The "Geometrical Equations:" Forgotten Premises of Felix Klein's *Erlanger Programm*

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Abstract

Felix Klein's *Erlanger Programm* (1872) has been extensively studied by historians. If the early geometrical works in Klein's career are now well-known, his links to the theory of algebraic equations before 1872 remain only evoked in the historiography. The aim of this paper is precisely to study this algebraic background, centered around particular equations arising from geometry, and participating on the elaboration of the *Erlanger Programm*. Another result of the investigation is to complete the historiography of the theory of general algebraic equations, in which those "geometrical equations" do not appear.

1 Introduction

Felix Klein's *Erlanger Programm*[1] is one of those famous mathematical works that are often invoked by current mathematicians, most of the time in a condensed and modernized form: doing geometry comes down to studying (transformation) group actions on various sets, while taking particular care of invariant objects.[2] As rough and anachronistic this description may be, it does represent Klein's motto to praise the importance of transformation groups and their invariants to unify and classify geometries.

The *Erlanger Programm* has been abundantly studied by historians.[3] In particular, [Hawkins 1984] questioned the commonly believed revolutionary character of the *Programm*, giving evidence of both a relative ignorance of it by mathematicians from its date of publication (1872) to 1890 and the importance of Sophus Lie's ideas[4] about continuous transformations for its elaboration. These ideas are only a part of the many facets of the *Programm*'s geometrical background of which other parts (non-Euclidean geometries, line geometry) have been studied in detail in [Rowe 1989b].

Beside this geometrical background, the historiography always mentions Camille Jordan, sometimes for his research on movement groups, more often for his *Traité des substitutions et des équations algébriques*:[5] Klein had been inspired by the *Traité*'s presentation of the equation

[1][Klein 1872], translated in English in [Klein 1893] and in French in [Klein 1974].

[2]However, this part about invariant objects is often neglected, if ever mentioned. [Perrin 2002] points out the importance of invariants in the *Erlanger Programm* and suggests a reinforcement of the incorporation of geometrical invariants in the French teaching of mathematics.

[3]Beside the references given above and below, see [Gray 1992; Gray 2005; Rowe 1983; Rowe 1985; Rowe 1992].

[4]About Lie's early works, see [Rowe 1989b] and the first chapter of [Hawkins 2000].

[5][Jordan 1870b].

theory within a group-theoretic framework and sought an analogous transformation group framework for geometry.[6] Whereas the historiography stops there on this algebraic point, this paper will investigate deeper into the connections between Klein and the theory of algebraic equations in the period preceding the publication of the *Erlanger Programm*. We will argue that Jordan's theory of substitutions as expressed in his *Traité des substitutions et des équations algébriques* is actually one part of a broader algebraic background, the one of the "geometrical equations"[7] The purpose of this investigation is obviously not to dismiss the historical works previously cited but rather to help give a more complete and accurate picture of Klein's pre-*Programm* mathematical background. Another goal of this investigation will be to broaden the historiography of the general theory of algebraic equations in which a discussion of the role of the "geometrical equations" has not yet appeared.

For the moment, let us go back to the *Erlanger Programm* itself and look for traces of equation theory in it. One obvious and explicit trace can be found in the concluding remarks: they confirm Klein's desire for an analogy between geometry and theory of equations we mentioned above:

> The further problems which we wished to mention arise on comparing the views here set forth with the so-called *Galois* theory of equations. [...]
>
> In the Galois theory, as it is presented for instance in *Serret's* "Cours d'Algèbre Supérieure" or in *C. Jordan's* "Traité des Substitutions," the real subject of investigation is the group theory or substitution theory itself, from which the theory of equations results as an application. Similarly we require a *theory of transformations*, a theory of the groups producible by transformations of any given characteristics. [Klein 1893, pp. 241-242]

Besides this explicit reference, other hints to equation theory appear in the *Programm*. One of them is a note in which Klein indicates that the terminology "groups of transformation" and its very concept come from the theory of substitutions.[8] Another is the idea to distinguish an element of space and to consider the transformations leaving it unaltered, which is to be linked to the notion of adjunction of an irrationality to an equation and its effect on its Galois group.[9]

