfred Clebsch was born in 1833 in Königsberg.[8] His academic path began in 1850 in the university of this city where he attended lectures of Otto Hesse, Friedrich Richelot, and Franz Neumann (who was the father of his childhood friend Carl Neumann). In 1854, he completed his doctoral thesis which was about mathematical physics and which he had prepared under the supervision of F. Neumann.

From 1854 to 1858, Clebsch taught mathematics in several high schools of Berlin and simultaneously began to publish works on mathematical physics and calculus of variations.[9] His research then stayed mainly focused on these topics between 1858 and 1863, a period of time during which he was a professor in the *Polytechnische Schule* of Karlsruhe. At that time, Clebsch also began to be interested in the geometry of algebraic curves and surfaces, especially through the works of Arthur Cayley, George Salmon, and James Joseph Sylvester.

In 1863, Clebsch joined the university of Giessen as professor of mathematics.[10] There, he met Paul Gordan who introduced him to the theory of Abelian functions and to the works of Bernhard Riemann. Later, their consequent joint work led to the publication of the book *Theorie der Abelschen Functionen*, (Clebsch & Gordan 1866). Surrounded by Alexander Brill, Ferdinand von Lindemann, Jacob Lüroth, and Max Noether, Clebsch marked his entrance into the domain of Abelian functions by introducing geometrical

[6]The paper was published in the fourth volume of the *Mathematische Annalen*, a journal which did include figures at that time (see for instance the 1873 volumes 6 and 7).

[7]We will see that, in the absence of figures, Clebsch often explicitly qualified as "geometric" some of the objects or points of view he introduced.

[8]The biographical elements of this paragraph come from two obituaries of Clebsch, (Neumann 1872; Brill et al. 1873).

[9]The authors of the obituary (Brill et al. 1873) divide Clebsch's research into six domains: mathematical physics; calculus of variations and the theory of first-order partial differential equations; the theory of curves and surfaces; the study of Abelian functions and their applications to geometry; representations of surfaces; invariant theory. According to these authors, the order of this list approximately reflects the chronology of Clebsch's interests, (Brill et al. 1873, p. 2).

[10]Clebsch was notably in competition with Richard Dedekind. The circumstances of his recruitment are described in (Dugac 1976, p. 132). About Clebsch's staying in Giessen, see (Lorey 1937, pp. 71-77).

points of view and by conversely explaining how to apply these transcendental functions to geometry—the corresponding research gave rise to a great memoir entitled *Ueber die Anwendung der Abelschen Functionen in der Geometrie*, (Clebsch 1864).[11] From this research, Clebsch notably brought out the notion of the genus of an algebraic curve and his work on this notion led him to study birational transformations of curves and surfaces. As a result, he engaged into a research domain which would then interest him until the end of his life: the theory of representations of surfaces.[12] Nearly at the same time, Clebsch launched into invariant theory, a domain which became absolutely major for him.

Succeeding to Riemann in Göttingen, Clebsch obtained a professorship there in 1868, and some of his Giessen students consequently followed him.[13] The next year, the first volume of the *Mathematische Annalen* appeared, a journal that he and C. Neumann cofounded. Clebsch's research was then almost entirely dedicated to the theory of representations of surfaces and the theory of invariants. His paper on the geometrical interpretation of the quintic equation was elaborated during this period of his life, and we will see that Clebsch did largely use elements coming from both these theories in this paper.

On November 7 1872, Clebsch suddenly died from a case of diphtheria. At the age of 39, Clebsch left over 100 publications[14] as well as a great mathematical fame: considered "as one of the leading German mathematicians" of the time, having "the gifts, the multiple talents, and the working power,"[15] he was a correspondent to the Academies of Berlin, Munich, Milan, Bologna, and Cambridge, a member of the *London Mathematical Society*, and was in close relation with mathematicians like Cayley, Luigi Cremona, and Camille Jordan among others.

1.2 A forgotten paper on the quintic equation

The paper of Clebsch on the quintic equation seems to have been completely forgotten by the historiography of the subject.[16] Indeed, this historiography usually presents the story

[11] This research is partially sketched in (Gray 1989, pp. 367-369) and (Houzel 2002, pp. 184-186). The book has been qualified by Igor Shafarevich as the "Zeugnis der Geburt der algebraischen Geometrie" and the "erster Schrei des Neugeborenen." (Shafarevich 1983, p. 136). Further, see also (Rowe 1989, p. 188), where David E. Rowe talks about the "fledging school at Giessen that specialized in algebraic geometry and invariant theory."

[12] In modern terms, to represent an algebraic surface on the plane means to find a birational map between this surface and the projective plane.

[13] Clebsch was again in competition with Dedekind, (Dugac 1976, pp. 133-134).

[14] A list of Clebsch's publications can be found in (Brill et al. 1873, pp. 51-55). It counts 107 items, among which four books he is (one of) the author(s) and two books he edited on the basis of works of Plücker and of Carl Gustav Jacob Jacobi. The other publications essentially divide into papers in the *Journal für die reine und angewandte Mathematik* (from 1856 to 1869) and in the *Mathematische Annalen* (from 1869 on).

[15] Quotations of Hesse dated 1862 and reported in (Dugac 1976, p. 133). Other similar comments from other mathematicians can be found in this reference and the ones that were quoted above. Other statements also emphasize Clebsch's pedagogical qualities.

[16] For reasons that will be explained below, the paper does appear in a few historical works related on the geometry of cubic surfaces.

of the quintic in the 19th century with the following sequence of names and contributions.[17]

In 1826, Niels Abel proved the impossibility to solve the general quintic equation with the help of radicals, thus answering to a question that had remained unsolved for a long time. Then, in 1858, Charles Hermite presented a solution of the quintic using elliptic functions, based on the Bring-Jerrard form $x^5+x+a=0$ and on the possibility of lowering the modular equation of level 5 that had been stated by Évariste Galois in 1832 and proved by Enrico Betti in 1853. Approximately at the same time as Hermite, Francesco Brioschi and Leopold Kronecker also solved the quintic with the help of the theory of elliptic functions, yet in a different manner from Hermite. From the middle of the 1870s on, Felix Klein eventually did his research on the icosahedron, a research meant to present a geometrical synthesis of the works on the quintic (including those of Hermite, Kronecker, and Brioschi). This led to the publication in 1884 of the famous *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*, (Klein 1884), a publication which closes the standard history of the quintic equation.[18]

As stated above, the paper of Clebsch on the quintic was essentially about presenting a geometric interpretation of the approaches of Hermite and Kronecker, which singularly resembles to the program that Klein later carried out in his *Vorlesungen über das Ikosaeder*. Yet it is Klein's research that has been retained in the historiography to the detriment of others, as illustrates well the only (vague) hint to the paper of Clebsch that I found: "In the 1870s, this subject [of the quintic equation] was extremely trendy and we will not review all the works that have been devoted to it, due to A. Clebsch, P. Gordan, L. Kiepert for instance, before F. Klein took an interest in it."[19] This presentation of Klein's approach as a major landmark outshining some of the preceding research will be discussed in my concluding remarks at the light of the analysis of Clebsch's mathematics in which we are about to enter now.

1.3 Structure of Clebsch's paper

Clebsch's article on the geometrical interpretation of the quintic divides in one introduction and 19 sections which can be grouped as follows. Sections 1 to 4 present the principles of the geometric interpretation and its application on one first example, that of the quartic equation. In the sections 5 to 9, Clebsch investigates geometrical properties of a certain quintilateral and associated curves. Next, he explains the interest of the study of a special curve $C=0$ (section 10), and this study is carried out in sections 11 to 14. Section 15 contains the geometrical interpretation of the Tschirnhaus transformation. In sections 16

[17] See for instance (Bottazzini 1994; Houzel 2002, ch. VII; Kiernan 1971).

[18] About the research on the icosahedron, see (Gray 2000). Moreover, some historians have analyzed in detail the works of Hermite and Kronecker on the quintic: see (Goldstein 2011) and (Petri & Schappacher 2004) respectively. I will come back at length on these historical researches in the rest of the paper.

[19] "Dans les années 1870, ce sujet était extrêmement à la mode et nous ne passons pas en revue tous les travaux qui y ont été consacrés, par A. Clebsch, P. Gordan, L. Kiepert par exemple, avant que F. Klein ne s'y intéresse." (Houzel 2002, p. 80).

to 18 Clebsch analyzes a special cubic surface he calls the "diagonal surface." In section 19, he eventually sums up the interpretation of Hermite's method and explains how to interpret Kronecker's one.

Like many papers of Clebsch, this one is a very rich text which can sometimes be hard to follow in reason of the sibylline writing, the technicality of some mathematical points, or the proliferation of results that do not serve the general argument but that Clebsch does not signal as such—parenthetically, the presence of such subsidiary results was announced as from the introduction of the paper: "With this [the geometrical overall view on the quintic], we get at the same time a number of purely geometrical results which seem fit to show the fecundity of the developed ideas and methods."[20] In what follows, I have tried to bring out the main lines of Clebsch's article with enough explanations for the reader to understand what it is about without being overwhelmed by a multitude of details. In particular, almost each of the subsidiary results will be passed over. Finally, to clarify the presentation I will here and there change the order of exposition defined by Clebsch.

2 General principles of Clebsch's geometrical interpretation

2.1 Quadratic substitutions

Clebsch started by considering an algebraic equation $f(\lambda) = 0$ of degree n,[21] and introduced what he called a "substitution" or a "transformation" of the unknown λ: it is a rational function

$$\xi = \frac{\varphi(\lambda)}{\psi(\lambda)},$$

meant to act on the equation $f(\lambda) = 0$, so that ξ designates the new unknown.[22] If we note $\lambda_1, \ldots, \lambda_n$ the roots of the starting equation, then the roots of the transformed equation are $\xi_1 = \varphi(\lambda_1)/\psi(\lambda_1), \ldots, \xi_n = \varphi(\lambda_n)/\psi(\lambda_n)$. In other words, if the factors of the first equation are noted $(\lambda - \lambda_i)$, then those of the transformed one are $(\varphi(\lambda_i) - \xi\psi(\lambda_i))$. Clebsch qualified the transformation $\xi = \varphi/\psi$ as "linear" when the polynomials φ and ψ are of degree 1, and as "superior" otherwise.

