On the past of a mathematical object

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1. Introduction

"We 'big picture' people rarely become historians," proclaims the 6-year-old boy Calvin after answering to his history test question on the significance of the Erie Canal: "To the cosmic sense, probably nil." [Watterson 1996, p. 13]. Choosing an appropriate spatiotemporal scale to put an historical event in perspective is indeed a smart way to escape a sticky school situation, but more virtuous reasons may also lead one to embed an event of the past into a bigger picture, a more sensible canvas than the whole universe being then adopted.

For instance, when investigating the history of a mathematical object, a common approach is to start by locating a definition of this object, which is chosen according to certain criteria and is used as a reference point for the rest of the inquiry. Drawing a bigger picture, then, may mean figuring out why and how the object was defined at the time. It may also mean understanding how it circulated afterwards, or from what it originated, two avenues which come within the scope of reconstituting the future and the past of the object.

In such reconstruction processes, the issue of identifying the researched object is crucial. The act of recognizing this object, possibly under the garments of transformed or earlier versions, and of establishing connections between such occurrences, indeed, is what founds the writing of a corresponding narrative. The pitfall of anachronistic identifications must therefore be avoided, so as not to produce pictures whose temporal extent and historical consistency would be dubious.

To rely on traces left by past mathematicians is an obvious solution, which still requires to be carefully reflected upon. In the case of the future of an object, a possibility to cope with the identification problem is to track this object thanks to features such as its name, its usual notation, or the references to the author who is credited for having defined it. On the contrary, one has to proceed a bit differently when trying to look backwards in time, since the

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issue is find footprints of something that does not exist yet, in the strict sense of the term.

This does not mean that investigating the past of an object is a vain question or a necessary impasse. Mathematical objects do not come from nowhere. They are often introduced to address specific problems which have their own histories, they enter into preexisting theories, they can be the outcome of mechanisms of reworking of other objects, and their name and notation may have already been used in related situations. But locating such elements and using them to build genealogies must be handled with caution, a particularly acute danger being to end up with illusory long-term histories.[1]

My aim in this chapter is to tackle such matters in the case of the objects that are the genera of algebraic curves, by offering and discussing concrete proposals for reconstituting their past.

In accordance with the previous lines, a point of reference must first be fixed. The chosen one consists in the definitions given by the German mathematician Alfred Clebsch in two papers published in 1865 in *Journal für die reine und angewandte Mathematik*, [Clebsch 1865b,a]. Although they are not formulated in the exact same way, these definitions perfectly match, and there is no reason for me to treat them differently in respect with the present issue. One of them reads as follows:

> The class of Abelian functions that is connected with an algebraic curve of the nth order is determined by the number $p = \frac{n-1 \cdot n-2}{2}$ if the curve has no double point and no cusp, and, in the 63^{rd} volume of this journal, I have given a number of results which rest on this remark. [...] Instead of classifying the algebraic curves in orders, and making subdivisions in them according to the number of double points and cusps that they contain, one can classify them into *genera* according to the number p; in the first genus are thus the curves for which $p = 0$, in the second one those for which $p = 1$, etc. Hence the different orders appear reciprocally as subdivisions of the genera [...].[2] [Clebsch 1865b, p. 43]

The genera defined by Clebsch are thus categories in a classification of algebraic curves, i.e. curves that can be defined by a polynomial equation. They are defined through the value of a number p coming from the theory of Abelian

[1] On this issue, see [Goldstein 1995, 2019], from which many of my reflections stem. Other works on the history of objects or theorems influenced me, such as [Sinaceur 1991; Brechenmacher 2010; Ehrhardt 2012].

[2] "Die Classe von Abelschen Functionen, mit welcher eine algebraische ebene Curve n^{ter} Ordnung zusammenhängt, wird durch die Zahl $p = \frac{n-1 \cdot n-2}{2}$ bestimmt, wenn die Curve keine Doppel- und Rückkehrpunkte besitzt, und ich habe im 63^{ten} Bande dieses Journal pag. 189 eine Reihe von Resultaten angeführt, welche sich auf diese Bemerkung stützen. [...] Statt die algebraischen Curven nach Ordnungen einzutheilen, und in diesen Unterabtheilungen zu machen nach der Anzahl der Doppel- und Rückkehrpunkte, welche dieselben aufweisen, kann man dieselben in Geschlechter eintheilen nach der Zahl p; zu dem ersten Geschlecht also alle diejenigen für welche $p = 0$, zum zweiten diejenigen, für welche $p = 1$, u.s.w. Dann erscheinen umgekehrt die verschiedenen Ordnungen als Unterabtheilungen in den Geschlechtern".

functions and given by the formula $p = \frac{(n-1)(n-2)}{2} - d$, where n is the order of the considered curve[3] and d is an integer whose value depends on the singularities on the curve.

Before proceeding, let me recall that only a few years after 1865, Clebsch and his contemporaries designated the number p itself as the genus of an algebraic curve, a practice that has persisted until today. However, such a shift in meaning is not easy to locate with precision and does not seem to have been accompanied by any precise (re)definition. I will thus neglect this phenomenon, my aim being anyway to discuss issues related to the reconstitution of the past of an object from a given point in time.

As is clear from the above quote, Clebsch's 1865 definition of the genera involve two mathematical domains, which appear together with hints at some pieces of their history: on one hand, the theory of algebraic curves, associated with the traditional classification in orders, and, on the other hand, the theory of Abelian functions, the past of which surfaces through a citation to an earlier paper of Clebsch, [Clebsch 1864b]. In what follows, we will see that these two facets correspond to two historical threads that have remained essentially distinct from one another before 1865, in the sense that they involve different mathematicians and mathematical contents, have their own timelines, and their own kinds of historical continuities and coherence.

More remarkable, perhaps, is the fact that these two threads also correspond to two manners of investigating the past of the genera. Indeed, one customary way of gaining a reasoned understanding of this past is to start with Clebsch's two 1865 articles, consider the publications that are cited therein, select those where the number p appears in one way or another, and start the process again with these publications. This allows to see p in a few texts published between 1865 and 1857, which corresponds to Bernhard Riemann's famous memoir on Abelian functions, [1857], and to recognize other versions of this number in earlier publications, in a sense that will be explained below. All these texts relate to the topic of Abelian functions, and, while only a handful of them involve algebraic curves, none mention any object called "genus."

An alternative way of doing consists in starting from a large set of publications related to a whole mathematical domain, and searching there elements that are relevant for the purpose. For example, when considering a large corpus made of texts on algebraic curves and dated before 1865, several notions of genera of algebraic curves appear, contrary to anything connected to the number p. Similarly, one could try to use a corpus on Abelian functions to spot publications that involve the number p in one way or another. This exercise yields the majority of the works found otherwise by the citation process, and it turns out to be quite perilous to add new texts: as show the first works, mathematicians of the time sometimes worked with what they saw as particular

[3] The order of a curve is the degree of its defining equation. As can be seen in the above quote, the term "order" also designated the category made of the curves of a given degree.

values of (earlier versions of) p, such as 1, which corresponds to the case where the Abelian functions are elliptic integrals. Hence, in the absence of any trace such as citations or a designation in the natural language – the number p is not called by any name –, it seemed too hazardous to select texts where the number 1 could be interpreted as having a relevant role in the history of the genera. Thus I chose to stick to recognition processes supported by the explicit traces that are the citations and the identity of the name "genus," which guarantees more solid foundations.

The next two sections are devoted to each of the mentioned historical threads, which correspond to tracking backwards the number p and the curve category called "genus," respectively. In both cases, I first make more explicit the processes of corpus formation and comment briefly on them. Then I present the corresponding narratives and analyze them in view of the "big picture" issue, insisting in particular on the historical complexity of each situation and on related historiographical questions. The concluding section finally compares these two pictures and reflects on what would be one unified big picture of the past of the genera of curves.