Now, if we want to look for Klein's works on algebraic equations, we can use his *Gesammelte mathematische Abhandlungen*. One can find a section called "Substitutionengruppen und Gleichungstheorie"—disctinct from the section devoted to the *Erlanger Programm*[10]—which contains only one article dated before 1872, year of the publication of the *Programm*. This article,

[6]For the mentions to movement groups, see [Rowe 1989b, p. 211] or Jean Dieudonné in the preface of the French translation of the *Programm*, [Klein 1974, p. x]. For the *Traité des substitutions et des équations algébriques*, see for instance [Hawkins 1984, p. 444] or [Birkhoff and Bennett 1988, p. 151]. Eventually, see also [Rowe 1989b, p. 211] where David Rowe discusses the possibility that Klein and Lie actually learned substitution theory from Jordan and his *Traité* while they were in Paris in 1870.

[7]See the end of this introduction for explanations about that expression.

[8]"Begriffsbildung wie Bezeichnung sind herübergenommen von der Substitutionentheorie", [Klein 1872, p. 5]. In [Klein 1893, p. 217], "Bezeichnung" is translated into "notation."

[9][Klein 1893, p. 218]. In German, Klein uses the verb "hinzufügen" (to add) for the process of distinguishing a given element of space, but he talks about the "Adjunction der Hauptgruppe invarianten Eigenschaften" (adjunction of invariant properties to the principal group), [Klein 1872, p. 8].

[10]The section "Substitutionengruppen und Gleichungstheorie" is in the second volume of the *Gesammelte mathematische Abhandlungen*, [Klein *Œuvres 2*]. In the preface of the first volume [Klein *Œuvres 1*], the editors point out that Klein actively participated to the editing process of his works, and especially to the organization of them. The existence of the different sections, their title and their content are therefore strongly marked by Klein's own choices.

entitled *Ueber eine geometrische Repäsentation der Resolventen algebraischer Gleichungen*, begins as follows:

> The general theory of algebraic equations is illustrated in the most beautiful way by a number of particular geometrical examples; I just think (see Camille Jordan. Traité des Substitutions. Paris 1870, p. 301ff.) of the problem of the inflection points of a curve of order three, of the 28 double tangents of a curve of order four, of the 27 lines on the surfaces of third degree etc., but also in particular of the cyclotomy.[11] [Klein 1871, p. 346]

Jordan's theory of substitutions is cited again but here with a hint to particular equations arising from geometrical situations. These special equations are the objects we will deal with in the present paper; like some of the mathematicians who will appear later, we will call them "geometrical equations." Let us remark that the cyclotomy mentioned by Klein will actually not appear in the corpus of investigation (see below). That being said, we will come back to the proximity expressed here between the cyclotomy and the "geometrical equations" in the concluding remarks.[12]

Before turning to the formation of the corpus, we will give here some mathematical explanations about the "geometrical equations." While these explanations may not faithfully respect the different approaches of the mathematicians we shall study in a moment; however they will be useful to the reader to get the idea of the label "geometrical equations." For a given geometrical situation, for instance the nine inflection points of a cubic curve, the corresponding geometrical equation is the algebraic equation ruling the configuration. Thus the nine inflection points equation is the algebraic equation of degree 9, the roots of which are the abscissas of the inflection points. For other situations like the twenty-seven lines upon a cubic surface, one cannot parametrize the geometrical objects with just one parameter: four parameters are required for a line in space. The twenty-seven lines equation is the equation obtained by eliminating 3 parameters out of the 4 in the system of equations expressing that a line is entirely lying in the surface. What the reader can remember is that the geometrical equation associated to a configuration of n objects is an algebraic equation of degree n; its roots correspond to the objects of the configuration and the relations existing between the roots correspond to the incidence relations existing between the objects.