Clebsch referred to two prior papers on such transformations, respectively due to Gordan and to himself, (Gordan 1870; Clebsch 1871d). These papers related to invariant theory; in their introduction, both Gordan and Clebsch recalled that the effect of linear transformations on algebraic forms was well-known, unlike the effect of superior transfor-

[20]"Dabei ergiebt sich zugleich eine Reihe bemerkenswerther rein geometrischer Resultate, welche geeignet scheinen, die Fruchtbarbeit [sic] der entwickelten Anschauungen und Methoden darzuthun." (Clebsch 1871b, p. 285).

[21]Throughout the whole paper, Clebsch did not comment on the nature of the coefficients (which could be rational, real, or complex for instance).

[22]Therefore, Clebsch's "transformations" are not transformations of the plane or the space like rotations or translations. Moreover, the significance of what he called "substitutions" differs from that of "substitutions" like in the phrase "the theory of substitutions," which refers to what is now called "permutations."

mations. Nevertheless, they also mentioned works of Hermite in which the French mathematician had introduced such transformations in his approach of the general equation of degree 4, (Hermite 1858b).

In the article we are studying, a special kind of superior transformations was at the core of Clebsch's considerations, namely the "quadratic" substitution $\xi = \varphi/\psi$ with φ and ψ of degree 2. For such a substitution, Clebsch introduced scalar numbers $x_1, \dots, y_3$ such that

$$\varphi(\lambda) = y_1 + \lambda y_2 + \lambda^2 y_3 \qquad \text{and} \qquad \psi(\lambda) = x_1 + \lambda x_2 + \lambda^2 x_3.$$

With this notation the factors of the transformed equation are given by

$$(y_1 - \xi x_1) + \lambda_i(y_2 - \xi x_2) + \lambda_i^2(y_3 - \xi x_3).$$

Clebsch's "geometrical interpretation"[23] then consists in considering x_1, x_2, x_3 and y_1, y_2, y_3 as homogeneous coordinates of two points x and y of the plane; these points were then called by Clebsch the "base points" of the substitution. Thus, to a quadratic substitution is associated a line (xy), and the roots ξ_i of the transformed equation correspond to the points of intersection of (xy) with the n lines defined by the equations $z_1 + \lambda_i z_2 + \lambda_i^2 z_3 = 0$.[24] According to Clebsch, this geometrical interpretation allowed "a clearer view on the essence of the quadratic substitution."[25]

Clebsch then remarked that the n lines $z_1 + \lambda_i z_2 + \lambda_i^2 z_3 = 0$ are tangent to the conic of equation $z_2^2 - 4z_1 z_3 = 0$,[26] and he summed up the whole:

> The set of all the equations into which a given equation is transformed by a quadratic substitution corresponds to the systems of points of intersection of the lines of a plane with the sides of a given multilateral, of which the sides touch a conic.[27] (Clebsch 1871b, p. 286)

In other words, the given equation $f(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n) = 0$ defines a multilateral

[23] "Aber an diese Darstellung knüpft sich eine geometrische Interpretation". (Clebsch 1871b, p. 285).

[24] Here, z_1, z_2, z_3 designate the homogeneous coordinates of the plane. The line (xy) has the parametric equations

$$\begin{cases} z_1 = y_1 - \xi x_1 \\ z_2 = y_2 - \xi x_2 \\ z_3 = y_3 - \xi x_3. \end{cases}$$

Its intersection with the line defined by $z_1 + \lambda_i z_2 + \lambda_i z^3 = 0$ is obtained for the parameter ξ satisfying the equality $(y_1 - \xi x_1) + \lambda_i(y_2 - \xi x_2) + \lambda_i^2(y_3 - \xi x_3) = 0$. According to the above, this parameter is the root ξ_i of the transformed equation.

[25] "Wir erhalten hierdurch eine deutlichere Einsicht in das Wesen der quadratischen Substitution". (Clebsch 1871b, p. 286).

[26] To check this point, one can substitute $z_1 = -\lambda_i z_2 - \lambda_i^2 z_3$ in the equation of the conic and see that one gets a (homogeneous) second degree equation having a discriminant equal to 0. This means that the line $z_1 = -\lambda_i z_2 - \lambda_i^2 z_3$ has a contact of order 2 with the conic, hence the tangency.

[27] "Die Gesammtheit aller Gleichungen, in welche eine gegebene durch eine quadratische Substitution übergeht, entspricht den Schnittpunktsystemen der Geraden einer Ebene mit den Seiten eines gewissen Vielseits, dessen Seiten einen Kegelschnitt berühren."

made of the n lines $z_1 + \lambda_i z_2 + \lambda_i^2 z_3 = 0$, all tangent to one conic; to a quadratic substitution $\xi = \varphi(\lambda)/\psi(\lambda)$ corresponds a line (xy); to the roots of the transformed equation correspond the points of intersection of (xy) with the polylateral.

Note that Clebsch used the term *Vielseit* that I translated by *multilateral* and not by *polygon* which I reserved for the translation of *Vieleck*—in the following pages, I will talk about quadrilaterals, quintilaterals, or n-laterals according to the number of sides. Indeed, Clebsch made a difference between these two objects that are dual one another: a polygon is a set of points (that are joined by lines) whereas a multilateral is a set of lines (that intersect in points).

Next, Clebsch investigated the effect of a change of the base points in his geometrical interpretation of quadratic substitutions. The result is that if a first substitution is given by two base points x, y, choosing two other points x', y' on the line (xy) as new base points corresponds to a supplementary linear substitution. More precisely, if the coordinates of x' and y' are given by

$$x'_i = \alpha x_i + \beta y_i \qquad \text{and} \qquad y'_i = \gamma x_i + \delta y_i,$$

then the linear substitution in hand is the homography[28]

$$\xi' = \frac{\gamma + \delta\xi}{\alpha + \beta\xi}.$$

This means that if a first quadratic substitution associated to a line of the plane has been operated on an equation, then choosing other base points on this line is equivalent to the operation of a linear substitution on the new equation. For Clebsch,

> It is very important [...] that the *characteristic* elements that are implied in the *superior* transformation appear separately from the potential influence of a subsequent *linear* transformation; and this property gives to the transformation in hand and to its geometrical meaning all their value.[29] (Clebsch 1871b, p. 287)

In the rest of the article, Clebsch endeavored to find quadratic substitutions that can transform a given equation into an equation of which some invariants vanish. To follow him, I first suggest a paragraph meant to recall some mathematical facts on algebraic forms and invariants. This paragraph is based on two books, (Clebsch 1872; Clebsch 1876). The former is Clebsch's book on the theory of binary forms, *Theorie der binären algebraischen Formen*; the latter, entitled *Vorlesungen über Geometrie*, was posthumously

[28] In Clebsch's paper, one reads $\xi' = \dfrac{\alpha + \beta\xi}{\gamma + \delta\xi}$, which seems erroneous.

[29] "Es ist von grosser Wichtigkeit, dass hierdurch die in der *höhern* Transformation liegenden *eigenthümlichen* Elemente gesondert erscheinen von dem Einfluss, welchen eine nachträgliche *lineare* noch ausüben kann; und diese Eigenschaft giebt der vorliegenden Transformation und ihrer geometrischen Deutung vorzugsweise ihren Werth."

edited by Ferdinand Lindemann on the basis of Clebsch's lectures on geometry, which included developments on invariant theory.

2.2 Forms and invariants

Let $f(x_1, x_2)$ be a binary form of degree n, i.e. a homogeneous polynomial of degree n. Such a form can always be written as follows:

$$f(x_1, x_2) = a_0 x_1^n + n a_1 x_1^{n-1} x_2 + \frac{n(n-1)}{2} a_2 x_1^{n-2} x_2^2 + \cdots + a_n x_2^n.$$

If a linear transformation is meant to act on f by changing x_1, x_2 into ξ_1, ξ_2, then one can write $f(x_1, x_2) = f'(\xi_1, \xi_2)$, where the coefficients a'_i of f' are functions of the coefficients a_i and of the coefficients of the linear transformation. Then an *invariant* of f is a homogeneous polynomial expression $I(a_1, \ldots, a_n)$ such that for all linear transformation, one has

$$I(a'_1, \ldots, a'_n) = r^k I(a_1, \ldots, a_n),$$

where r is the determinant of the linear transformation and k is an integer depending only on this transformation. The *degree* of the invariant I is its degree when seen as a polynomial.

Take for instance the quadratic form $f(x_1, x_2) = a_0 x_1^2 + 2a_1 x_1 x_2 + a_2 x_2^2$. By a mere computation, one sees that the action of the linear transformation

$$\begin{cases} x_1 = \alpha_{11}\xi_1 + \alpha_{12}\xi_2 \\ x_2 = \alpha_{21}\xi_1 + \alpha_{22}\xi_2 \end{cases}$$

on f induces the equality $f(x_1, x_2) = a'_0\xi_1^2 + 2a'_1\xi_1\xi_2 + a'_2\xi_2^2$ in which the new coefficients are $a'_0 = a_0\alpha_{11}^2 + 2a_1\alpha_{12}\alpha_{21} + a_2\alpha_{22}^2$, etc. Now, one can easily check that

$$a'_0 a'_2 - {a'_1}^2 = r^2(a_0 a_2 - a_1^2),$$

which plainly means that the discriminant $\Delta = a_0 a_2 - a_1^2$ is an invariant (of degree 2) of the quadratic form f.

Let us now turn to the symbolic notation of forms and invariants that had been introduced by Aronhold from 1849 onwards and that Clebsch had adopted and used since the end of the 1850s.[30] For a binary form $f(x_1, x_2) = a_0 x_1^n + n a_1 x_1^{n-1} x_2 + \cdots + a_n x_2^n$, the symbolic notation consists in writing $f = (b_1 x_1 + b_2 x_2)^n$, or even shorter $f = b_x^n$, and in stipulating that in the development of the binomial $(b_1 x_1 + b_2 x_2)^n$, one has to formally

[30] For some points of a history of the theory of invariants (including this symbolic notation), see (Fisher 1966; Parshall 1989). Besides, (Kung & Rota 1984) contains a modern presentation, along with a modern justification, of the symbolic notation and the subsequent computations.

substitute each of the terms $b_1^n, b_1^{n-1}b_2, \ldots, b_2^n$ by $a_0, a_1, \ldots, a_n$ respectively. The use of the letter b being purely formal, it is possible to use any other letter instead or even at the same time. For instance, one can write $f = b_x^n = c_x^n$, the substitution rules being also true for the c coefficients, i.e. $c_1^n = a_0, c_1^{n-1}c_2 = a_2$, etc. The interest of the symbolic notation of a binary form lies in the possibility to use it to define invariants of the form. The elementary constituents of these invariants are symbolic determinants noted $(bc) = b_1c_2 - b_2c_1$; raising this expression to the power n gives

$$(bc)^n = b_1^n \cdot c_2^n - n \cdot b_1^{n-1}b_2 \cdot c_1c_2^{n-1} + \cdots + (-1)^n \cdot b_2^n \cdot c_1^n.$$

and one has to replace all the formal symbols $b_1^{n-k}b_2^k$ and $c_1^{n-k}c_2^k$ by a_k.