2. Tracking a number

2.1 Formation of the corpus

As explained above, the corpus considered in this section is obtained by selecting the papers that are cited in the two 1865 articles by Clebsch and where p can be found, and by repeating the operation from the obtained references. This process has been performed three times. The reason to this is that a fourth layer only adds texts that deal exclusively with elliptic functions, a situation which more complicated to handle with respect to the issue of recognizing p through particular values.

The corpus thus gathered is made of 29 texts by 13 authors. These texts have been published over a century, between 1766 and 1865, although most of them date from after the mid-1820s. They are represented, together with their explicit[4] citation links, in Figure 1.

As this graph immediately suggests, the situation is quite entangled, and cannot be reduced to a linear sequence of texts nicely articulated with one another and going from one chronological boundary to the other. That said, there exist other features, which cannot be guessed from the appearance of the graph and contribute to the complexity of the picture. On one hand, the citations themselves are of different kinds, some being precise technical borrowings while others acknowledge previous works that must be surpassed,

[4]Implicit references are important to take into account, too, but may be much more difficult to grasp. An example will be seen below, in the case of [Jacobi 1832].

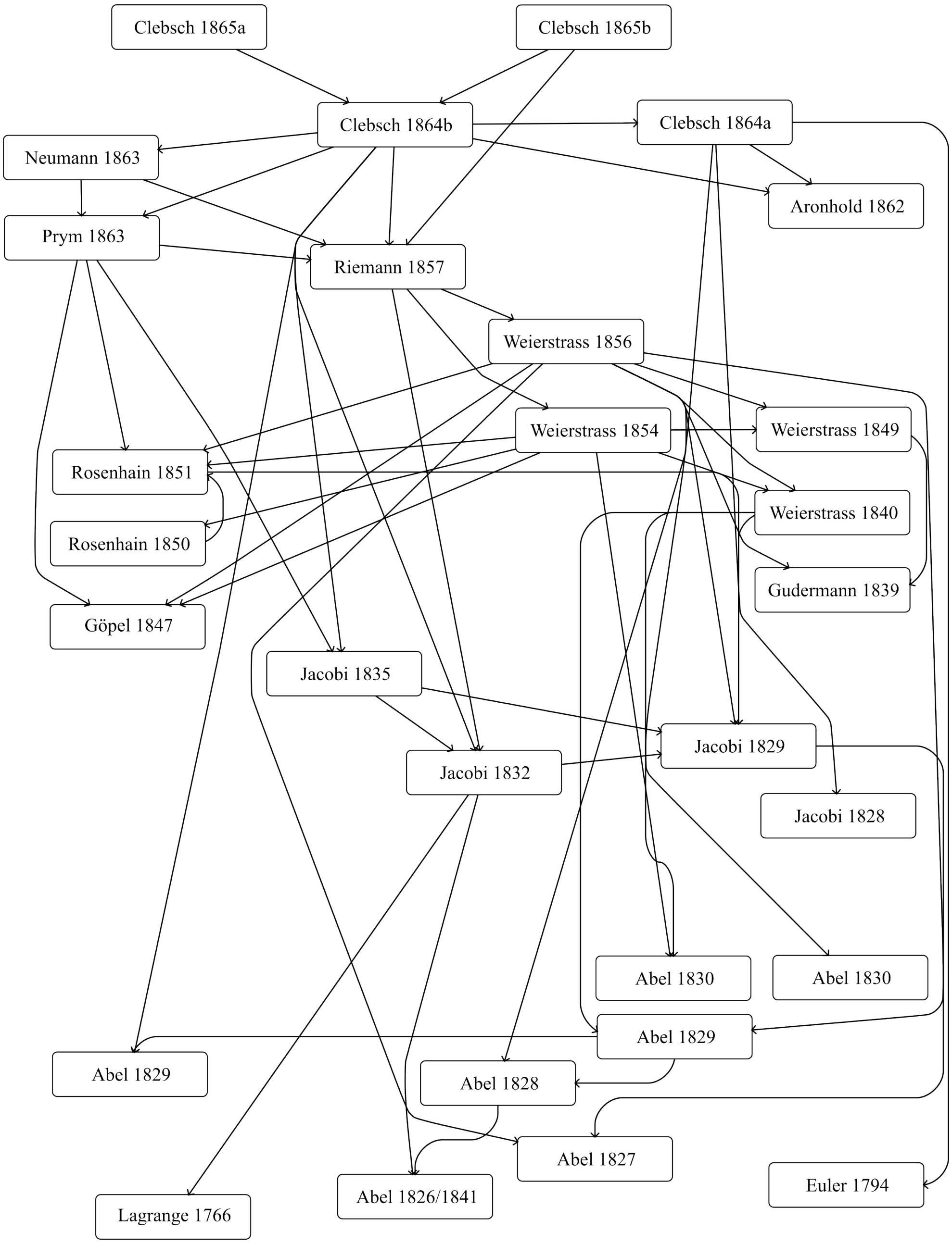

Figure 1: Graph of explicit citations between the texts of the corpus on Abelian functions.

for example,[5] and they do not necessarily concern p directly. On the other hand, delving into the texts and focusing on how the number p is involved reveals a variety of thematic threads, of which some are absent from certain texts while others may coexist in given publications.

For reasons of space, I cannot describe all the texts of the corpus and all their links. I will thus confine myself to a selection that illustrates the above-mentioned phenomena concerning the number p. This selection is made of texts which form an apparently continuous chain of citations, with a limited number of ramifications. It contains the two 1865 papers by Clebsch where the genera of curves are defined and the 1864 memoir of the same that is cited therein, [Clebsch 1864b, 1865b,a], Riemann's memoir on Abelian functions, [Riemann 1857], as well as texts by Carl Weierstrass [1854, 1856], Georg Rosenhain [1851], Carl Gustav Jacob Jacobi [1832], and Niels Henrik Abel [1826/1841, 1828]. I will describe these texts chronologically, concentrating on the occurrences of (earlier versions of) p, on Abel's addition theorem and the inversion problem, and on the connections between the texts.[6]

2.2 Local descriptions...

The first text in our chain is the broad memoir that Abel wrote and sent to the French Academy of Science in 1826, but was published only posthumously, in 1841, [Abel 1826/1841]. It is part of the corpus because both Abel himself and Jacobi mentioned the 1826 version in published papers. The memoir was devoted to what would later be called Abelian functions, that is, functions "of which the derivatives can be expressed by the means of algebraic equations, all the coefficients of which are rational functions of one and the same variable."[7] Such a function was denoted by

$$\psi(x) = \int f(x, y)\, \mathrm{d}x,$$

where f is a rational function and y is a function of x defined implicitly by a polynomial equation $\chi(x, y) = 0$. Abel presented these functions as generalizations of rational and elliptic integrals. These integrals, indeed, correspond to the case where $\chi(x, y) = y^2 - p_0(x)$ for a polynomial p_0 of degree 1 or 2, or 3 or 4, respectively. They were taken as examples in the last section of the

[5] On the need to examine carefully the role of the citations and, more generally, on their use in the history of mathematics, see [Goldstein 1999].

[6] Almost all the selected texts have been analyzed in other historical works, without however focusing on the facets that interest me. See for instance [Gray 1989] or [Houzel 2002], as well as other references that will be given below. Furthermore, my main objective here is historiographical, which is not the case of such past research.