2 The geometrical equations: corpus

In order to form my investigation corpus, we used the section entitled "Geometrische Gleichungen" of the chapter of the *Encyklopädie der mathematischen Wissenschaften* devoted to the Galois theory and its applications.[13] We could have used the *Répertoire Bibliographique des Sciences*

[11] "Die allgemeine Theorie der algebraischen Gleichungen wird in schönster Weise durch eine Anzahl besonderer geometrischer Beispiele illustriert; ich erinnere nur (Vgl. Camille Jordan. Traité des Substitutions. Paris 1870, S. 301 ff.) an das Problem der Wendepunkte der Kurven dritter Ordnung, an die 28 Doppeltangenten der Kurven vierter Ordnung, an die 27 Linien auf den Flächen dritten Grades usw., dann aber namentlich auch an die Kreistheilung."

[12] The role of the cyclotomic equation as a model for the theory of algebraic equations has been studied by historians: see in particular [Neumann 2007] for its role in connection with Carl Friedrich Gauss's *Disquisitiones Arithemticae* and [Boucard 2011] for Louis Poinsot's views on it.

[13] [Hölder 1899, pp. 518-520].

Mathématiques,[14] and indeed its section devoted to the Galois theory and the equation theory contains a paragraph named "Applications to the theory of particular equations: equation of the inflection points of a cubic, of the 27 lines of a cubic surface; modular equations, etc." However, all the references given there and concerning the geometrical equations are already in the *Encyklopädie*, so we only considered this last work. Furthermore, no use could have been made of the *Jahrbuch über die Fortschritte der Mathematik* and the *Catalogue of scientific papers* because these classification journals do not contain any section devoted to the geometrical equations.

In order to form the corpus from the section "Geometrische Gleichungen" of the *Encyklopädie*, we first selected every reference that actually deals with algebraic equations. Indeed, some of the given references are only about geometrical results whereas the others are about the corresponding geometrical equation;[15] the latter are the ones that interest us for the present paper. Secondly, we repeated the same process (i.e. selecting the references dealing with geometrical equations) for every text obtained in the first step. From all the texts thus gathered, we finally removed those dated after 1872, the date of the *Erlanger Programm*.

The corpus obtained is made of 13 texts; one of them is a book, the others are research articles or notes to the French or the Berliner Academies. These texts, except for the earliest one written by Otto Hesse [Hesse 1847], belong to a quite close time period covering 1863, 1864 and then every year from 1868 to 1872. The two papers of 1863 and 1864 are Ernst Eduard Kummer's: [Kummer 1863; Kummer 1864]. Then we have two articles of Alfred Clebsch [Clebsch 1868; Clebsch 1871], Jordan's *Traité des substitutions et des équations algébriques* [Jordan 1870b], three short publications [Jordan 1869a; Jordan 1869b; Jordan 1869c] which are more or less extracts of the *Traité* that preceded its publication and an additional note [Jordan 1870a], two papers of Klein [Klein 1870; Klein 1871] of which the second is the one we talked about in the introduction, and finally, an article of Lie, [Lie 1872].

Let us remark that all of the authors of this corpus are German, except for Jordan[16] and Lie. Furthermore, strong links exists between them. Clebsch learned analytic geometry with Hesse during his study in Königsberg between 1850 and 1854, and Hesse's research on algebraic equations stimulated Clebsch's interest in this subject.[17] Clebsch was friends with Jordan and they exchanged mathematical ideas in particular about special algebraic equations:

> By the multiple relations he formed with Camille Jordan, [Clebsch's] attention had always been aimed to everything connected to curious groupings of roots of an equation. Reciprocally one has essentially to thank him if Camille Jordan had been able to devote a particular chapter of his great work (Traité des substitutions et des équations algébriques. [...]) to the "equations of the geometry."[18] [Brill *et al.* 1873, p. 47]

[14] The *Répertoire Bibliographique des Sciences Mathématiques* had been published first in 1894, but classified retrospectively mathematical papers dated as from 1800, and even 1600 for history and philosophy of science. See [Nabonnand and Rollet 2002].

[15] For instance, [Cayley 1849] and [Jordan 1870b] are both cited for the equation of the twenty-seven lines. The latter actually deals with it, whereas the former is the article commonly designated as the first publication mentioning and proving the existence of the twenty-seven lines on a cubic surface. For historical information about the twenty-seven lines, see [Lê 2011].

[16] For whatever it is worth, Jordan is qualified as "almost German" in [Klein 1979, p. 318].