To illustrate this, take again the quadratic form $f(x_1, x_2) = a_0x_0^2 + 2a_1x_1x_2 + a_2x_2^2$, symbolically noted $f = b_x^2 = c_x^2$. Since $(b_1x_1 + b_2x_2)^2 = b_1^2x_1^2 + 2b_1b_2x_1x_2 + b_2^2x_2^2$, the rules of symbolic substitutions are

$$\begin{cases} b_1^2 = a_0 \\ b_1b_2 = a_1 \\ b_2^2 = a_2 \end{cases} \qquad \text{and similarly} \qquad \begin{cases} c_1^2 = a_0 \\ c_1c_2 = a_1 \\ c_2^2 = a_2. \end{cases}$$

We now consider the symbolic expression $(bc)^2$ of which the expanded expression is

$$(bc)^2 = b_1^2c_2^2 - 2b_1b_2c_1c_2 + b_2^2c_1^2.$$

Applying the symbolic substitution rules,[31] one gets $(bc)^2 = a_0a_2 - 2a_1a_1 + a_2a_0 = 2\Delta$. Since $\Delta = (bc)^2/2$, this proves that the discriminant of the quadratic form f is a rational function of symbolic expressions (bc).

In fact, in a paper published in 1861, Clebsch had proven that every invariant of a binary form (of any degree) can be symbolically represented as a linear combination of products of symbolic determinants (bc), (Clebsch 1861). In other words, Clebsch had shown that every invariant I of a binary form $f = b_x^n = c_x^n$ can be written

$$I = \sum C \prod (bc),$$

where the coefficients C are constants numbers.

At that time, Clebsch used a geometrical interpretation of binary forms which can be explained as follows. Such a form $f(x_1, x_2)$ defines n (possibly infinite) ratios x_1/x_2 thanks to the equation $f(x_1, x_2) = 0$. Each of these ratios can then be seen as a point on a

[31] It is important to emphasize that one can only replace what is replaceable: for instance, b_1 does not appear alone in the substitution rules, and hence cannot be substituted—the only coefficients implying b_1 that can be replaced are b_1^2 and b_1b_2.

projective line with coordinates $(x_1 : x_2)$—Clebsch would talk about "series of points" on a line. Reciprocally, n points on a projective line define a binary form of degree n, so that one can talk about the invariants of a series of points on a line.

To end this section, it should be added that Clebsch also knew a symbolic calculus for ternary forms, i.e. forms depending on three variables. These forms are symbolically denoted by $f = (b_1 z_1 + b_2 z_2 + b_3 z_3)^n$ or $f = b_z^n = c_z^n$ and again, there are symbolic determinants (bcu) defined by

$$(bcu) = \begin{vmatrix} b_1 & c_1 & u_1 \\ b_2 & c_2 & u_2 \\ b_3 & c_3 & u_3 \end{vmatrix}.$$

The rules of symbolic substitution (for instance in order to compute $(bcu)^n$) are exactly the same as those for binary forms: one has to expand as much as possible and then to replace everything that can be replaced.

2.3 Making invariants vanish

Let us get back to Clebsch's article on the geometrical interpretation of the quintic. An equation $f(\lambda) = 0$ of degree n being given, Clebsch explained that it is possible to associate a binary form $f(x_1, x_2)$ by homogenizing the equation, i.e. by replacing the unknown λ by x_1/x_2 and then by multiplying the whole by x_2^n. In this way, one can talk about the invariants of the equation $f(\lambda) = 0$, which are the invariants of the binary form $f(x_1, x_2)$. An important point is that the invariant character of them always refers to linear transformations. So if a quadratic substitution is operated on an equation, its invariants can change. As stated above, this was essential to Clebsch who wanted the transformed equations to have certain invariants equated to zero.[32]

Clebsch erected as a theorem the following geometrical interpretation:

> All the lines that cut a given n-lateral [coming from an algebraic equation,] in such a way that the system of points of intersection has a vanishing invariant $[J]$ of degree χ, envelop a curve $(J = 0)$ of class $\chi n/2$. [...] The curve $J = 0$ has the sides of the n-lateral as χ-times tangents.[33] (Clebsch 1871b, p. 291)

I do not want to transcribe Clebsch's demonstration of this theorem but I will explain its content, which will be useful to understand the rest.

[32] This idea of looking for superior transformations making invariants vanish had already been brought to light by Hermite a few years earlier, in the cases of the equations of degree 4 and 5, (Goldstein 2011, pp. 248-249). For the fifth degree in particular, after some research Kronecker and Brioschi had made, Hermite had elaborated an extensive memoir in which he had sought to unify the diverse approaches of the quintic through invariant theory, (Hermite 1865-66).

[33] "Alle Geraden, welche ein gegebenes n-Seit so schneiden, dass für das Schnittpunktsystem eine gewisse Invariante χ^{ten} Grades verschwindet, umhüllen eine Curve $(J = 0)$ der Classe $\chi n/2$. [...] Die Curve $J = 0$ hat die Seiten des n-Seits zu χfachen Tangenten."

Firstly, remember that one can talk about the (homogeneous) coordinates of a line in the plane: they are the coefficients u_1, u_2, u_3, defined up to a common multiplier, that appear in an equation $u_1z_1+u_2z_2+u_3z_3 = 0$ of the line. The diverse curves of the plane can then be described by these so-called *tangential* coordinates instead of the usual *punctual* coordinates z_1, z_2, z_3. For instance, to define a curve by the equation $u_1^2 + u_2u_3 = 0$ means to define it as the envelop of all the lines whose coordinates satisfy this equation—in other words, a line of the plane is tangent to the curve if and only if its coordinates satisfy the equation. This equation is then called a *tangential equation*, and its degree is the *class* of the curve.

As for Clebsch's theorem, let $f = a_x^n = b_x^n$ be the symbolic notation of the binary form associated to the given algebraic equation of roots $\lambda_1, \ldots, \lambda_n$. As we saw, this equation corresponds to a n-lateral of which the sides are respectively defined by $z_1+\lambda_iz_2+\lambda_i^2z_3 = 0$. The product of the left-hand side of these equations gives a ternary form in z_1, z_2, z_3, of degree n, symbolically noted $f = a_z^n = b_z^n$.[34]

Take now a line of coordinates u_1, u_2, u_3 and consider the system of its n points of intersection with the n-lateral—in the geometrical interpretation, the line represents a quadratic substitution and the system of points represents the transformed equation. Then Clebsch's result is that the quadratic substitutions making an invariant $J = \sum C \prod(ab)$ of the starting equation vanish correspond to the lines of which the coordinates satisfy[35]

$$J = \sum C \prod(abu) = 0,$$

being understood that this symbolic notation derives from that of the ternary form f. This expression can be seen as a polynomial expression in u_1, u_2, u_3, so that the equation $J = 0$ can be interpreted as the tangential equation of a curve. Therefore, a quadratic substitution makes an invariant J vanish if and only if its representative line is tangent to the curve $J = 0$. Further, Clebsch's theorem specifies that if J is of degree χ, then the curve $J = 0$ is of class $\chi n/2$ and has the sides of the n-lateral as χ-times tangents.

3 The example of the quartic equation

Following Clebsch, let us look at the example of the quartic equation to clarify all this. To bring things into perspective, his order of exposition is here inverted: I begin by recalling some results on the quartic equation that had been obtained by Hermite in the 1850s before turning to the geometrical interpretation.[36]

[34] Clebsch used the letter f to designate both the binary form and this ternary form.

[35] Here again, Clebsch used the same letter J twice. On one hand, it designates an invariant of the binary form f; on the other hand, it denotes an invariant of the ternary form f.

[36] The article cited by Clebsch is (Hermite 1858b). Hermite's research on algebraic equations of degree 4 is explained and discussed in (Goldstein 2011).

Hermite had remarked that the every quartic equation of the form

$$x^4 - 6Sx^2 - 8Tx - 3S^2 = 0 \tag{1}$$

can be solved with the help of a special equation linked to the theory of elliptic functions, namely the *modular equation* associated to the transformation of order 3 of elliptic functions.[37] The issue had then been to bring the general quartic equation into such a form. Hermite had proceeded in two steps. The first one was to prove that an equation $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ of which the quadratic invariant $i = ae - 4bc + 3c^2$ vanishes can be brought into the form (1) thanks to a mere translation of the variable. The second step was to show that every equation of degree 4 can be transformed into an equation having its invariant i equal to zero. For that purpose, Hermite had found an adequate polynomial transformation $y = \varphi(x)$ of the unknown x.[38] Further, Hermite had cared to prove that the coefficients of this transformation could be expressed with the coefficients $a, b, \ldots, e$ of the starting equation through rational procedures and square roots. In other terms, the coefficients of $y = \varphi(x)$ imply the *adjunction* of square roots. The fact that such *irrationals* (the adjoined roots) are of the second degree (they are square roots) was important because it agreed with the solution of the quartic equation with radicals.[39]

Let us return to Clebsch's treatment of the quartic equation. Clebsch used the symbolic notation of the associated binary form $f = a_x^4 = b_x^4$ as well as that of its quadratic invariant, $i = (ab)^4$. As explained above, the quartic equation gives rise to a quadrilateral in the plane, and Clebsch asserted that, up to a change of the plane coordinates, its sides can be represented by the equations[40]

$$\begin{aligned} z_1 + z_2 + z_3 &= 0 \\ -z_1 + z_2 + z_3 &= 0 \\ z_1 - z_2 + z_3 &= 0 \\ z_1 + z_2 - z_3 &= 0. \end{aligned}$$

The product of the four left-hand sides of these equations, corresponding to the ternary

[37] Further explanations on the modular equation are given below, in the case of the quintic.

[38] Let us note that Clebsch's use of rational transformations can be seen as a trace of his geometrical approach, whereas Hermite's use of polynomial transformations rather refers to an algebraic manner.

[39] Hermite's attention on adjunctions of square roots refers to the incorporation in his research of elements coming from Galois' works. (Goldstein 2011, p. 250).