[7] "[Les fonctions] dont les dérivées peuvent être exprimées au moyen d'équations algébriques, dont tous les coefficients sont des fonctions rationnelles d'une même variable", [Abel 1826/1841, pp. 176–177].

memoir, together with the case where p_0 is of degree 5 or 6, which leads to hyperelliptic integrals.[8]

The main result in Abel's memoir was the addition theorem. It was introduced as encompassing the known theorems according to which the sum of rational integrals is a rational integral, and that the sum of elliptic integrals is an elliptic integral whose argument is determined algebraically in function of the given data.[9] One form of the addition theorem given by Abel states that if $\psi(x_1), \ldots, \psi(x_\alpha)$ are α values of an Abelian function ψ, there exist an integer μ and an algebraic-logarithmic function v of quantities associated with the x_i, such that[10]

$$\psi(x_1) + \cdots + \psi(x_\alpha) = v - (\psi(x_{\alpha+1}) + \cdots + \psi(x_\mu)).$$

The number $\mu - \alpha$ corresponds to what would coincide with p decades later, at least under some conditions on the equation $\chi(x, y) = 0$. Abel highlighted that this number is "very remarkable,"[11] and he devoted many pages to investigate it. In particular, in the final examples of the paper, Abel found that $\mu - \alpha = 0$, 1 or 2 for rational, elliptic, or hyperelliptic integrals, respectively, [Abel 1826/1841, pp. 256–260].

The lack of response of the French Academy of Sciences to the submission of his 1826 memoir prompted Abel to write an article that would be published shortly after in Crelle's *Journal für die reine und angewandte Mathematik*, [Abel 1828]. In the introduction, Abel first stated the general addition theorem, referring to manuscript of 1826. Then he explained that his present aim was to prove the theorem in the case of general hyperelliptic integrals

$$\psi(x) = \int \frac{r(x)}{\sqrt{R(x)}}\, \mathrm{d}x,$$

where r is a rational function and R a polynomial.[12] In fact, for some intermediary results, Abel was led to consider integrals having the specific form

$$\int \frac{\delta_0 + \delta_1 x + \cdots + \delta_{m-2} x^{m-2}}{\sqrt{R(x)}}\, \mathrm{d}x,$$

[8] More generally, hyperellptic integrals correspond to $\chi(x, y) = y^2 - p_0(x)$ with p_0 of degree > 4. Abel did not use the phrase "hyperelliptic integrals." The terminology was not settled at the time, and the corpus even allows to follow part of its evolution. Nevertheless, I will elude this issue for the sake of simplicity.

[9] The theorem on elliptic integrals is usually attributed to Leonhard Euler, including by several authors in the corpus, as will be exemplified below.

[10] Abel also showed how to replace the sign $-$ by a sign $+$, and how to extend the theorem so as to treat linear combinations of values $\psi(x_i)$ with rational coefficients. On this addition theorem and on other results that have been called "Abel's theorem," see [Kleiman 2004].

[11] "Dans cette formule le nombre des fonctions $[\psi(x_{\alpha+1}), \ldots, \psi(x_\mu)]$, est très-remarquable." [Abel 1826/1841, p. 210].

[12] With the 1826 notations, it corresponds to $\chi(x, y) = y^2 - R(x)$ and $f(x, y) = \frac{r(x)}{y}$.

with $R(x)$ of degree $2m-1$ or $2m$. Retrospectively, one can see in this formula the general expression of the integrals of the first kind associated with $\chi(x,y)=y^2-R(x)$, that is, integrals that remain finite everywhere – a related fact is that the numerator of the integrated fraction depends on $m-1$ coefficients.

Further, these integrals helped Abel prove the addition theorem: if R is of degree $2m-1$ or $2m$, and if $x_1,\dots,x_{\mu_1}$, $x'_1,\dots,x'_{\mu_2}$ are any variables, there exist algebraic functions $y_1,\dots,y_{m-1}$ of them, such that

$$\psi(x_1)+\cdots+\psi(x_{\mu_1})-\psi(x'_1)-\cdots-\psi(x'_{\mu_2})=v+\varepsilon_1\psi(y_1)+\cdots+\varepsilon_{m-1}\psi(y_{m-1}),$$

where v is an algebraic-logarithmic function and the ε are ±1. In particular, and although Abel did not write it explicitly, comparing this formulation of the theorem with the 1826 version shows that $\mu-\alpha=m-1$ in the present case.

While the memoir of 1826 could not be read by most mathematicians before 1841, the version published in Crelle's journal did circulate. Even if Jacobi did not cite the latter explicitly in his paper belonging to our corpus, the many attributions to Abel of results that are contained in it leaves little doubt on the fact that he drew upon it.[13] Jacobi's general framework was hyperelliptic integrals, which he presented based on known results on rational and elliptic integrals. In particular, Jacobi underscored the importance of the addition theorem, which he explicitly attributed to Abel and presented as the generalization of Euler's theorem on elliptic integrals

$$\Pi(x)=\int_0^x\frac{\mathrm{d}x}{\sqrt{X}},$$

where X is a fourth-degree polynomial. After having stated Euler's theorem, Jacobi turned to particular hyperelliptic integrals

$$\Pi(x)=\int_0^x\frac{A+A_1x}{\sqrt{X}}\,\mathrm{d}x,$$

with X of degree 5 or 6, and then to general hyperelliptic integrals

$$\Pi(x)=\int_0^x\frac{A+A_1x+\cdots+A_{m-2}x^{m-2}}{\sqrt{X}}\,\mathrm{d}x,$$

with X of degree $2m-1$ or $2m$. For these last integrals, the addition theorem was stated under the form that for any variables $x,x_1,\dots,x_{m-1}$, there exist algebraic functions $a,a_1,\dots,a_{m-2}$ of them such that

$$\Pi(x)+\Pi(x_1)+\cdots+\Pi(x_{m-1})=\Pi(a)+\Pi(a_1)+\cdots+\Pi(a_{m-2}).$$

[13] On the other hand, Jacobi explicitly mentioned the 1826 memoir in a footnote, explaining that it had been sent to the Academy of Sciences and that it should be published, [Jacobi 1832, p. 397].

Jacobi then highlighted that this theorem implies that the sum of any number of $\Pi(x_i)$ can be expressed by a sum of $m-1$ values $\Pi(b_i)$: this result corresponds to the addition theorem contained in [Abel 1828] and stresses the role of the number $m-1$, whose notation is the same as Abel's.

The number $m-1$ also appeared in original results by Jacobi, among which the statement of the so-called inversion problem. The initial question was to find the reciprocal function of $\Pi(x)$, i.e. to find $\lambda(u)$ such that $u = \Pi(x)$ if and only if $x = \lambda(u)$. Jacobi first recalled that it had already been solved for rational and elliptic integrals. Passing to hyperelliptic integrals and using the addition theorem, he explained that the adequate way to tackle the inversion was to introduce several variables. Thus, if X is of degree 5 or 6, instead of considering the single equation $u = \Pi(x)$, the idea was to invert the system

$$\begin{cases} u = \Phi(x) + \Phi(y) \\ v = \Phi_1(x) + \Phi_1(y), \end{cases} \tag{1}$$

where $\Phi(x) = \int \frac{\mathrm{d}x}{\sqrt{X}}$ and $\Phi_1(x) = \int \frac{x\,\mathrm{d}x}{\sqrt{X}}$. Similarly, Jacobi indicated that the inversion problem for hyperelliptic integrals with X of degree $2m-1$ or $2m$ consisted in inverting a system of $m-1$ equations in $m-1$ unknowns.

If Jacobi indicated the right way of tackling the inversion problem, to carry it out effectively was left to some his successors. It was tackled in particular in the case of hyperelliptic integrals corresponding to a polynomial X of degree 5 or 6 in a paper that Rosenhain sent to the French Academy of Sciences in 1846 at the occasion of the *concours* of this year. The paper was successful and was published five years later, [Rosenhain 1851]. Jacobi's name was associated with the inversion problem only through vague references; explicit citations concerned mostly the *Fundamenta nova*, [Jacobi 1829], from which Rosenhain took the idea to consider θ-functions to solve the problem. Jacobi, indeed, had used θ-functions of one variable to study elliptic functions, and Rosenhain managed to define θ-functions of two variables to solve the inversion system (1).