[17] [Neumann 1872, p. 550] and [Brill *et al.* 1873, p. 47].

[18] "Später wurde [Clebsch's] Aufmerksamkeit durch die vielfachen Beziehungen, in die er mit Camille Jordan getreten war, immer wieder auf Allesn was mit merkwürdigen Gruppirungen von Wurzeln einer Gleichung im Zusammenhange steht, hingelenkt. Umgekehrt hat man es ihm hauptsächlich zu verdanken, wenn Camille Jordan

At the beginning of 1869, Klein met Clebsch in Göttingen; the latter would become Klein's "professor and paternal friend."[19] A few month later Klein went to Berlin, where his friendship relation with Lie began. He also met Kummer, who was professor in Berlin at that time but was disappointed by and even despised his attitude, along with Karl Weierstrass's and Leopold Kronecker's.[20] In summer 1870 Klein and Lie went to Paris. They met in particular Jordan and discovered the *Traité des subtitusion et des équations algébriques*, designated as a "book with seven seals" by Klein.[21] Let us finally add that Klein wrote an obituary of Hesse in 1875, which suggests that Klein knew Hesse's works quite well.

The different mathematicians of our corpus possessed thus numerous personal links. Moreover this impression of great coherence is reinforced by the inter-textual links existing between the texts of the corpus. These links can be seen in figure 1, where a continuous arrow is put if an explicit citation is given: for instance, [Jordan 1870b] refers to [Clebsch 1868]. The dashed arrows mean implicit references clear enough to guess what is the cited text: for instance, [Jordan 1870b, p. VI] mentions the "famous memoirs of Mr. Hesse on the inflection points of the curves of the third order." We also represented in the same block Jordan's *Traité des substitutions et des équations algébriques* and the three notes [Jordan 1869a; Jordan 1869b; Jordan 1869c] that just preceded its publication.

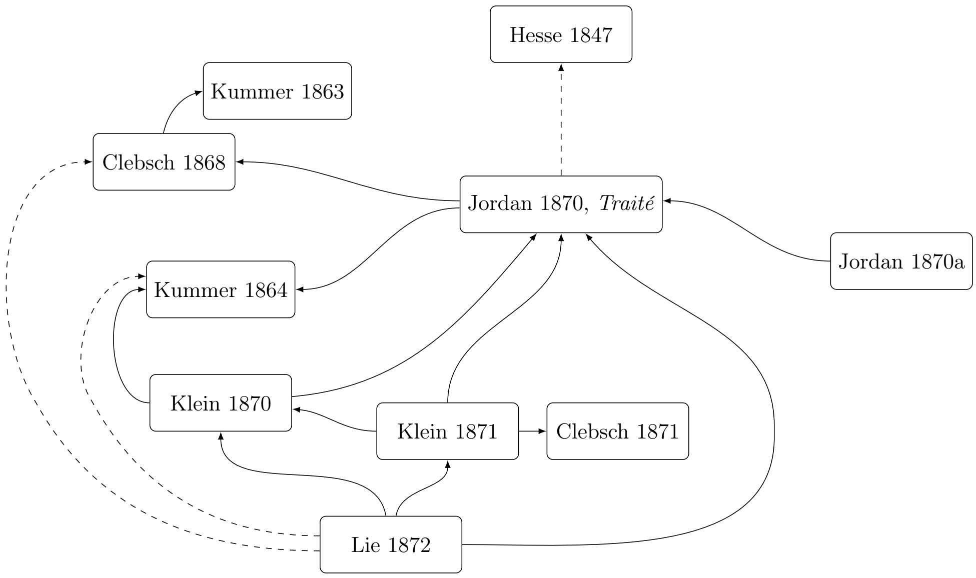

Figure 1: Graph of citations. The dashed lines are implicit references. The block "Jordan 1870, *Traité*" represents the *Traité des substitutions et des équations algébriques* and the three notes that preceded its publication.

The aim of the graph is not to make a proper investigation of the inter-textual links in our

im Stande war, in seinem grossen Werke (Traité des substitutions et des équations algébriques. Paris, Gauthier-Villars 1870) ein besonderes Capitel den 'Gleichungen der Geometrie' zu widmen."