[40] To choose a projective frame of the plane is equivalent to choose a triangle of reference, of which the sides become the axis of vanishing of the coordinates. To bring the equations of the sides of a quadrilateral into the form announced by Clebsch, it is sufficient to choose the triangle formed by the three diagonals of the quadrilateral as the triangle of reference (a quadrilateral is a set of 4 lines; these lines intersect in six points and thus define three diagonals).

form f representing the quadrilateral, is

$$f = z_1^4 + z_2^4 + z_3^4 - 2z_1^2 z_2^2 - 2z_2^2 z_3^2 - 2z_3^2 z_1^2.$$

Denoting it symbolically by $f = a_z^4 = b_z^4$, Clebsch's geometrical interpretation theorem states that to find a substitution making i vanish is equivalent to find a tangent to the curve of tangential equation $i = (abu)^4 = 0$. In his memoir, Clebsch announced that this equation is

$$i = \frac{8}{3}(u_1^4 + u_2^4 + u_3^4 - u_1^2 u_2^2 - u_2^2 u_3^2 - u_3^2 u_1^2) = 0.$$

To check this point, we have to make the expression $(abu)^4$ explicit, so we first have to find the substitution rules of the symbolic notation. Since

$$f = a_z^4 = (a_1 z_1 + a_2 z_2 + a_3 z_3)^4 = z_1^4 + z_2^4 + z_3^4 - 2z_1^2 z_2^2 - 2z_2^2 z_3^2 - 2z_3^2 z_1^2,$$

we expand the expression $(a_1 z_1 + a_2 z_2 + a_3 z_3)^4$:

$$\begin{aligned}(a_1 z_1 + a_2 z_2 + a_3 z_3)^4 &= a_1^4 z_1^4 + a_2^4 z_2^4 + a_3^4 z_3^4 + \\ &+4a_1^3 a_2 z_1^3 z_2 + 4a_1^3 a_3 z_1^3 z_3 + 4a_1 a_2^3 z_1 z_2^3 + 4a_2^3 a_3 z_2^3 z_3 + 4a_1 a_3^3 z_1 z_3^3 + 4a_2 a_3^3 z_2 z_3^3 + \\ &+ 6a_1^2 a_2^2 z_1^2 z_2^2 + 6a_2^2 a_3^2 z_2^2 z_3^2 + 6a_3^2 a_1^2 z_3^2 z_1^2 + \\ &+ 12a_1^2 a_2 a_3 z_1^2 z_2 z_3 + 12a_1 a_2^2 a_3 z_1 z_2^2 z_3 + 12a_1 a_2 a_3^2 z_1 z_2 z_3^2.\end{aligned}$$

The comparison of the two preceding expressions now yields the following substitution rules:

$$\begin{cases} a_1^4 = a_2^4 = a_3^4 = 1 \\ a_1^3 a_2 = a_1^3 a_3 = a_1 a_2^3 = a_2^3 a_3 = a_1 a_3^3 = a_2 a_3^3 = 0 \\ a_1^2 a_2^2 = a_2^2 a_3^2 = a_1^2 a_3^2 = -1/3 \\ a_1^2 a_2 a_3 = a_1 a_2^2 a_3 = a_1 a_2 a_3^2 = 0. \end{cases}$$

Furthermore, as explained above, similar rules can be obtained by replacing at once all the letters a by letters b.

Let us now turn to the symbolic determinant (abu). By definition, one has

$$(abu)^4 = \begin{vmatrix} a_1 & b_1 & u_1 \\ a_2 & b_2 & u_2 \\ a_3 & b_3 & u_3 \end{vmatrix}^4 = \big((a_1 b_2 - a_2 b_1)u_1 + (a_3 b_1 - a_1 b_3)u_2 + (a_1 b_2 - a_2 b_1)u_3\big)^4.$$

The rules of symbolic calculus indicate that we have to completely expand this power before proceeding to the substitutions. For instance, in the expression of $(abu)^4$, the coefficient of u_1^4 is

$$a_1^4 b_2^4 - 4a_1^3 a_2 b_1 b_2^3 + 6a_1^2 a_2^2 b_1^2 b_2^2 - 4a_1 a_2^3 b_1^3 b_2 + a_2^4 b_1^4,$$

which is, according to the substitution rules, equal to

$$1 \cdot 1 - 4 \cdot 0 \cdot 0 + 6 \cdot (-\frac{1}{3}) \cdot (-\frac{1}{3}) - 4 \cdot 0 \cdot 0 + 1 \cdot 1 = \frac{8}{3}.$$

Doing the same for the coefficients of $u_1^3 u_2$, $u_1^2 u_2^2, \ldots$, we do find the expression that Clebsch had announced:

$$i = \frac{8}{3}(u_1^4 + u_2^4 + u_3^4 - u_1^2 u_2^2 - u_2^2 u_3^2 - u_3^2 u_1^2).$$

Clebsch then remarked that this expression can be factorized:

$$i = \frac{8}{3}(u_1^2 + \varepsilon u_2^2 + \varepsilon^2 u_3^2)(u_1^2 + \varepsilon^2 u_2^2 + \varepsilon u_3^2),$$

the number ε being a primitive cubic root of unity. This factorization proves that the curve $i = 0$ is made of the conics defined by the tangential equations $u_1^2 + \varepsilon u_2^2 + \varepsilon^2 u_3^2 = 0$ and $u_1^2 + \varepsilon^2 u_2^2 + \varepsilon u_3^2 = 0$ respectively.[41] Clebsch then concluded:

> With this, one can attach a geometrical clothing to the solution of the equation of degree 4 like Hermite [...][42] did, (Hermite 1858b). This solution relies on the fact that one can change, by a superior transformation, the biquadratic equation into another for which i vanishes [...]. Indeed, one only needs to take the quadrilateral $f = 0$ associated to the equation of order 4 and to build [the curve $i = 0$]. According the above, this equation splits into 2 conics, which can be done with the help of a quadratic equation; each tangent of such a conic then gives a biquadratic equation for which $i = 0$ and which is thus solved by a *pure* cubic equation.[43] (Clebsch 1871b, p. 296)

It is interesting to underscore that Clebsch had no care for effectiveness. Indeed, to explicitly find the quadratic substitution making i vanish is a point that was not mentioned—this feature puts him in opposition to Hermite who always looked for effective procedures. Besides, we see here that Clebsch's "geometrical clothing" consisted in interpreting the coefficients of his quadratic substitution and the invariants meant to vanish in terms of lines and curves. So it was not about geometrically interpret elements like the transformation $y = \varphi(x)$ that Hermite had used in his research. For Clebsch, Hermite's contribution

[41] Let us recall that the degree of one of these equations is the class of the associated curve. Here we have curves of class 2, which correspond to the curves of degree 2, i.e. conics.

[42] In this quotation, besides the works of Hermite, Clebsch also mentioned those of Gordan, (Gordan 1870). The latter had studied the vanishing of another invariant of the equation of degree 4.

[43] "Man kann hieran in geometrischem Gewande die Lösung der Gleichung 4$^{\text{ten}}$ Grades knüpfen, wie Hermite (Comptes Rendus t. 46. p. 961) und Gordan (Borchardt's Journal Bd. 71, p. 164) dieselbe gegeben haben. Diese Lösung beruht darauf, dass man die biquadratische Gleichung durch eine höhere Transformation in eine solche verwandelt, für welche i [...] verschwindet [...]. In der That braucht man nur das zu der Gleichung 4$^{\text{ter}}$ Ordnung gehörige Vierseit $f = 0$ zu nehmen, und [die Curve i=0 zu bilden. Diese] Gleichung zerfällt nach dem Obigen in 2 Kegelschnitte, eine Zerlegung, welche mit Hülfe einer quadratischen Gleichung ausgeführt wird; jede Tangente eines solchen Kegelschnittes mliefert dann eine biquadratische Gleichung, für welche $i = 0$ und welche also durch eine *reine* cubische Gleichung gelöst wird."

rather lied in the fact that he had solved the quartic equation by proving that it was possible to make its invariant i vanish (which in turn allowed to use the theory of elliptic functions).

However, it was important to Clebsch to control the irrationals that are implied in his geometrical interpretation: these irrationals must fit with the methods he interpreted. In the case in hand, since the curve $i = 0$ is made of two conics, the geometrical interpretation only brings in square roots (corresponding to the cubic root of unity ε); to find a tangent to one of these conics then introduces no supplementary irrational.[44] Hence the only introduced irrationals were square roots, just like in Hermite's approach.

4 The quintic equation

After the example of the quadrilateral, Clebsch came to the general equation of the fifth degree. As I mentioned at the beginning of the paper, Clebsch referred to Hermite's and Kronecker's researches on the subject. Let us here recall the main points of their works, which will allow us to compare with Clebsch's approach afterwards.[45]

These works were linked with the theory of transformation of elliptic functions, which consists in searching y, ℓ, and M in function of x and k, so that

$$\frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-\ell^2y^2)}} = \frac{1}{M}\frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k^2x^2)}}.$$

When y is searched into the form $y = U(x)/V(x)$ with U, V coprime polynomials of degree n and $n-1$ respectively, the transformation is said to be of order n. In that case, ℓ is linked to k by an equation of degree $n+1$ called the *modular equation*; M is also linked to k by an equation of degree $n+1$ called the *multiplier equation*.

In 1832, Galois had announced that if n is equal to 5, 7, or 11, the modular equation has a reduced equation of degree n. This result, that had been stated without demonstration, had been proven by Betti in 1853 with considerations of decompositions of the group of the modular equation. In modern terms this group is $\mathrm{PGL}(2, \mathbf{F}_n)$, which reduces to $\mathrm{PSL}(2, \mathbf{F}_n)$ after the adjunction of a square root. The lowering of the modular equation seen by Galois then corresponds to the existence of a (non-normal) subgroup of $\mathrm{PSL}(2, \mathbf{F}_n)$ of index n.

The special case $n = 5$ is the one on which Hermite had based his approach of the quintic. The French mathematician had gone further than Betti by explicitly searching the form of the reduced equation of degree 5 corresponding to the subgroup of index 5—the results had been published in 1858, (Hermite 1858a). For this purpose, he had found a

[44] In modern terms, if a conic is given by the vanishing of a quadratic form, its tangents are obtained thanks to the associated bilinear form. All these reflections appear very vaguely in Clebsch's memoir.