Later, Weierstrass investigated and solved the problem for general hyperelliptic integrals. The main lines of the results were first indicated in a short paper, [Weierstrass 1854], before a more complete version was published, [Weierstrass 1856]. While acknowledging Rosenhain's contribution, Weierstrass explained that his predecessor's approach could not be generalized adequately, so that he developed a new way to handle the problem. Jacobi's name was mentioned a number of times, most often without explicit reference; on the contrary, the 1828 paper by Abel was cited for the proof of the addition theorem for hyperelliptic integrals.

Interestingly, Weierstrass did not follow Abel's or Jacobi's notations. He fixed a polynomial $R(x) = A_0(x-a_1)(x-a_2)\cdots(x-a_{2\rho+1})$ for a given integer

ρ, the inversion problem being then encapsulated in the differential system

$$\begin{cases} \mathrm{d}u_1 = \frac{1}{2}\frac{P(x_1)}{x_1-a_1}\cdot\frac{\mathrm{d}x_1}{\sqrt{R(x_1)}} + \frac{1}{2}\frac{P(x_2)}{x_2-a_1}\cdot\frac{\mathrm{d}x_2}{\sqrt{R(x_2)}} + \cdots + \frac{1}{2}\frac{P(x_\rho)}{x_\rho-a_1}\cdot\frac{\mathrm{d}x_\rho}{\sqrt{R(x_\rho)}} \\ \mathrm{d}u_2 = \frac{1}{2}\frac{P(x_1)}{x_1-a_2}\cdot\frac{\mathrm{d}x_1}{\sqrt{R(x_1)}} + \frac{1}{2}\frac{P(x_2)}{x_2-a_2}\cdot\frac{\mathrm{d}x_2}{\sqrt{R(x_2)}} + \cdots + \frac{1}{2}\frac{P(x_\rho)}{x_\rho-a_2}\cdot\frac{\mathrm{d}x_\rho}{\sqrt{R(x_\rho)}} \\ \quad\vdots \\ \mathrm{d}u_\rho = \frac{1}{2}\frac{P(x_1)}{x_1-a_\rho}\cdot\frac{\mathrm{d}x_1}{\sqrt{R(x_1)}} + \frac{1}{2}\frac{P(x_2)}{x_2-a_\rho}\cdot\frac{\mathrm{d}x_2}{\sqrt{R(x_2)}} + \cdots + \frac{1}{2}\frac{P(x_\rho)}{x_\rho-a_\rho}\cdot\frac{\mathrm{d}x_\rho}{\sqrt{R(x_\rho)}}, \end{cases}$$

where $P(x) = (x-a_1)(x-a_2)\cdots(x-a_\rho)$. The number ρ thus appears both as the number of equations and variables in the inversion problem, and as the number of constants in the polynomials $\frac{P(x)}{x-a_j}$. In other words, it corresponds to what was denoted by $m-1$ in Abel [1828] and in Jacobi [1832]. Without having the possibility of being assertive on this point, the change of notation may be seen as a trace of a shift of focus: this number playing an important role in these matters, the symbols are adapted to make it appear as an entity in its own right.

Only one year after Weierstrass's paper was published that of Riemann [1857], whose framework was completely different. Like Abel, Riemann studied general Abelian functions, that is, primitives of algebraic functions s defined implicitly by a polynomial equation $F(s,z) = 0$. The investigation made use of the surfaces that would soon bear Riemann's name.[14] Riemann studied these surfaces first in a setting independent of Abelian functions. In particular, he characterized them with their connectivity order, a surface being $(n+1)$-ply connected if it can be separated in two pieces by making n cross-cuts. In the special case of a Riemann surface associated with an algebraic function s, Riemann then showed that the number n is necessarily even, the connectivity order being thus of the form $2p+1$. The number p thus introduced is the one which will be eventually used to define the genera in Clebsch's papers.

This number was pivotal in Riemann's whole theory of Abelian functions. For instance, Riemann proved that it is equal to the maximal number of linearly independent integrals of the first kind. This resulted from the writing of such an integral as

$$\int \frac{\varphi(s,z)}{\frac{\partial F}{\partial s}}\,\mathrm{d}z,$$

where φ is a polynomial having p independent coefficients – this encompasses the elliptic and hyperelliptic integrals considered by Abel, Jacobi, and Weierstrass, for which the number of coefficients was denoted by $m-1$ and ρ. However, contrary to the latter, the number p cannot be read directly on the writing of the polynomial $F(s,z)$. Instead, Riemann established a formula expressing p in function of numbers associated with F: if n is its degree with respect to

[14]On Riemann surfaces, see for instance [Scholz 1980]. As is well known, these surfaces were first introduced in Riemann's doctoral dissertation, [Riemann 1851].

s and if w is the number of pairs (s,z) such that $F(s,z) = \frac{\partial F}{\partial s}(s,z) = 0$ and $\frac{\partial F}{\partial z}(s,z) \neq 0$, then

$$p = \frac{w}{2} - (n-1),$$

provided a supplementary technical condition is satisfied.

As for Jacobi's inversion problem, Riemann first acknowledged that Weierstrass had solved it in the hyperelliptic case. However, he added that he mostly knew what Weierstrass had sketched in the 1854 paper, and thus that the correspondence "not only in the results but also in the methods leading to them will for the most part only be revealed by the promised detailed presentation" given in [Weierstrass 1856].[15] Riemann did not restrict himself to hyperelliptic integrals: he expressed the inversion problem as the issue of finding, for each system of complex numbers $(e_1, \ldots, e_p)$, some values $\eta_1, \ldots, \eta_p$ such that

$$(e_1, \ldots, e_p) \equiv \left(\sum_{\nu=1}^{p} u_1(\eta_\nu), \ldots, \sum_{\nu=1}^{p} u_p(\eta_\nu) \right),$$

where $u_1, \ldots, u_p$ form a system of linearly independent integrals of the first kind, and where the congruence sign refers to their periods. The problem was then solved with the help of θ-functions of p variables, the theory of which was developed in the same paper.

Finally, Abel's addition theorem also appeared in Riemann's paper, but there was no question in proving it again. Instead, Riemann reformulated it and used it, "after Jacobi [1832], for the integration of a system of differential equations,"[16] which refers to a theorem of Jacobi's that I chose not to mention.

A number of Riemann's results were taken by Clebsch in his 1864 memoir devoted to the application of Abelian functions to geometry, [Clebsch 1864b]. Clebsch explained that Riemann's paper contained everything that was needed for such applications, but that the difficulties to read it had refrained mathematicians to do so. At the same time, he asserted that instead of conceiving integrals as Riemann did, one should adopt the viewpoint of Jacobi [1832], which seems to refer to a refusal of using Riemann surfaces.

Clebsch was first and foremost interested in the study of algebraic curves. At the beginning of his work, he considered a plane algebraic curve defined by

[15] "[I]n wie weit zwischen den späteren Theilen dieser Arbeiten und meinen hier dargestellten eine Uebereinstimmung nicht bloss in Resultaten, sondern auch in den zu ihnen führenden Methoden stattfindet, wird grossentheils erst die versprochene ausführliche Darstellung derselben ergeben können." [Riemann 1857, p. 116]. The exact reference to Weierstrass's 1856 paper was given in Riemann's paper but, oddly enough, at the end of the sentence where was described the 1854 one (which was also cited a few lines before). That Riemann wrote that he would compare the methods and results later seems to indicate that he added the reference quite late, or even that the journal editor added himself the exact bibliographic data shortly before publication.