[19] [Courant 1925, p. 199].

[20] [Rowe 2013, pp. 2-3].

[21] [Klein *Œuvres 1*, p. 51].

corpus, but rather to see how interconnected our texts are. In particular, the *Traité des substitutions et des équations algébriques* appears as a central work for the subject of the geometrical equations. Let us also notice that Klein is connected to every other protagonist, except for Hesse—however, Klein wrote Hesse's obituary and deals with [Hesse 1847] in it. These connections confirm the coherence of our corpus as an algebraic background for Klein in his pre-*Erlanger Programm* times. We now describe each of the texts in order to get an idea of this background.

3 Hesse and the nine inflection points: 1847

[Hesse 1847] deals with the solubility of certain algebraic equations of degree 9. More precisely, the equations at hand are those for which the roots are linked three by three by symmetrical relations: there exists a function θ such that for every couple of roots x_χ, x_λ, the quantity $x_\mu = \theta(x_\chi, x_\lambda)$ is another root of the equation and such that $x_\lambda = \theta(x_\mu, x_\chi)$ and $x_\chi = \theta(x_\lambda, x_\mu)$. In the introduction, Hesse places his article in the lineage of Niels Abel's works about the resolution of particular algebraic equations, recalling a conjecture that Abel gave in [Abel 1830]: if an equation is of prime degree and if its roots are linked three by three so that one of them can be rationally expressed by the two others, then the equation is solvable by radicals.[22]

Hesse also indicates in a footnote that the study of these special equations of degree 9 has been suggested by his former teacher Carl Gustav Jacob Jacobi after the latter had been acquainted with Hesse's research on cubic curves:

> I [Hesse] bring out the following extract of a letter of January 1844 written in Rome by Professor Jacobi to whom I communicated the first results of my research about the inflection points. "You could also probably solve the general problem: to solve an equation of degree nine, when a given rational and symmetric function $F(x, x')$ of every two roots x, x' always gives a third root x'' such that if $F(x, x') = x''$, then $F(x', x'') = x$ and $F(x'', x) = x'$. For this is the case here for the equations of the inflection points of a curve of the third order. So would be revealed a new class of solvable algebraic equations, totally different from those to which Abel applied the method of Gauss." With this hint, I undertook the present research on equations of the ninth degree.[23] [Hesse 1847, p. 202]

The mentioned research on the inflection points are probably those published in [Hesse 1844]. In this research, Hesse proves in particular that the inflection points of a curve of order n are the intersections of this curve with another of degree $3(n-2)$ which was later called the "Hessian"

[22] The formulation of this conjecture is quite typical of Abel and Galois: see for instance [Abel 1829], an another conjecture in [Abel 1830] and [Galois 1846, p. 395].

[23] "Aus einem vom Januar 1844 aus Rom datirten Schreiben des Herrn Professor Jacobi, dem ich die ersten Resultate meiner Untersuchung über die Wendepunkte mitgetheilt hatte, hebe ich folgende Stelle heraus. 'Sie werden wahrscheinlich auch das allgemeine Problem lösen können: eine Gleichung neunten Grades aufzulösen, wenn eine gegebene rationale symmetrische Function $F(x, x')$ je zweier Wurzeln x, x' immer wieder eine dritte Wurzel giebt, in der Art, dass wenn $F(x, x') = x''$, auch $F(x', x'') = x$, $F(x'', x) = x'$ ist. Denn dieses ist hier bei den Gleichungen der Wendepunkte der Curven dritter Ordnung der Fall. Es würde sich so eine neue Classe von auflösbaren algebraischen Gleichungen eröffnen, welche von denen, auf die Abel die Gauss'sche Methode ausgedehnt hat, total verschieden sind.' Auf diese Andeutung hin habe ich die vorliegende Untersuchung der Gleichung neunten Grades unternommen."

of the first curve.[24] In the particular case $n = 3$, he deduces the (already known) fact that every cubic curve has nine inflection points. Hesse goes further and proves that the twelve lines containing three by three the inflections points form four triangles, each of them containing all of the nine inflection points.[25]