[45] I will use (Goldstein 2011) for what is about Hermite and (Petri & Schappacher 2004) for Kronecker. Also see (Gray 2000; Houzel 2002) where other approaches (including Brioshi's) are described. Like Clebsch, I will completely omit Brioshi's contribution on the topic.

function of the roots of the modular equation taking exactly 5 values under the action of $\mathrm{PSL}(2, \mathbf{F}_5)$. These values $z_1, \ldots, z_5$ were of the form $z_i = \Phi(\omega + 16i)$, where ω is the quotient of the periods of the elliptic functions and Φ a function (of which the form was explicitly known) depending of quantities linked to the same functions. An important point was that these formulas give rise to series developments. Hermite had deduced from these developments that the five z_i are the roots of the equation

$$\Phi^5 - \alpha\Phi - \beta = 0, \tag{2}$$

where α and β are quantities depending (in an explicit way) of quantities linked to the elliptic functions. This equation of degree 5 is the reduced equation of the modular equation.

The relation with the general equation of the fifth degree came from its so-called "Jerrard form."[46] The mathematician George Birch Jerrard had indeed proved that the general quintic can be brought into the form

$$y^5 - y - A = 0 \tag{3}$$

with the use of an adequate Tschirnhaus transformation $y = a + bx + cx^2 + dx^3 + ex^4$, of which the coefficients imply only square and cubic roots of the coefficients of the quintic. To solve the quintic, Hermite had then identified the forms (2) and (3), and had thus explicitly expressed the roots y of the latter in function of the roots z_i of the former.[47] In this way, the equation of degree 5 was solved with the help of elliptic functions.

Kronecker's research had been shared with Hermite in a letter of 1858, of which an extract had then been published, (Kronecker 1858). Unlike Hermite though, Kronecker neither used the Jerrard form of the quintic, nor based his approach on the modular equation: instead, he considered the multiplier equation associated to the transformation of order 5 of elliptic functions.

His starting point had been to consider a cyclic function

$$f = f(\nu, x_0, x_1, x_2, x_3, x_4)$$

depending on the roots $x_0, \ldots, x_4$ of the quintic and of a parameter ν. In modern terms, this means that f is unchanged under the action of a cyclic subgroup of the reduced group of the quintic $\mathrm{PSL}(2, \mathbf{F}_5) \simeq \mathfrak{A}_5$. From f, Kronecker had exhibited five other cyclic func-

[46] One can also find the names "Bring-Jerrard form" or "Tschirnhaus-Jerrard form" as Clebsch would write. See (Beauville 2012) for a modern point of view linking this form with the *essential dimension* of the symmetric group $\mathfrak{S}_5$.

[47] The equation $\Phi^5 - \alpha\Phi - \beta = 0$ can be brought into the form $\Psi^5 - \Psi - \alpha^{-5/4}\beta = 0$ when putting $\Phi = \alpha^{1/4}\Psi$. The Jerrard form $y^5 - y - A = 0$ being given, the next technical point is to find elliptic functions giving rise to some constants α, β satisfying $A = \alpha^{-5/4}\beta$. This had been proven by Hermite with the help of an equation of degree 4 with coefficients in $\mathbf{Q}[\sqrt{5}]$.

tions $f_0, \dots, f_4$; the six functions $f, f_0, \dots, f_4$ then correspond to the six cyclic subgroups of order 5 of $\mathfrak{A}_5$. Since these functions are cyclic, the solution of the equation of degree 6 on which they depend bring the quintic into a pure equation, i.e. an equation of the form $x^5 = A$, which is solvable with radicals. In other terms, once f is known, the roots x_i can be deduced from it with the help of radicals.

The equation on which $f, f_0, \dots, f_4$ depend is the one that Kronecker had linked to the theory of elliptic functions. Indeed, he had indicated that it is possible to determine (with the help of square roots) a parameter ν such that

$$f^2 + f_0^2 + f_1^2 + f_2^2 + f_3^2 + f_4^2 = 0.$$

Thanks to this condition, Kronecker had explicitly computed the form of the equation of which the cyclic functions depend. More specifically, he had proven that the six functions satisfy the equation

$$f^{12} - 10\phi f^6 + 5\psi^2 = \psi f^2,$$

where ϕ and ψ are rational functions of the coefficients of the quintic and of some square roots. The equation in f had finally been linked to the multiplier equation, and Kronecker had consequently expressed $f, f_0, \dots, f_4$ with the help of elliptic functions.

As I wrote earlier, after Kronecker and Brioschi completed their research on the quintic, Hermite had sought a point of view which would unify the different approaches. He achieved this through the theory of invariants, and in particular through the study of vanishing of particular invariants, (Hermite 1865-66).

4.1 Jerrard form and C invariant

Let us now turn to Clebsch's research. Clebsch gave a list of invariants and covariants[48] of the quintic, referring to works he and Gordan had done earlier, (Clebsch & Gordan 1867). Among them, the most important for what follows is an invariant of degree 12 that Clebsch noted C. Other invariants A, B and linear covariants α, δ also played a role, yet in a more auxiliary way.[49]

Clebsch recalled that if the invariant C vanishes, then the equation of the fifth degree can be brought into a Jerrard form by a linear substitution—as he specified himself, he had published this result shortly before, (Clebsch 1871a). More precisely, Clebsch's result

[48] A *covariant* of a binary form $f(x_1, x_2)$ is a polynomial expression $K(a_0, \dots, a_n, x_1, x_2)$ implying the coefficients of f and the variables x_1, x_2, such that for any invertible linear substitution acting on x_1, x_2, one has $K(a'_0, \dots, a'_n, \xi'_1, \xi'_2) = K(a_0, \dots, a_n, x_1, x_2)$, the notations being those previously given. In particular, a covariant is said to be *linear* when the variables x_1, x_2 are linearly implied.

[49] All these invariants and covariants were already listed in (Hermite 1865-66). A difference with Hermite is that Clebsch defined the invariants and covariants by means of the symbolic notation. For instance, denoting a quintic by $f = a_z^5 = b_z^5$, he first defined a covariant $i = (ab)^4 a_z b_z$ and the invariant A was then given by the symbolic relation $A = (ii')^2$.

is that when the C invariant is zero, the linear substitution[50]

$$x = \frac{\delta - B\alpha}{\delta + \frac{B}{2}\alpha}$$

has the effect of transforming the quintic into the equation

$$x^5 - \frac{B}{4A^2}(5x+1) = 0,$$

which is "almost" a Jerrard form.

Basing himself on these previous results, Clebsch summed up the problem:

> It is well-known that Hermite's solution of the equations of the fifth degree is grounded on a solution of the equation [in its Jerrard form] thanks to elliptic functions. According to the above, as soon as the C invariant of an equation of the fifth degree vanishes, the equation is brought into this form and hence is solved in the Hermitian sense. But if C is not zero for the given equation, then one can formulate the problem of solving the equation of the fifth degree as follows: *to bring, with the help of a superior transformation, the equation into any other for which C vanishes.* But in our geometrical interpretation, this is nothing else than *having to find any tangent to the curve $C = 0$.*[51] (Clebsch 1871b, p. 318)

Clebsch also specified that such a tangent must be found with the help of square or cubic roots, since the use of a Tschirnhaus transformation bringing the quintic into the Jerrard form imply these kinds of radicals only.

He remarked further that once the tangent is found, it is possible to induce the supplementary linear transformation

$$x = \frac{\delta - B\alpha}{\delta + \frac{B}{2}\alpha}$$

by choosing adequate base points on this tangent. As we saw, this would then have the effect to bring the starting equation into the form $x^5 - \frac{B}{4A^2}(5x+1) = 0$.

Like in the case of the quartic equation, what is perceived by Clebsch as geometric is here particularly clear: it is about replacing the search of a quadratic substitution by the search of a tangent to a curve and of suitable points on this tangent. Moreover, we can see that Clebsch considered the quintic solved "in the Hermitian sense" as soon as it has the

[50] Here, x is the new unknown, the starting one being implicitly contained in the variables of the covariants α and δ.

[51] "Bekanntlich beruht Hermite's Auflösung der Gleichungen 5^{ten} Grades auf einer Lösung der Gleichung (14) mit Hülfe der elliptischen Functionen. Sobald die Invariante C einer Gleichung 5^{ten} Grades verschwindet, ist durch das Vorige die Züruckführung der Gleichung auf diese Form, also ihre Lösung im Hermite'schen Sinne. Ist bei der gegebenen Gleichung aber C von Null verschieden, so kann man die Aufgabe, die Gleichung 5^{ten} Grades zu lösen, darin setzen: *dass mittelst einer höheren Transformation die Gleichung in eine solche mit verschwindendem C verwandelt werden soll.* Aber in unserer geometrischen Interpretation heisst dies nichts anderes, *als dass irgend eine Tangente der Curve $C = 0$ gefunden werden soll.*" Clebsch's emphasis.

Jerrard form. Thus all the intermediate steps of Hermite's method were completely left out, and they were not geometrically interpreted *a fortiori*. Rather, the very possibility to solve the Jerrard quintic with elliptic functions was accepted as such, and the geometrical interpretation concerned what allowed these functions to enter the picture.

4.2 The curve $C = 0$

Aiming at finding a tangent to the curve $C = 0$, Clebsch extensively studied the latter and determined its genus in particular.[52] Let us recall that the *genus* of an algebraic curve is an integer depending on its degree and on its possible singularities. More specifically, the genus of a smooth curve of degree n is the number

$$p = \frac{(n-1)(n-2)}{2},$$

and in the case of a singular curve, one has to deduct quantities depending on the number and the nature of the singularities. The name *Geschlecht*, translated in English by "genus," is usually attributed to Clebsch; as written in our introduction, it appeared in his works dated from the beginning of the 1860s, in which he had presented a way to apply Abelian functions to geometry.[53]

In the case of the curve $C = 0$, Clebsch used a dual version of the genus formula written above, based on the class of the curve instead of its degree. On one hand, since the invariant C is of degree 12, Clebsch's theorem of geometrical interpretation stated that the curve $C = 0$ is of class $5 \cdot 12/2 = 30$. On the other hand, Clebsch proved that this curve has 12 double inflection tangents, that is, 12 lines that touch the curve at two different inflection points. From the knowledge of the class of $C = 0$ and the number of double inflection tangents, Clebsch eventually computed the genus p of the curve and found $p = 4$.

4.3 Completion of the first geometrical interpretation

This genus computation was important to Clebsch because it enabled him to deploy his past research on birational maps[54] to successively transform the curve $C = 0$. In this process, a crucial point was the possibility to find a birational map between the projective plane and any cubic surface, i.e. any algebraic surface defined by a polynomial equation of degree 3. Such a possibility had been proven in a 1866 paper called *Die Geometrie auf*

[52] In the process Clebsch proved many intermediate results, as well as a large number of properties that he did not exploit for his geometrical interpretation of the quintic.