[16] "Ich benutze nun nach Jacobi (dieses Journals Bd. 9 Nr. 32 §. 8) das Abel'sche Additionstheorem zur Integration eines Systems von Differentialgleichungen". [Riemann 1857, p. 137].

an equation $f(x_1, x_2, x_3) = 0$ of degree n between homogeneous coordinates of the plane, which he saw as the locus of the intersection points of the lines of two pencils whose parameters s and z are linked by an equation $F(s, z) = 0$.[17] The similarity with Riemann's notations is no coincidence: it prepared the effective mathematical transfer, especially the transformation of the formula $p = \frac{w}{2} - (n-1)$. Indeed, having linked the curve $f = 0$ with an equation $F(s, z) = 0$ allowed Clebsch to interpret w as the class of the curve, that is, as the number of tangents that can be drawn to it from any (generic) point of the plane. The combination of the well-known equality $w = n(n-1) - d$, where d is the number of double points of $f = 0$,[18] with Riemann's formula eventually yielded

$$p = \frac{(n-1)(n-2)}{2} - d.$$

The number p can be seen in many other places of Clebsch's paper. For instance, it was involved in the homogeneous writing of the integrals of the first kind:

$$\int \Theta \cdot \frac{\sum \pm c_1 x_2 \, \mathrm{d}x_3}{c_1 \frac{\partial f}{\partial x_1} + c_2 \frac{\partial f}{\partial x_2} + c_3 \frac{\partial f}{\partial x_3}},$$

where $\sum \pm c_1 x_2 \, \mathrm{d}x_3$ is the determinant whose elements are the c_i, x_j, and $\mathrm{d}x_k$, and where Θ is a polynomial having p independent coefficients. Just like in Riemann, this writing was associated with the fact that p is the maximal number of independent integrals of the first kind, a system of which being denoted by $u_1, \ldots, u_p$.

On the other hand, contrary to Riemann, Clebsch did not deal with Jacobi's inversion problem. One of his main results is what he presented as a consequence of Abel's addition theorem:[19] mn points $x_1, \ldots, x_{mn}$ of the curve $f = 0$ are the intersection points of this curve with a curve of order m if and only if

$$\begin{cases} u_1(x_1) + \cdots + u_1(x_{mn}) \equiv 0 \\ \quad \vdots \qquad\qquad\quad \vdots \\ u_p(x_1) + \cdots + u_p(x_{mn}) \equiv 0, \end{cases}$$

modulo the periods of the Abelian functions. As can be seen, p occurs here as the number of equations, which corresponds to the number of integrals of the first kind.

[17] For more details on Clebsch's research on this point, see [Lê 2020, pp. 79–84].

[18] This expression of w had been proved by Poncelet in the 1820s. We will encounter the definition of this concept of class in the next section. In the case studied by Clebsch, its validity comes from Riemann's supplementary condition mentioned above, which amounts to the fact that the curve $f = 0$ has only nodes as singularities.

[19] The reference given by Clebsch was [Abel 1829], and not [Abel 1826/1841] or [Abel 1828]. The cited paper contains what Steven L. Kleiman [2004] called "Abel's elementary function theorem."

We finally arrive at Clebsch's publications of 1865 where the genera of curves are defined, [Clebsch 1865b,a]. The main reference for this definition was the memoir on the application of Abelian functions to geometry, a reference that was used above all for the formula $p = \frac{(n-1)(n-2)}{2} - d$. This formula was taken as such, without being reworked, and served to gather algebraic curves into genera, as has been seen in our introduction. Riemann's memoir was also cited in [Clebsch 1865a], when Clebsch presented the property of p being invariant under birational transformations of a curve as being "another clothing of Riemann's theorem."[20] This invariance was used to derive results on algebraic curves, which I will not report here.

2.3 … of a global picture

In the narrative that has been written, the number p, or some of its earlier versions, can be recognized at each step, whether in the very writing of the hyperelliptic integrals, the addition theorem or the inversion problem. However, the process of identifying of these versions and making links between them takes different forms according to the cases. In the passage from Riemann to Clebsch, the comparison is made quite easy by a range of features: Clebsch adopted Riemann's same notations for p, $F(s, z)$, w, etc., cited him explicitly for the formula for p, and worked at the same degree of generality for Abelian functions, in the sense that the grounding equation $F = 0$ was given by a polynomial of any form and any degree. On the contrary, no such elements are available to connect explicitly Riemann with his predecessors, or to connect some of them with one another. In particular, as mentioned, the fact that the numbers $\mu - \alpha$, $m - 1$, ρ, and p do not bear any name in the natural language contributes to the difficulty of the situation.

Hence it is the responsibility of the historian to identify Riemann's p and Abel's $\mu - \alpha$, for instance. In general, such a process is often performed via the understanding of the mathematical content, and it may be more or less immediate – and it can be done more or less explicitly in the final historical outcome. This is where features such as citations between texts, even if vague or not directly linked to the investigated object, are useful safeguards. It is with this in mind that I warned against the hazard of recognizing the number 1 as a particular value of p in texts that would deal with elliptic functions all the while being disconnected from sources where p or its earlier versions appear explicitly.[21] In any case,

These observations thus provide some perspective on the actual circulation of the number under scrutiny in our selection of texts, even though these texts form a continuous chain of citations. The above analysis also shows that these

[20] "In der That ist dieser Satz nur eine andere Einkleidung desjenigen, welchen Herr Riemann dieses Journal Band 54 pag. 133 gegeben hat." [Clebsch 1865a, p. 98].

[21] The case of $p = 0$, which is associated with rational (or trigonometric) functions, would be even more problematic because of the greater number of possible concerned texts.

citations may incarnate a spectrum of positions, from the plain adoption to the claim of breaks in approach, by way of ambivalent attitudes, with one author refusing a part of a work while still borrowing results. Moreover, the addition theorem and the inversion problem are examples of specific questions that are important to take into account to understand the evolution of p but have their own dynamics in terms of first occurrences, proofs, uses, and reformulations. The succession of the chosen texts, finally, does not reflect any process such as a progressive rise in generality or in rigor.

All this contributes to drawing a bigger picture into which the episode of the definition of the genera of curves fits, and which must be thought of with the necessary nuances and caution. In particular, even if having extracted a sequence of texts could have first been interpreted as a way to provide an easy narrative, it is not the case. To employ a mathematical metaphor, the situation at hand is locally linear (and has been locally depicted), but globally much more complex.

Of course, an even bigger picture could be obtained by taking into account elements that have been ruled out above, such as entire texts, or topics that are connected to p and are tackled in the publications considered above, as illustrates Riemann's counting of $3p - 3$ constants on which depends each birational class of equations $F(s, z) = 0$ for which $p > 1$. Such avenues, which will not be explored here, illustrate the richness of the situation and, thus, the difficulties to which one is confronted when searching for a global description.

3. Tracking a name

I now turn to the second path, associated with the name "genus" and with algebraic curves.

3.1 Formation of the corpus

Let me first recall that none of the references of Clebsch's 1865 papers introducing the genera of curves deal with a notion bearing the same name. This is why the angle of attack must be changed, for instance by starting from a large corpus of texts related to algebraic curves and then searching in it what can be relevant for the purpose. This corpus is constructed in the following way.

Because the introduction of algebra in geometry made by René Descartes and Pierre Fermat changed to a great extent the ways to deal with curves, I chose the year 1637, when *La Géométrie* has been published and the *Isagoge* has been written, as the lower bound of my time interval. The corpus thus extends approximately over two centuries and a half, from 1637 to 1865.