We now describe Hesse's approach of the special equations of degree 9 which are the objects of [Hesse 1847]. Three roots x_χ, x_λ, x_μ are called "conjugated" if they are linked with the relations described above. Hesse then proves that there are exactly 12 triplets of conjugated roots and he represents them on a table:

$x_1x_2x_3$	$x_4x_5x_6$	$x_7x_8x_9$
$x_2x_4x_7$	$x_3x_5x_8$	$x_1x_6x_9$
$x_5x_7x_1$	$x_6x_8x_2$	$x_4x_9x_3$
$x_8x_1x_4$	$x_9x_2x_5$	$x_7x_3x_6$

He then forms functions $y_{\chi\lambda\mu} = (\alpha - x_\chi)(\alpha - x_\lambda)(\alpha - x_\mu)$ of a new unknown α, where x_χ, x_λ, x_μ are conjugated roots. The twelve functions thus obtained are again represented on a table:

y_{123}	y_{456}	y_{789}
y_{247}	y_{358}	y_{169}
y_{571}	y_{682}	y_{493}
y_{814}	y_{925}	y_{736}

Hesse carries on the process and introduces the functions $z = (\beta - y_{\chi\lambda\mu})(\beta - y_{\chi'\lambda'\mu'})(\beta - y_{\chi''\lambda''\mu''})$ of an other new unknown β, where the y's are taken from a line of the last table. This gives him four functions z_1, z_2, z_3, z_4 depending thus of an equation of degree 4, and he proves that the coefficients of this equation are rational functions of the coefficients of the original equation.

Since this last equation is of degree 4, it is solvable by radicals (with the coefficients of the original equation): Hesse indicates that one can find one of its roots, say z_1, as an entire function of β. Then the equation $z_1 = 0$ is of degree 3 in β and so its roots y_{123}, y_{456}, y_{789} can be found by radicals. Finally, the equation $y_{123} = 0$ is of degree 3 and is as such solvable by radicals. But its roots are x_1, x_2, x_3; and similarly, the remaining roots $x_4, \ldots, x_9$ can be found by radicals too.

Hesse then turns to the inflection points equation. He defines this equation as the result of the elimination of one variable between the equations of a cubic curve and of what it was later called its Hessian. In order to prove that the inflection point equation is of the kind of those he investigated previously, Hesse refers to a paper of Jean-Victor Poncelet, [Poncelet 1832], to recall that the nine points are lined up three by three. This geometrical property is used to prove that the roots of the inflection points equation satisfy relations like those described in the title of the paper: three roots are linked by the θ relations when the points they represent are lined up. Finally, since the inflection points equation belongs to the class investigated by Hesse, it is solvable by radicals.

Let us remark that Hesse has proceeded in two distinct steps. The first one is the algebraic investigation of a geometrically disembodied equation; the second one is the incarnation, *via* the alignment of the inflection points, of this equation into the nine points equation. It is interesting

[24] Let us recall that a first curve has an homogeneous equation $f(x_1, x_2, x_3) = 0$, then its Hessian is the curve given by the equation $H = 0$ where $H = \det(\frac{\partial^2 f}{\partial x_i \partial x_j})_{1 \leq i,j \leq 3}$.

[25] It was already known that the inflection points are lined up three by three. The *Encyklopädie* and Hesse himself (as we shall see later) refer to [Poncelet 1832].

to compare the descriptions of this process made by two of the authors we shall deal with later: Clebsch and Klein. In a passage of Julius Plücker's obituary, Clebsch evokes cubic curves and Hesse:

> As he found [the four triangles] (Crelle's Journal, vol. 28, 1844), Hesse could deduce the true nature of the problem. So appeared the wonderful character of this class of algebraically solvable equations of degree 9 which bear Hesse's name and for which the inflection points form the first example.[26] [Clebsch 1872, p. 22]

The existence of the four triangles as a way to reveal the nature of the equation is also emphasized in Hesse's obituary, written by Klein:

> Hesse seized the problem of the algebraic determination of the nine inflection points. Because one can sort in four triangles the twelve lines on which the points lie three by three, the resolution of the equation of the ninth degree depends on an equation of the fourth degree.[27] [Klein 1875, p. 48]

Clebsch and Klein both put the four triangles as the core of the resolution of the nine inflection points equation. Yet, even if one can indeed recognize the importance of the incidence information coded in the triangles for the formation of resolvents, Hesse never mentions them in his paper. Thus Clebsch and Klein seem to derive algebraic results directly from geometrical properties. This is an important feature of their treatment of the geometrical equations, and we shall encounter it again many times.