[53] At that time, Cayley would rather talk about the "deficiency" of a curve. See (Dieudonné 1974; Gray 1989; Houzel 2002). Moreover, let us recall that a different notion of genus existed, coming from the theory of algebraic forms as exposed in Gauss' *Disquisitiones Arithmeticae*. On this point, see (Lemmermeyer 2007).

[54] A *rational map* is a map defined almost everywhere and rationally transforming coordinates; a *birational map* is an invertible rational map of which the inverse is also rational.

den Flächen dritter Ordnung, where Clebsch talked about the "representation of a cubic surface on the plane," (Clebsch 1866). He had constructed a birational map defined on the whole plane excepted for six points placed in general position and called the *fundamental points* of the representation—to each of the fundamental points corresponds not a point, but an entire line contained in the cubic surface. Clebsch had also studied how the curves of the plane and the curves on the surface transformed through the representation.[55] In particular, Clebsch had shown that each line joining two fundamental points and each conic going through five of these points are transformed into a line contained in the cubic surface. This yielded twenty-seven lines, which is the total number of lines contained in every (smooth, complex) cubic surface.[56]

These results were applied to the curve $C = 0$ in the following way. First, since this curve is of genus 4, Clebsch proved that it is birational to a plane curve of order 6 with 6 double points,[57] say Γ. Since the curve $C = 0$ was described by tangential coordinates, he asserted that this birational transformation maps each tangent of $C = 0$ to a point of Γ. Next, Clebsch considered the six double points of Γ as the fundamental points of the representation of a certain cubic surface on the plane containing Γ: in this representation, the curve Γ is sent on a space curve γ which is the complete intersection of the cubic with a certain quadratic surface.[58] Clebsch then projected on the plane E "this quadric surface from one of its points (which can be found with the help of just one quadratic equation) in the usual way"[59]; this means that he stereographically projected the quadric from one of its points. According to Clebsch, this projection would send the space curve γ on a plane curve Γ' of order 6 with 2 triple points.

Hence there exists a transformation associating to each tangent to the curve $C = 0$ a point of the curve Γ', and this transformation is almost birational since a square radical is introduced by the choice of a point on the quadric.

Now, let us recall that Clebsch's aim was to find a tangent to the curve of $C = 0$ with the help of square and cubic roots. For that purpose, Clebsch wrote that the two triple points of Γ' "split" with the help of a quadratic equation and that any line passing through one of them intersects Γ' in three additional points "which split by means of a

[55]See (Lê 2013, pp. 59-60) for deeper explanations on Clebsch's results and proofs.

[56]The existence of 27 lines on every smooth, complex cubic surface had been stated and proved by Cayley and Salmon, (Cayley 1849; Salmon 1849). See (Lê 2015c) for elements of a history of the twenty-seven lines theorem, and especially its role in the encounters of groups, algebraic equations, and geometry in the second half of the 19th century.

[57]A curve of order 6 with 6 double points is a curve of genus $(6-1)(6-2)/2-6=4$, which is equal to the genus of $C = 0$. However, it must be underlined that the equality of the genera of two curves does not imply the birational equivalence of the curves (whereas the reciprocal is true). Clebsch proved this birational equivalence by an approach which could be nowadays qualified as a computation of dimensions of spaces of curves.

[58]A paragraph of the paper (Clebsch 1866) in which Clebsch had proven the existence of plane representations of cubic surfaces is devoted to such space curves.

[59]"[Man bildet] diese Fläche 2^{ter} Ordnung sodann von einem ihrer Punkte (dessen Auffindung nur die Lösung einer quadratischen Gleichung fordert) auf die gewöhnliche Weise ab", (Clebsch 1871b, p. 327).

cubic equation." He concluded that "to each of these points finally corresponds a tangent to $C = 0$"[60], and these words marked the end of his first geometrical interpretation.

To recapitulate, this interpretation was grounded on the fact that a quintic can be brought into a Jerrard form (and hence can be solved "in the Hermitian" sense) when its C invariant vanishes. In Clebsch's geometrical terms, it was thus about finding any tangent to the curve $C = 0$ with the help of square and cubic roots, which was achieved thanks to the transformation of $C = 0$ into another curve Γ'. It finally remained to select a point on the latter, which corresponds to a tangent to $C = 0$. Note that Clebsch used thrice quadratic and cubic equations "splitting" points to control the introduced irrationals and ensure they are square or cubic roots.[61]

Clebsch did not stop there. He then expressed his will to find "the formulas of the transformation and so the analytic solution of the question." For this he went through a second interpretation, implying the Tschirnhaus transformation itself.

4.4 Geometrical interpretation of "the Tschirnhaus method"

According to Clebsch, this geometrical interpretation of the Tschirnhaus method was "not so direct"[62] as the previous one concerning the quadratic substitution. He recalled that this method consists in considering the transformation

$$\xi = a + b\lambda + c\lambda^2 + d\lambda^3 + e\lambda^4, \tag{4}$$

where a, b, c, d, and e should be chosen so that the coefficients of the second, third, and fourth powers in the quintic $f(\lambda)$ vanish. The roots of the transformed equation being noted $\xi_1, \ldots, \xi_5$, these conditions are equivalent to

$$\sum_{i=1}^{5} \xi_i = 0, \qquad \sum_{i=1}^{5} \xi_i^2 = 0, \qquad \sum_{i=1}^{5} \xi_i^3 = 0.$$

Replacing all the ξ_i thanks to the equality (4), these conditions can be rationally expressed with the coefficients $a, \ldots, e$ and with symmetric functions of $\lambda_1, \ldots, \lambda_5$, which are also

[60] "Jedem dieser Punkte endlich entspricht eine Tangente von $C = 0$."(Clebsch 1871b, p. 327).

[61] Such "splitting" equations are part of a bigger family, that of "geometrical equations," which are algebraic equations associated to the determination of diverse geometrical configurations. For Clebsch among other mathematicians, these equations participated to an intuitive, geometrical understanding of the theory of substitutions, especially during the period 1868-1872, (Lê 2015a). See also (Lê 2015b), where the special organization of the activities involving geometrical equations is characterized as a *cultural system*.

[62] "Auch diese Jerrard'sche Modification der Tschirnhausen'schen Methode ist, wenngleich nicht so direct wie die quadratische Substitution, einer Art geometrischer Deutung fähig". (Clebsch 1871b, p. 328).

rational functions of $a, \dots, e$. This gives a system

$$\begin{cases} \Phi(a, b, c, d, e) = 0 \\ \Psi(a, b, c, d, e) = 0 \\ \mathrm{X}(a, b, c, d, e) = 0, \end{cases}$$

where Φ, Ψ, X are homogeneous polynomial functions of order 1, 2, 3 respectively.

As Clebsch explained, the linear function Φ allows to consider a, b, c, d, e as pentahedral coordinates of space which means that it allows to linearly eliminate one of these quantities so that the remaining four can be seen as homogeneous coordinates of space. The equations $\Psi = 0$ and $\mathrm{X} = 0$ then become the respective equations of a quadric and a cubic surface. Their intersection is a space curve of order 6, and to each of its points corresponds an adequate Tschirnhaus transformation.

To establish a link with his first geometrical interpretation, Clebsch next sought to find a quadratic substitution having the same effect than a Tschirnhaus transformation: with the previous notations, it is equivalent to find quadratic polynomials $\varphi(\lambda)$ and $\psi(\lambda)$ such that

$$\frac{\varphi(\lambda_i)}{\psi(\lambda_i)} = a + b\lambda_i + c\lambda_i^2 + d\lambda_i^3 + e\lambda_i^4$$

holds for every root λ_i. It is thus sufficient to find φ and ψ such that for any λ,

$$\varphi(\lambda) = (a + b\lambda + c\lambda^2 + d\lambda^3 + e\lambda^4)\psi(\lambda) + (p + q\lambda)f(\lambda),$$

the scalars p and q being arbitrary constants. Putting as above

$$\varphi(\lambda) = y_1 + \lambda y_2 + \lambda^2 y_3 \qquad \text{and} \qquad \psi(\lambda) = x_1 + x_2\lambda + x_3\lambda^2,$$

and noting $f = \alpha + \beta\lambda + \gamma\lambda^2 + \delta\lambda^3 + \varepsilon\lambda^4 + \zeta\lambda^5$, the searched condition is equivalent to

$$\begin{cases} y_1 = ax_1 + p\alpha \\ y_2 = ax_2 + bx_1 + p\beta + q\alpha \\ y_3 = ax_3 + bx_2 + cx_1 + p\gamma + q\beta \\ 0 = bx_3 + cx_2 + dx_1 + p\delta + q\gamma \\ 0 = cx_1 + dx_2 + ex_1 + p\varepsilon + q\delta \\ 0 = dx_3 + ex_2 + p\zeta + q\varepsilon \\ 0 = ex_3 + q\zeta. \end{cases}$$

This system allows to rationally determinate $x_1, \dots, y_3$ as functions of $a, b, \dots, e$, or conversely to express $a, \dots, e$ as rational functions of $x_1, \dots, y_3$. The formulas thus obtained connect the two geometrical interpretations of Clebsch: to a point of pentahedric coordinates (a, b, c, d, e) in the interpretation of the Tschirnhaus method corresponds a couple of points x, y in the interpretation of the quadratic substitution, and reciprocally. Further, both interpretations make appear a space curve of order 6, which I here note $\mathscr{C}$, and which is the complete intersection of a cubic surface and a quadric surface. But now, the advantage—that is what Clebsch announced—is that these surfaces have equations: $\Phi = \Psi = 0$ and $\Phi = \mathrm{X} = 0$, and the representation formulas can be deduced from the previous system.

More precisely, the Cramer formulas yield

$$\begin{aligned} \rho x_1 &= M_1 & \qquad \sigma y_1 &= N_1 \\ \rho x_2 &= M_2 & \qquad \sigma y_2 &= N_2 \\ \rho x_3 &= M_3 & \qquad \sigma y_3 &= N_3, \end{aligned}$$

where $M_1, \dots, N_3$ are homogeneous functions of $a, b, \dots, e$ and where ρ, σ are constants. These expressions of x_1, x_2, x_3 on one hand, and of y_1, y_2, y_3 on the other hand, are the formulas of representation on the plane of the two surfaces $\Phi = \Psi = 0$ and $\Phi = \mathrm{X} = 0$ respectively. Hence to each point $(a, \dots, e)$ of the space curve rationally corresponds two points x, y such that the line (xy) is tangent to $C = 0$. One can finally deduce the formulas of representation of the curve $C = 0$:[63]

$$\begin{aligned} \tau u_1 &= M_2 N_3 - M_3 N_2 \\ \tau u_2 &= M_3 N_1 - M_1 N_3 \\ \tau u_3 &= M_1 N_2 - M_2 N_1. \end{aligned}$$

In other words, these formulas rationally associate to a point $(a, \dots, e)$ of the space curve $\mathscr{C}$, a line of coordinates u_1, u_2, u_3 which is tangent to $C = 0$. As a result, in order to find a tangent to $C = 0$ with square and cubic radicals, it is sufficient to find a point of $\mathscr{C}$ with square and cubic radicals.