To collect publications on algebraic curves, I first used the *Catalogue of scientific papers*, which allows to survey the period 1800–1865. Specifically, the *Catalogue* contains a section devoted to "Algebraic Curves and Surfaces of degree higher than the second," and I considered all the papers referenced to in the subsections on generalities and on plane curves, which represents 368 articles. For earlier publications, I drew, on one hand, upon Jeremias David Reuss' *Repertorium commentationum*, of which the chapter devoted to mathematics contains sections on algebraic curves: 74 articles, published between 1694 and 1795, are thus retained. To complete the corpus, I eventually gathered all the primary references listed in Carl B. Boyer's *History of analytic geometry* and dated between 1637 and 1799, which adds 80 new references.[22]

Then I investigated these 522 texts, searching in particular for any notion called *Geschlecht*, which is the German original word employed by Clebsch. In order not to restrict myself to German publications, words used at different times as explicit foreign equivalents of *Geschlecht* have also been taken into account: the Latin *genus*, the French *genre*, the Italian *genere*, the English "genus," as well as a few other ones.[23]

The examination of these words reveals two important points. First, they all refer to categories of curves, and there are mainly four different notions of genera that have successively been proposed, due to Descartes (1637), Newton (1704), Euler (1748) and Cramer (1750), before that of Clebsch was defined in 1865.[24] Second, the context of using genera of curves is always that of curve classifications, and, except for the case of Descartes, they are accompanied by considerations of other taxons of curves, such as orders, classes, and species. Although special attention is paid to genera, to include these categories in the study thus helps build a more coherent narrative and understand phenomena that concern genera. I will elaborate on this issue after describing the most salient points of this narrative.[25]

3.2 Classifications of curves

The first genera that can be found in our corpus are defined by René Descartes in *La Géométrie*, [Descartes 1637]. In the second book of this text, Descartes

[22]A historical work not reduced to being a topical bibliography, Boyer's book, [Boyer 1956], is different in nature than the *Catalogue* and Reuss' *Repertorium.* The main reason that I chose to rely on it is that Boyer used bibliographies made by mathematicians of the past to which I could not access. Of course, my narrative has been constructed independently from Boyer's, who, besides, dealt with a different historical issue.

[23]Let me recall that the investigation follows here a nominal way. Thus the question is not to locate concepts which would correspond mathematically to genera but would have a completely unrelated name. That said, it turns out that the corpus does not bear any trace of such phenomena.

[24]A few other technical or semi-technical meanings of the genus of curves can be found, and will be accounted for below. Non-technical meanings are not retained as relevant for the present study: for instance, the French *genre* is used in phrases such as "*ce genre de considérations*," which can be translated by "this kind of considerations," or "this type of considerations."

[25]A similar study is proposed in [Lê 2023], although the corpus is a bit different and no special emphasis is made on the genera.

proposed to "distinguish [curves] in order in certain genera"[26] by using the equations defining them. More precisely, if $n > 1$, the nth genus encompassed the curves whose equation is of degree $2n - 1$ or $2n$. As for the first genus, it consisted only of curves with an equation of the second degree because Descartes did not recognize straight lines as curves – significantly, he called the latter "curved lines."

Descartes justified this way of classifying curves only by alluding to the fact that the "difficulties" of the fourth degree can be "reduced" to the third degree, and that those of the sixth degree can similarly be reduced to the fifth degree.[27] This justification referred to the possibility of reducing equations with one unknown of degree 4 to equations of degree 3, a possibility that Descartes apparently believed to be true also for higher degrees and for equations with two unknowns.[28]

The classification proposed by Descartes was received in different ways in the 17th century. Fermat, for one, criticized it in his *Dissertatio tripartita* and preferred to classify curves degree by degree, calling "species" the resulting families of curves.[29] Another mathematician who adopted such a negative position decades later is Jacob Bernoulli [1695]. On the other hand, Descartes' genera were fully integrated in the works of other mathematicians, such as Frans van Schooten [1657] and Jacques Ozanam [1687].

In the corpus under scrutiny, only one 17th-century reference contains a notion of genus of curves that is not related to Descartes'. This is a book by John Craig, where genera are used to divide the totality of curves, be they algebraic or not. Specifically, algebraic curves themselves form the first genus, while transcendent curves are classified into the other genera according to further criteria, [Craig 1693, p. 42]. Such a definition, however, does not seem to have been taken up by later mathematicians.

This contrasts with the notion introduced by Isaac Newton in his famous *Enumeratio linearum tertii ordinis*, published in 1704 as an appendix to the treatise *Opticks*, [Newton 1704].[30] At the very beginning of this text, Newton asserted that lines can be divided into orders "according to the dimensions of the equation expressing the relation between absciss and ordinate, or, which is the same thing, according to the number of points in which they can be cut by

[26] "[D]istinguer [les lignes courbes] par ordre en certains genres" [Descartes 1637, p. 319]. On *La Géométrie*, see [Bos 2001; Serfati 2005; Herreman 2012, 2016].

[27] "[I]l y a règle générale pour réduire au cube toutes les difficultés qui vont au quarré de quarré, & au sursolide toutes celles qui vont au quarré de cube" [Descartes 1637, p. 323].

[28] See for instance [Bos 2001, p. 356]. Another interpretation of Descartes' grouping of curves is given in this reference, and is linked with the issue of constructing curves associated with the problem of Pappus and with the Cartesian classification of geometrical problems.

[29] The *Dissertatio* is edited in [Fermat 1891, pp. 118–132]. According to [Mahoney 1999, p. 130], it has probably written at the beginning of the 1640s. See also a 1657 letter from Fermat to Kenelm Digby where the arguments of the *Dissertatio* are taken up and made more explicit, [Fermat 1999, pp. 491–497].

[30] On the *Enumeratio*, see [Guicciardini 2009, pp. 109–136].

a straight line."[31] The nth order was thus made up of the lines defined by an equation of degree n. Furthermore, echoing the distinction between lines and curves, Newton proposed a parallel classification of curves, the corresponding categories being genera. More precisely, the nth genus of curves was that of the curves defined by an equation of degree $n + 1$, so that "[a] curve of the second genus is the same as a line of the third order."[32]

The title of his work made clear that Newton's aim was to classify the lines of the third order. In fact, once past this title and the very first lines, Newton mainly used the vocabulary of curves in the text, where he divided the curves of the second genus in 72 species. Such a terminology thus refers to the usual articulation between genera and species, the orders not being common categories of classification at the time.[33]

Many publications from the first half of the 18th century used the same classifying vocabulary. They often involved both lines and curves, even though a certain prevalence of the former can be observed. For instance, James Stirling's *Lineae Tertii Ordinis Neutonianae* only dealt with lines and their orders, [Stirling 1717], while François Nicole also mentioned curves and their genera in his *Traité des lignes du troisième ordre, ou des courbes du second genre*, [Nicole 1731]. Such works were direct continuations of Newton's *Enumeratio*, but orders and genera appeared in other type of publications. One of them is a 1705 book by Nicolas Guisnée, entitled *Application de l'algèbre à la géométrie*, where the double classification was adopted although Newton was not referred to, [Guisnée 1705]. Other examples are Edmund Stone's *New Mathematical Dictionary*, [Stone 1726], and Colin Maclaurin's *Treatise of algebra*, [MacLaurin 1748], where the English words used to refer to what Newton designated as genera are "genders" and "kinds," respectively. Maria Gaetana Agnesi [1748], for her part, also spoke both about orders of lines and genera of curves, the Italian words being *ordine* and *genere*. But she also made use of the latter term in a less technical sense, like when she described the curve defined by $a^{m-1}x = y^m$ as a "curve of the genus of the parabolas," a phrase which recalls the taxonomic connotation of the word.[34]

No publication, however, used the vocabulary of curves and genera without involving that of lines and orders. Moreover, in many cases the terminology of curves appeared only in the titles of the papers or of their sections, or were involved in a reduced way in the statement of theorems and their proofs, to the benefit of lines and orders.[35] Lines and orders are therefore the objects

[31] "Lineae Geometricae secundum numerum dimensionum aequationis qua relatio inter Ordinatas & Abscissas definitur, vel (quod perinde est) secundum numerum punctorum in quibus a linea recta secari possunt, optimè distinguuntur in Ordines." [Newton 1704, p. 139]. The given English translation comes from [Talbot 1860].

[32] "Curva secundi generis eadem cum Linea Ordinis tertii." [Newton 1704, p. 139].

[33] On this point, see [Lê 2023, pp. 96–100].