4 Kummer and the quartic surfaces: 1863-1864

The two references of Ernst Eduard Kummer that appear in our corpus are dated 1863 and 1864. These years correspond to the beginning of what has been called Kummer's "third period" devoted to geometry.[28] Kummer's first geometrical papers were published in 1862 and 1863 (but some are dated 1860); they are investigations on ray systems but also on models of such systems and of particular surfaces.[29]

The two references in our corpus deal with special algebraic surfaces of order 4: some containing conic sections and some having 16 singular points—such surfaces would be called "Kummer surfaces" later on. Like many papers on geometry of that time, they do not explicitly aim at a particular result and are (more or less coherent) collections of geometrical results such as the existence of singularities, the incidence relations between elements of the surfaces, the study of particular intersections with other objects, the discussion of particular cases of surfaces, etc. A

[26]"Indem Hesse diese fand (Crelles Journ. Bd. 28, 1844), vermochte derselbe die wahre algebraische Natur des Problems zu erschliessen. Es zeigte sich der wunderbare Character jener Classe algebraische lösbarer Gleichungen 9. Grades, welche Hesse's name führen, und für welche die Wendepuncte das erste Beispiel bilden."

[27]"Andererseits ergriff Hesse das Problem der algebraischen Bestimmung der neun Wendepuncte. Weil man die zwölf Linien, auf welchen dieselben zu drei vertheilt liegen, in vier Dreiecke ordnen kann, hängt die Lösung der betr. Gleichung neunten Grades von einer Gleichung vierten Grades ab."

[28]See [Lampe 1892-93] and André Weil in the preface of [Kummer *Œuvres*, vol. 2]. The first and second periods concern respectively function theory and number theory. According to Emil Lampe, Kummer had found his ground ideas on geometry from Carl Friedrich Gauss's *Disquisitiones generales circa superficies curvas*. In a letter dated of July 1862, Kummer described to Kronecker his interest for George Salmon's treatise on geometry of three dimensions which he was studying "with diligence and pleasure," see [Kummer *Œuvres*, vol.1, p. 100]

[29]About mathematical models, Klein and the Kummer surface, see [Rowe 2013].

typical example of such a result is the following. In [Kummer 1863], Kummer proves that if a quartic surface contains a conic section with multiplicity 2 and exactly two other double points such that their junction line do not intersect the conic section, then every plane containing these double points intersect the quartic surface in two conic sections.

After other geometrical results in the same vein, one geometrical equation appears near the end of [Kummer 1863]. Just before, Kummer proves that every quartic surface having a double conic section is intersected by every double tangent plane in a pair of conics. He follows considering the general form of the equation of the surface: $\phi^2 = 4p^2\psi$ (where ϕ, ψ are quadratic forms and p is a linear form), which is equivalent to $(\phi + 2\lambda p^2)^2 = 4p^2(\psi + \lambda\phi + \lambda^2 p^2)$ for all parameters λ. Then Kummer proves that if $\psi + \lambda\phi + \lambda^2 p^2 = 0$ is the equation of a cone, every tangent plane to this cone is tangent to the quartic surface in two contact points and intersect it additionally in a couple of conics. He follows:

> The easy-to-develop condition that $\psi + \lambda\phi + \lambda^2 p^2 = 0$ represents a cone leads to an equation of the fifth degree for the constant λ, the five roots of which give five cones.[30] [Kummer 1863, p. 335]

The end of the paper is a discussion on the roots of this equation of degree 5. For instance, he indicates that if the equation has imaginary roots, then so are the corresponding tangent plans.

Here the geometrical equation is not studied for itself, but it is rather used to find the number of particular cones. We will see in what follows that this role for the geometrical equation differs from the other texts of the corpus.