Clebsch settled this last point using again "geometrical equations":

> If one considers, with Hermite, the equation
>
> $$x^5 - ax - b = 0 \qquad (1)$$
>
> as being directly solved by elliptic functions, then it only matters to bring the quintic into this form. So we only need to determine any point of the space curve of order 6,

[63] For that purpose, remember that a line $u_1 z_1 + u_2 z_2 + u_3 z_3 = 0$ contains two points $(x_1 : x_2 : x_3)$ and $(y_1 : y_2 : y_3)$ if and only if $(u_1 : u_2 : u_3) = (x_2 y_3 - x_3 y_2 : x_3 y_1 - x_1 y_3 : x_1 y_2 - x_2 y_1)$. Let us underscore that did not make explicit any of the formulas of representation.

which is done by intersecting a generator of the surface $\Psi = 0$ with the diagonal surface [$\mathrm{X} = 0$]. For that purpose, a quadratic equation and a cubic equation are to be solved; the first one to find a generator of the surface of the second order; the other one to determine the intersection points of this surface with the diagonal surface. The point of the space curve of order 6 found this way gives a tangent to $C = 0$, and it has been seen how the system of the points of intersection on this tangent leads to the form (1). If one knows the points of intersection of this system, then the sides of the quintilateral are split and the equation of the fifth degree is solved.[64] (Clebsch 1871b, p. 341)

These sentences ended Clebsch's geometrical interpretation of Hermite's solution of the quintic.

Here again, a special attention was given to the irrationals which are introduced in the different geometrical steps and controlled by "geometrical equations." Our analysis of the second geometrical interpretation also confirms what we stated earlier about the first one: Clebsch did not use Hermite's steps of the solution of the quintic by the means of elliptic functions, as the search for an explicit writing of reduced equation of order 5 of the modular equation and of its roots. On the contrary, he considered that this solution was assured once and for all, as soon as the Jerrard form is found.

As previously written, Clebsch also wanted to geometrically interpret Kronecker's method of solving the quintic. To transcribe his approach, I will now be briefer, the ideas being of the same vein as the ones I have been depicting until now. The first step is actually connected to what we just saw, since it is about examining the properties of the surface $\Phi = \mathrm{X} = 0$.

4.5 The diagonal surface

In order to investigate more closely the surface defined by the equations $\Phi = \mathrm{X} = 0$, Clebsch changed the pentahedric coordinates. Indeed, he came back to $\xi_1, \xi_2, \ldots, \xi_5$, linked by the relation $\sum \xi_i = 0$, so that the equation of the cubic surface was then $\sum \xi_i^3 = 0$.

The planes $\xi_i = 0$ are the five faces of the pentahedron of the cubic surface[65]; each of them is cut by the others in a quadrilateral. Each of these quadrilaterals have three

[64] "Wenn man nach Hermite die Gleichung $x^5 - ax - b = 0$ (1) als durch elliptischen Functionen unmittelbar gelöst betrachtet, so kommt es nur darauf an, die Gleichung 5$^{\text{ten}}$ Grades in diese Form zu bringen. Man hat dann nur einen belibiegen Punkt der Raumcurve 6$^{\text{ter}}$ Ordnung zu ermitteln, was geschieht, indem man eine Erzeugende der Fläche $\Psi = 0$ mit der Diagonalfläche schneidet. Dazu ist eine quadratische und eine cubische Gleichung zu lösen; erstere, um eine Erzeugende der Fläche 2$^{\text{ter}}$ Ordung zu finden; die andere, um die Durchschnitte derselben mit der Diagonalfläche zu bestimmen. Der gefundene Punkt der Raumcurve 6$^{\text{ter}}$ Ordnung giebt eine Tangente von $C = 0$, und wie das Schnittpunktsystem auf dieser zu der Form (1) führt, ist in §11. gezeigt worden. Kennt man die Schnittpunkte dieses Systems, so sind auch die Seiten des Fünfseits getrennt, die gegebene Gleichung 5$^{\text{ten}}$ Grades gelöst."

[65] In 1851, James Joseph Syvester had asserted that every cubic form $F(x, y, z, w) = 0$ can be brought to the form $F = a_1 z_1^3 + a_2 z_2^3 + a_3 z_3^3 + a_4 z_4^3 + a_5 z_5^3$ where $z_1, \ldots, z_5$ are linear forms in x, y, z, w satisfying the condition $z_1 + z_2 + z_3 + z_4 + z_5 = 0$. The *pentahedron* of the cubic surface defined by $F = 0$ is the set of the five planes respectively defined by $z_i = 0$.

diagonals, that Clebsch proved to be some of the twenty-seven lines of the cubic surface.[66] For this reason, Clebsch called the surface X = 0 the "diagonal surface of the pentahedron" or shorter the "diagonal surface."[67]

Therefore, for this particular cubic surface, there are 15 lines that are "immediately known"[68] once the faces of the pentahedron are known: these are the previously described diagonals, which are the mutual intersections of the pentahedron faces. Clebsch found the 12 missing lines using again the sides of the pentahedron. To be more accurate, he proved that if ω is a primitive fifth root of unity, all the points of space having for pentahedral coordinates $(\omega^{\alpha_1}, \omega^{\alpha_2}, \ldots, \omega^{\alpha_5})$ or the conjugates $(\omega^{-\alpha_1}, \omega^{-\alpha_2}, \ldots, \omega^{-\alpha_5})$, where $(\alpha_1, \alpha_2, \ldots, \alpha_5)$ is any permutation of $\{1, 2, \ldots, 5\}$, are points belonging to the diagonal surface, and he further showed that all the lines joining these points two by two are completely included in the surface.[69] Now, there are 12 points of coordinates $(\omega^{\alpha_1}, \omega^{\alpha_2}, \ldots, \omega^{\alpha_5})$—some permutations α give the same points—which yields the 12 missing lines. From that, Clebsch finally deduced a result on another "geometrical equation," namely the twenty-seven lines equation:

> The equation of degree 27 on which the 27 lines of the surface depend only demands, for the diagonal surface, the solution of the equation of degree 5 occurring for the pentahedron, as well as the determination of fifth roots of unity.[70] (Clebsch 1871b, p. 333)

Through a study of the incidence relations existing between the lines of the diagonal surface, Clebsch also came to the conclusion that one of the double-sixes of the surface[71] is "rationally known." Afterwards, he considered a representation of the diagonal surface corresponding to one of the two halves of this rational double-six. He finally proved, thanks to the incidence relations of the twenty-seven lines of the diagonal surface that the six fundamental points of the representation can be assembled into ten Brianchon

[66] For instance, he proved that on the face $\xi_5 = 0$, the equations of the diagonals are of the type

$$\begin{cases} \xi_1 + \xi_2 &= 0 \\ \xi_3 + \xi_4 &= 0 \\ \xi_5 &= 0. \end{cases}$$

It is then easy to check that each quintuplet $(\xi_1, \ldots, \xi_5)$ satisfying these conditions also satisfy the equation $\xi_1^3 + \cdots + \xi_5^3 = 0$ of the surface.

[67] "Ich werde diese specielle Fläche deswegen die *Diagonalfläche des Pentaeders* nennen." (Clebsch 1871b, p. 333). Afterwards, Clebsch shortened the name and wrote "die Diagonalfläche."

[68] "Man sieht, dass auf dieser Fläche sofort 15 der 27 Geraden bekannt sind."(Clebsch 1871b, p. 333).

[69] To check this point, remark that the line joining $(\omega^{\alpha_1}, \ldots, \omega^{\alpha_5})$ and $(\omega^{-\alpha_1}, \ldots, \omega^{-\alpha_5})$ is parameterized by $(\chi, \lambda) \mapsto (\chi\omega^{\alpha_1} + \lambda\omega^{-\alpha_1}, \ldots, \chi\omega^{\alpha_5} + \lambda\omega^{-\alpha_5})$. A simple computation proves that these coordinates satisfy the equation $\sum \xi_i^3 = 0$.

[70] "Die Lösung der Gleichung 27^{ten} Grades, von welcher die 27 Geraden der Fläche abhängen, erfordet bei der Diagonalfläche nur die Lösung der beim Pentaeder auftretenden Gleichung 5^{ten} Grades und die Bestimmung von fünften Wurzeln der Einheit."

[71] A *double-six* is a set of twelve lines among the twenty-seven contained in a cubic surface, satisfying particular incidence relations. In his paper on the plane representation of cubic surfaces, Clebsch had proven that the lines corresponding to the six fundamental points build the half of a double-six.

hexagons: this means that these six points can be joined one another in ten different ways so that they form hexagons of which the diagonals are concurrent.

In the note published in the *Nachrichten von der Königlichen Gesellschaft der Wissenschaften und der Georg-Augusts-Universität* that had announced the results of his paper on the quintic, Clebsch underscored the difficulty to clearly understand the configuration of the 27 lines of a cubic surface. He also stressed that the diagonal surface offers a simpler case and thus wished for the creation of a concrete model of the surface:

> For the actual representation of the system of the 27 lines of a surface of the third order, which form a very entangled system, the diagonal surface offers a simple and easy-to-construct example which exhibits with no major modification the largest part of the properties of the general system. A production of handy models of this surface should therefore be recommended.[72] (Clebsch 1871c, p. 342)

I do not want to go deeper in the details of the production of models, but the reader should note that plaster models of the diagonal surface have indeed been produced afterwards (see figure 1).[73] So Clebsch's work of the quintic made appear a particular surface when interpreting coefficients as coordinates of space, but the research on the quintic was not supported by a visual tool. On the contrary, the process of constructing of a model was nourished by this research and the actual building has been realized later.

Figure 1 – Plaster model of the diagonal surface. Source: (Fischer 1986).