[34] "$a^{m-1}x = y^m$, curva del genere delle parabole." [Agnesi 1748, p. 940].

[35] This is thus the exact opposite to Newton's case.

that were mainly used in practice during in the first half of the 18th century.

The coexistence of lines and curves was explicitly abolished in two major books published in the middle of that century. One of them is Leonhard Euler's *Introductio ad analysin infinitorum*, whose second volume contained chapters devoted to the theory of algebraic curves [Euler 1748]. Euler first defined the orders of "curved lines" via the degree of the equations. Further, to encompass the case of straight lines, he explained that since it would be inappropriate to qualify the latter as curves, he would only speak about "lines" to refer to both cases, [Euler 1748, p. 26].

Genera in the Newtonian sense thus disappeared together with the distinction between lines and curves, but Euler did introduce genera to divide lines of a given order. For instance, in the chapter devoted to the lines of the third order, Euler first explained that they can be classified into species, according to the number and the nature of their infinite branches. After having brought out 16 such species, he emphasized that they are not the same as Newton's 72 species, and he showed how these 72 can be distributed into the 16. At the end, he added:

> Most of these species are so extensive that they each include quite considerable varieties, if we consider the shape they present in a finite space. It is for this reason that Newton multiplied the number of species, in order to distinguish one from another the curves that offer notable differences in this space. It will therefore be more appropriate to call *Genera* what we have designated as *Species*, and to refer to *Species* the varieties they contain.[36] [Euler 1748, p. 126]

In other words, Euler adjusted the terminology to fit with the great number of categories he had to deal with, and what he had first called "species" were renamed "genera."

A similar situation occurred in Gabriel Cramer's *Introduction à l'analyse des lignes courbes algébriques*, [Cramer 1750].[37] In the preamble of the book, Cramer recalled that "Algebra alone provides the means to distribute Curves into Orders, Classes, Genera & Species" and that "it is to the illustrious Newton that Geometry is most indebted for this distribution."[38] Later, Cramer defined the orders of lines through the degree of their defining equation. He then explicitly recalled the old distinction between lines and curves:

[36] "Species autem hae plerumque tam late patent, ut sub unaquaque varietates fatis notabiles contineantur; si quidem ad formam, quam Curvae habent in spatio finito, respiciamus. Hancque ob causam Newtonus numerum specierum multiplicavit, ut eas Curvas, quae in spatio finito notabiliter discrepant, a se invicem secerneret. Expediet ergo has, quas *Species* nominavimus, *Genera* appellare, atque varietates, quae sub unoquoque deprehendantur, ad *Species* referre."

[37] On this book, see [Joffredo 2017, 2019].

[38] "[L]'Algèbre seule fournit le moyen de distribuer les Courbes en Ordres, Classes, Genres & Espèces" and "c'est à l'illustre Newton que la Géométrie est surtout redevable de cette distribution" [Cramer 1750, p. VIII]

> Mr. Newton distinguishes between the Orders of Lines & the Genera of Curves. Since the first Order contains only the straight Line [...], he calls Curves of the first Genus the Lines of the second Order, Curves of the second Genus the Lines of the third Order, & so on. However reluctant one may be to deviate from the denominations established by this Great Man, it seemed to me that this expression was too cumbersome in terms of expression, & I decided to say indifferently Curves or Lines of the second Order, Curves or Lines of the third Order, &c.[39] [Cramer 1750, p. 53]

Just like in Euler, the notion of genus that Newton had defined thus vanished with the disappearance of the difference between curves and lines. Although neither of these authors explained why they favored lines and orders, it is most likely that these objects survived because their enumeration corresponds exactly to the degrees of the equations.

Genera of lines can still be found in Cramer's text, as subcategories of each order that reflect properties related to the infinite branches. In particular, Cramer classified curves of the third order into 14 genera, and he systematically indicated their correspondence to Newton's species. For instance, the curves with three concurrent asymptotic lines were gathered in one genus, which "contains the nine species of redundant hyperbolas whose asymptotes intersect at one point. Newton, Nb. 4."[40]

Comments on the appropriate classification scale appeared when Cramer treated the case of curves of the fourth order. Imitating the approach he used in the third-order case, Cramer was first led to consider eight cases of equations of curves. The three first cases yielded 1, 6, and 9 genera, respectively. But when Cramer treated the next case, he indicated that the number of genera was too great to be completely listed: "It would be impossible to enumerate all the genera of curves included in this IVth Case: but they can be reduced to five Classes."[41] In other words, since the genera became too numerous to be enumerated exhaustively, Cramer decided to introduce a new type of curve categories, situated between the fourth order and its genera.

Strikingly, no technical use of the word "genus" can be found in the texts of our corpus between Euler's and Cramer's books and 1833, when Euler's genera were mentioned again. On the contrary, the orders of lines (or curves,

[39]"Mr. Newton distingue les Ordres des Lignes & les Genres des Courbes. Comme le premier Ordre ne renferme que la Ligne droite [...], il appelle Courbes du premier Genre, les Lignes du second Ordre, Courbes du second Genre, les Lignes du troisième Ordre, & ainsi de suite. Quelque répugnance qu'on ait à s'écarter des dénominations établies par ce Grand Homme, il m'a paru que cette distinction génoit trop l'expression, & je me suis déterminé à dire indifféremment, Courbes ou Lignes du second Ordre, Courbes ou Lignes du troisième Ordre, &c.".

[40]"Ce Genre contient les neuf espèces d'Hyperboles redondantes dont les trois asymptotes se croisent en un point. Newton, N.º 4." [Cramer 1750, p. 362].

[41]"On ne saurait énumérer tous les genres des courbes comprises dans ce IVe Cas : mais on peut les réduire à cinq Classes." [Cramer 1750, p. 379].

the difference being abandoned) were absolutely commonplace in the texts that have been published during this period. Their use engendered many comments at the occasion of a specific episode, during which new "classes" of curves were introduced.

This episode is the famous duality controversy, which, at then end of the 1820s, opposed Joseph-Diez Gergonne and Jean-Victor Poncelet.[42] Let me briefly recall that Gergonne's principle consisted in associating a dual theorem with any given theorem, the two being related by the exchange of the words "points" and "lines," and of associated verbs and adjectives. However, Gergonne made the mistake to do as if curves of order $n > 1$ were replaced by curves of the same order in this process: this is one of the points that Poncelet used against him. When he reworked his theory, Gergonne [1827/1828] introduced the notion of "class," a curve being of the nth class if n tangents can be drawn to it from a given point of the plane. The principle of duality could then be corrected by making curves of the nth class correspond to curves of the nth order.

Classification issues were part and parcel of Gergonne's and Poncelet's arguments. On one hand, Gergonne emphasized that orders and classes provided two dual ways of classifying curves which were dual to one another. On the other hand, Poncelet blamed Gergonne for having "admitted simultaneously two essentially different classifications for curves," and he insisted that he himself did not "shy away from the difficulty of preserving [to the classification] of curves [its] legitimate and universally [accepted] definition."[43] If Poncelet thus refused the classification itself, he was still interested in evaluating the number of tangents that can be drawn to a curve, and he found that this number is $n(n-1)-d$ if the curve is of order n and has only d nodes as singularities.

In spite of Poncelet's hostility, the notion of class defined by Gergonne was adopted quite quickly by the 19th-century geometers. In the corpus, the first one to use it is Julius Plücker, who published many papers on algebraic curves during the 1830s,[44] including those where he completed Poncelet's formula for the class-number of a curve.

Genera are mentioned in one of these papers, which is a presentation of the book *System der analytischen Geometrie*, [Plücker 1833, 1835]. As this presentation explained, one of the aims of the book was to rework the classification of the curves of the third order. Plücker rooted the question in

[42] On this controversy, see [Lorenat 2015] and the references given on p. 547, as well as [Etwein, Voelke, and Volkert 2019].