[72] "Für die wirkliche Darstellung des Systems der 27 Geraden einer Oberfläche dritter Ordnung, welche ein sehr verwickeltes System bilden, giebt die Diagonalfläche ein einfaches und leicht construirbares Beispiel, welche zugleich die grösste Zahl der Eigenschaften des allgemeines Systems ohne zu grosse Modificationen aufweist. Es dürfte sich daher zu Herstellung bequemer Modelle diese Fläche besonders empfehlen."

[73] See (Lê 2015c, pp. 64-69). About models of surfaces in the second half of the 19th century, see also (Polo-Blanco 2007; Rowe 2013). The scattered appearances of Clebsch's article on the quintic in histories of geometry that I mentioned in my introduction are linked to this diagonal surface.

4.6 Interpretation of Kronecker's approach

As explained earlier, Kronecker had proven that the quintic can be brought into a pure equation $z^5 = A$ thanks to the multiplier equation (of degree 6) associated to the transformation of order 5 of elliptic functions. Clebsch carried out his geometrical interpretation of this approach in two steps. The first one was to find an equation of degree 6 of which the resolution brings the quintic into a pure equation; the second one was to prove that this equation of degree 6 is analogous to the multiplier equation.

For the first point, Clebsch based his approach on the study of a curve associated to a certain invariant B of the quintic. He proved that this curve breaks down in two curves $B_1 = 0$ and $B_2 = 0$, each of them having six double tangents. Then he demonstrated that if one uses one of these double tangents as the basis for a quadratic substitution, the quintic transforms into a pure equation. Consequently, it remained to show that the "splitting equation" of the six double tangents to $B_1 = 0$ is the same as the multiplier equation.

For that purpose, Clebsch started by proving that the splitting equation of the six tangents is analogous to the equation splitting the six vertices of the rational Brianchon hexagon found in the representation of the diagonal surface. Again, invariant properties were put to the front: referring to the paper (Clebsch 1871a), Clebsch indicated that if an equation of degree 6 has two of its invariants (noted a and c) equal to zero, then it can be transformed into the multiplier equation thanks to a linear substitution. It was thus about proving that this was the case for the equation splitting the six vertices. Clebsch symbolically noted this equation $\varphi = \alpha_u^6 = 0$, a ternary form in u_1, u_2, u_3 of which each factor represents one of the vertices of the hexagon.[74]

Clebsch had then the idea to translate by duality the principle of geometrical interpretation of the beginning: an equation $\varphi = \alpha_u^6 = \beta_u^6 = 0$ defines a hexagon; to any quadratic substitution acting on this equation corresponds a point of the plan, and the substitutions making an invariant $a = (\alpha\beta)^6$ vanish correspond to the points belonging to the curve of punctual equation $a = (\alpha\beta x)^6 = 0$; moreover, the vertices of the hexagon are double points of this curve.

Now, in the present case, since the six vertices have been taken as the fundamental points of the representation of the diagonal surface, each curve containing these points with mutliplicity 2 correspond to a curve of order 6 contained in the diagonal surface, namely a curve which is the complete intersection of the diagonal surface with a certain quadric surface.[75] Just like above, one can find the points of this curve thanks to quadratic and cubic equations. Therefore, such points yield points of the plane which correspond to quadratic substitutions making the invariant a vanish. Finally, Clebsch proved that

[74] In tangential coordinates $(u_1 : u_2 : u_3)$, the linear equation $z_1u_1 + z_2u_2 + u_3z_3 = 0$ represents the point of punctual coordinates $(z_1 : z_2 : z_3)$.

[75] Once again, this result had already been proven in (Clebsch 1866).

in the case of a Brianchon hexagon, the vanishing of the invariant a implied that of the invariant c, which thus completed the geometrical interpretation of Kronecker's method.

Here, nothing from what Kronecker had done to prove the connection between the multiplier equation and the solution of the quintic even appears in Clebsch's interpretation—this obliteration seems even more radical when compared to the case of the interpretation of Hermite's approach, where at least the crucial Jerrard form prominently appeared in the reasoning of Clebsch. But just like in the Hermitian case, the geometrical interpretation went through a use of invariants which established the connection between the quintic and the multiplier equation.

5 A glimpse at Clebsch's geometry, in the shadow of the icosahedron

The interpretation of the method of Kronecker closed the paper of Clebsch, and the latter concluded: "So, really all the elements of the solution of the equations of the fifth degree are here grouped and linked in a geometrical image."[76]

Clebsch thus asserted to have put together everything connected to the solution of the quintic. However, even a glance at the (cursory) chronology sketched in the introduction of the present paper proves that, besides the missing intermediate steps of Hermite and Kronecker, other points are lacking: in particular, the names and the contributions of Abel, Galois, Betti, or Brioschi do not appear in Clebsch's article. How can this be explained? Firstly, we saw that Clebsch took for granted Hermite's approach, that is, the possibility to solve the quintic with elliptic functions as soon at it has the Jerrard form. In other words, elliptic functions were plainly accepted by Clebsch as a way to express the solutions of the quintic; as such, there was no need for him to reconsider the issue of solving the quintic with radicals and hence to mention Abel. Moreover, we saw that Clebsch did not discuss the intermediary steps of Hermite's approach like the (crucial) one consisting in forming a reduced equation of the modular equation. Therefore the contributions of Galois and Betti, which are linked to the existence of this reduced equation, may have been cast out of the picture because they were seen as transitional elements in Hermite's solution of the quintic. Let us also add that beside this solution with elliptic functions, Clebsch omitted another facet of the contribution of Hermite: the one where the latter offered an unifying picture of the works of himself, of Kronecker, and of Brioschi with help of the theory of invariants.

Apart those missing names and contributions, other remarkable absentees from the article of Clebsch are worth a comment. These absentees are mathematical objects that one could have expected to find in a 1871 paper on the quintic equation, namely groups

[76]"So finden sich denn wirklich alle Elemente der Auflösung der Gleichungen 5^{ten} Grades hier in einem geometrischen Bilde zusammengefasst und verbunden", (Clebsch 1871b, p. 345).

of substitutions.[77] Now, this observation may partially explain why Clebsch's geometrical interpretation of the quintic has been forgotten by the historiography. Indeed, the traditional history of algebraic equations (at least after Galois' works) has been constructed in a large part through the retrospective prism of group theory.[78] Clebsch elaborated his research on the quintic at the very beginning of the 1870s, a period during which the theory of algebraic equations underwent sharp changes because some other works insisted more and more on groups—for instance, Camille Jordan's celebrated *Traité des substitutions et des équations algébriques* was published in 1870, just before the paper of Clebsch. The absence of groups in this paper thus made it out of accordance with the new disciplinary tendencies of the time. Since these tendencies have later been used to select what should be kept in the history of algebraic equations, it could explain why Clebsch's contribution eventually disappeared from the picture.

Another possible reason of the oblivion of Clebsch's paper in the historiography bears upon Felix Klein's 1884 *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*. Similarly to Clebsch's one, this work consisted in exposing the theory of the quintic equation within a geometrical frame. Nevertheless Klein's approach was different from Clebsch's, for it was grounded on the interpretation of the five roots of the quintic as coordinates of the 3-dimensional space, and on the isomorphism between their group of substitutions and the group of space transformations leaving a regular icosahedron unaltered. Some of the elements that Clebsch developed in his paper on the quintic do appear in Klein's book, but they are treated in diverse ways. For instance, the idea to interpret the conditions $\sum \xi_i = \sum \xi_i^3 = 0$ as the equations of a cubic surface is not imputed to Clebsch, but the latter is referred to for the mere study of this particular surface. In other words, Clebsch is not mentioned for his general method of geometrical interpretation, but for technical tools that eventually appear as disconnected from the quintic.[79] Doing so, Klein gave the impression that the only geometrical interpretation of the quintic was his own—this maneuver is also particularly flagrant in the chapter on the history of the quintic equation, where Klein completely eluded Clebsch's contribution. Since it is Klein's research on the icosahedron that eventually survived in the historiography as *the* geometric approach of the quintic, one thus understands why Clebsch remained invisible with his research to the topic.[80]

[77] Groups of transformations are not to be found in Clebsch's article either. More generally, there are neither groups of substitutions, nor groups transformations in all the publications and manuscripts of Clebsch that I read. In fact, some passages of his correspondence indicate that Clebsch had difficulties to understand these objects. See (Lê 2015b, p. 18).

[78] The effects of this prism on the reception of Galois' works themselves have been thoroughly analyzed in (Ehrhardt 2012).

[79] For the mentions of the diagonal surface, see (Klein 1884, pp. 166, 218, 226). Moreover, let use note that the article of Clebsch is cited as an inspirational source in (Klein 1871). In this paper, which is cited in the *Vorlesungen über das Ikosaeder*, Klein expressed for the first time the general idea to represent roots of an equation and their substitutions by points of the space and associated linear transformations.

[80] The conclusions of this paragraph and the preceding one match with the fate of Hermite's research

The main objective of my paper was to see how Clebsch used geometry to cope with an algebraic problem. As we saw, geometry appeared through many objects and techniques: coefficients of quadratic substitutions as well as coefficients of the Tschirnhaus transformations were interpreted as coordinates of lines, curves, or surfaces; the vanishing of invariants were linked to conditions of tangencies; the study of curves and surfaces implied particular associated points or lines, genus computation, and representations of surfaces; finally, the appearing irrationals were geometrically controlled with the help of "geometrical equations." These objects and techniques, which were not accompanied by any figure, any diagram, or any concrete surface model, formed the material of the "geometrical clothing" of the theory of the quintic that Clebsch wanted to sew.

Does this geometrical clothing provide a clearer, more intuitive frame for the theory of the quintic equation? Of course, the possible answers to this question depend on when and whom it is asked: the modern reader, unused with all the ways of doing we described above, may still be skeptical. However, we see by contrast that the constituents of the arsenal that Clebsch deployed throughout his paper are tools that he (and his close colleagues) perceived as geometric and intelligible. Excavating the article of Clebsch was meant to discuss the diverse facets that geometry can have. It led us to unveil and understand parts of the research of this almost forgotten mathematician, yet presented by some of his pairs as one of the "greatest representative" of "the German algebraico-geometric science" of his time.[81]

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analyzed in (Goldstein 2011).

[81] These words come from the obituary of Hesse written by M. Noether: "Innerhalb zweier Jahre hat die deutsche algebraisch-geometrische Wissenschaft ihre beiden grössten Vertreter verloren: seinem so früh dahingeschiedenen Schüler Alfred Clebsch ist der Altmeister Otto Hesse jetzt nachgefolgt." (Noether 1875, p. 77).

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