[43] "[I]l ne s'est agi de rien moins que de torturer le sens des mots, en admettant simultanément deux classifications essentiellement distinctes pour [les] courbes." Further: "je n'ai pas reculé devant la difficulté de conserver aux classifications des courbes [...] leur définition légitime et universellement admise." [Poncelet 1828, pp. 300, 302].

[44] The adoption of the notion of class by Plücker and other mathematicians is described in [Lê 2023, pp. 110–116].

Newton's *Enumeratio* and declared that Euler, though having made progress in the theory of infinite branches, did not succeed to settle down the issue. The genera that Euler had defined were thus briefly evoked in this discussion – the presentation paper, written in French, used the word *genres*, while the book itself referred to these categories by the term *Geschlechter*. However, no genera were involved in Plücker's own classification, which consisted in dividing the third order into species only.

Only a very few occurrences of genera occur in the corpus between 1833 and 1865. They either correspond to concepts defined by past mathematicians, or concepts that are used in a less technical way, in the image of Agnesi's example given above. For instance, Joseph Dienger devoted a paper published in 1847 to a "curve deriving from the ellipse, of the genus Conchoid"[45] and Ernest De Jonquières, in an article of 1861, was led to consider a "curve of the fourth order, of the genus of the *lemniscates*."[46] Finally, a paper published in 1863 by a certain F. Lucas also employed the term "genus" as he referred to Newton's works on the classification of third-order curves, [Lucas 1863].

Such a paucity shows that when Clebsch defined the *Geschlechter* in 1865, the word was, so to speak, available to be endowed with a new technical definition. Moreover, we saw that when Clebsch did so, the idea of introducing a new classification of curves was clear. A notable difference between his *Geschlechter* and the other notions of genus that we encountered is that these genera were not presented as subcategories of orders, quite the opposite. The idea was to divide the totality of algebraic curves into genera, each of them being then subdivided according to the orders. This classification program was thus different from Gergonne's, for whom the distribution in classes was not supposed to replace that in orders, but to offer another way to conceive the division of curves.

3.3 Continuities and discontinuities

If the episode of Clebsch defining the genera is thus embedded into a narrative that extends over almost two centuries and a half, it is true that there is no direct, effective link made by Clebsch between this episode and the previous ones – his mention of the classification by orders appears more as a general mathematical fact than a historical reference *per se*.

More generally, the genera proposed by Descartes, Netwon, Euler, Cramer, and Clebsch, have no direct mathematical link to one another. They cannot be seen as different instances of one and the same concept which would correspond to different degrees of generality of the associated algebraic curves, for example. The situation is thus distinct from that of the numbers p, ρ and $m-1$, the

[45]"Note sur une Courbe dérivant d'une ellipse (du genre Conchoïde)", [Dienger 1847, p. 234]. The words that are here between parentheses are in a footnote in the original paper.

[46]"[La courbe...] est une courbe du quatrième ordre, du genre des *lemniscates*." [Jonquières 1861, p. 211].

first becoming equal to the other two if the framework of Abelian functions in which it is defined is particularized. Nor are they concepts, each of which would have been developed from the previous one following a discussion of its relevance and an attempt to replace it. Significantly, if Cramer did refer explicitly to Newton to explain why the latter's genera should be abandoned, his own genera referred to something else, and the bridge with Newton was made on the level of species.

From this point of view, isolating the works of the five above-mentioned mathematicians, focusing on their definition of genera solely, and juxtaposing their description would yield quite a discontinuous result, both in terms of chronology and intellectual dynamics. Greater coherence is achieved by taking into account both other works that contain a notion of genus (be it a simple adoption of Descartes' notion, for example, or a semi-technical one) and works involving other categories of curves, namely orders, classes, and species. In doing so, the historian weaves a more complete and continuous historical fabric related to the classifications of curves, in which the episodes of the definitions of the genera can be advantageously included.

As in the previous section, this bigger picture could obviously made more complex. For instance, most of the texts that have been kept silent above do not contain any new notion of genera, orders, classes, or species, but they could be used to get a view on the range of mathematical questions in which these categories have been involved, or to analyze the process of their banalization. And questions revolving around the tension between the categories of curves and the numbers that characterize them, or the mathematical links between these numbers, could also be investigated further.

4. Bigger pictures

The two pictures obtained by following the paths of the number p and of the name "genus" are essentially disjointed in terms of content, which reflects the historical divide between the theory of Abelian functions and the theory of algebraic curves. Most mathematicians, indeed, only contribute to one of these pictures, the three exceptions being Clebsch, Jacobi, and Siegfried Aronhold, among whom only the first has papers that belong to both corpora. The mathematical questions, and the techniques that are deployed to address them, are also proper to both pictures, up these three mathematicians' cases – Clebsch's interpretation of the number of ramification points as the class of a curve nicely illustrates this point.[47]

The two narratives have also their own timelines and their own pace, with one extending over two centuries and a half while the other is mainly

[47] Aronhold and Jacobi mix together elliptic functions and algebraic curves in papers that belong to the corpus on Abelian functions, [Jacobi 1828; Aronhold 1862]. Their works are analyzed and compared to Clebsch's approach, [Clebsch 1864a], in [Lê 2018].

concentrated in forty years. As explained in the introduction, this difference does not merely stems from a convention of corpus formation. It reflects an asymmetry in the process of identifying earlier versions of mathematical objects and in the limits to which the historian is confronted when doing so. In particular, by providing an apparent greater stability, the nominal path seems to allow a more straightforward writing of a long-term history. It must be remembered, however, that this stability is not based on the sameness[48] of an object but on the common framework of classifying algebraic curves, a framework that is general enough to give rise to different ways of approaching it, and to answers having little in common apart from the names of the considered categories of curves. This eventually echoes the differences between the intellectual dynamics that underlie the succession of the diverse concepts of genera and that of the numbers $\mu - \alpha$, $m - 1$, ρ, and p: contrary to the former, the latter can be seen as being the same, provided the technical frameworks with which they are associated are adjusted to one another.

As disconnected as the two pictures that have been obtained can be, both have their own kind of coherence, and there can be no question of raking them, by asserting that one would be more relevant or more significant than the other. Quite simply, they refer to two types of connection of a newly defined object with its past, and it is important to be clear on this point in the historical account.

This takes us eventually back to the issue of drawing one picture into which the episode of the 1865 definition of the genera can be embedded, a picture which would correspond to the reconstruction of the past of these genera.

Considering all this, I think it would be misguided to aim at writing a final result consisting in one unified narrative, well-ordered chronologically and integrating all the above elements. In fact, what has been proposed in this chapter corresponds better to what should be done to draw a bigger picture, in my view: the construction of corpora on the basis of explicit criteria, their systematic study in view of determined objectives, the comparison of the obtained results, and a reflection on the possibility of merging them or not. In particular, I do not see what has been produced as a mere juxtaposition of two narratives, even though these narratives are related to quite autonomous historical situations.

Of course, the result is still a partial picture, since other entire tracks could be followed and yield other pictures with their own content, dynamics, and chronologies. For instance, one could try tracing back the topological interpretation of the number p as the half of the number of sections required to disconnect a surface, or researching the past of the number $\frac{(n-1)(n-2)}{2}$, which would lead to the path of questions on curve singularities.

But my point is that big pictures are not meant to stay out of range of the historian of mathematics, as long as sound methodological groundings underpin

[48] I borrow this term from [Goldstein 2019].

their drawing and significant details are still taken into account to delineate clearly the specificity of each author, of each text. I have tried to illustrate this in the case of the reconstruction of the past of a mathematical object. Considering other objects, searching for their future or trying to thicken the comprehension of the episode of their definition in synchronicity rather than in diachronicity would probably require to address other questions. In this respect, this chapter presents itself not so much as a way of settling the question by inferring general laws, but rather as a case study aimed at stimulating historiographical and methodological reflection.

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