We now turn to the second paper of Kummer, [Kummer 1864]. It begins with the proof of the existence of quartic surfaces with 16 singular points and the proof that this number is the maximal number of singularities on a surface of order four. Kummer proves that each of the 16 singular plans contains exactly 6 singular points and that each singular point is contained in exactly 6 singular plans. A paragraph is also devoted to the description of a wire model of a surface with 16 singularities.

Kummer then indicates how to link those surfaces with the theory of caustics and proves the following result:

> The complete ray system of order 12 and class 28 which has a general surface of the fourth order for caustic surface consists, when this caustic has 16 singular points, firstly of 16 ray systems, of which each consists only of all the lines lying on a plane; secondly of four ray systems of order 2 and class 2; and thirdly of one ray system of order 4 and class 4.[31] [Kummer 1864, p. 258]

Kummer goes on to specify that when the plane in which the 12 rays of the four systems of class 2 and of the system of class 4 is a tangent plane to the surface, then the 12 rays merge two by two. The six rays thus obtained become then tangent lines to the surface which all meet in another point on the surface. He deduces immediately:

[30]"Die leicht zu entwickelnde Bedingung, daß $\psi + \lambda\phi + \lambda^2 p^2 = 0$ eine Kegelfläche darstellt, führt auf eine Gleichung fünften Grades für die Constante λ, deren fünf Wurzeln fünf Kegelflächen geben."

[31]"Das vollständige Strahlensystem 12ter Ordnung und 28ter Klasse, welches eine allgemeine Fläche vierten Grades zur Brennfläche hat, besteht, wenn diese Brennfläche vierten Grades 16 singuläre Punkte hat, erstens aus 16 Strahlensystemen, deren jedes nur aus allen in einer Ebene liegenden graden Linien besteht, zweitens aus vier Strahlensystemen zweiter Ordnung und zweiter Klasse, und drittens aus einem Strahlensysteme vierter Ordnung und vierter Klasse."

> The equation of degree 6 with which are determined on the most general surface of fourth degree the six tangents having a common contact point and touching once the surface in another point falls, for the quartic surface with 16 singular points, into four factors of degree 1 and one factor of degree 2 which can be rationally expressed with the coordinates of the common contact point.[32] [Kummer 1864, p. 259]

This is the only appearance of a geometrical equation in [Kummer 1864]. This equation is not at the core of the investigation. Instead, it appears more like an incidental remark following geometrical properties: the equation splits into four factors of degree 1 and one factor of degree 2 because the six lines group themselves into four "systems" of one line and one "system" of two lines. The secondary role of the geometrical equation can be confirmed from a letter from Kummer to Kronecker:[33] this letter is indeed all about the result concerning the ray systems but does not mention anything about the corresponding geometrical equation.

The geometrical equations have thus an auxiliary role in Kummer's papers. However, they show Kummer's ability to transfer geometrical properties to algebraic results and to work on them in order to find geometrical results. We now continue the description of the corpus with another text dealing with particular quartic surfaces.

5 Clebsch and the quartic surfaces with double conic section: 1868

The first article of our corpus written by Clebsch is devoted to special quartic surfaces, those containing a double conic section, [Clebsch 1868]. According to the authors of Clebsch's obituary, this article belongs to the group of works concerning the theory of surface representations.[34] Such works had begun in 1865 with a paper in which Clebsch proved the possibility of representing a cubic surface on a plane.[35] Afterward he treated other examples: the so-called Steiner surface, the ruled cubic surfaces and finally, the quartic surfaces with double conic section which are the object of Clebsch's paper that we are studying here. One of the main goals of the development of the representation of surfaces theory was to study the "geometry on the surfaces,"[36] that is the investigation of special points and curves lying upon the surfaces.

In order to establish the representations of the cited special surfaces, Clebsch used previous works on those surfaces; in the case of [Clebsch 1868], the main reference is [Kummer 1863] which we described in the preceding section. In the first paragraph of this paper, Clebsch defines the quartic surfaces he is about to study, gives the formulas of its representation on a plane and affirms that there is exactly 16 lines lying upon the surface.

[32]"Die Gleichung sechsten Grades, durch welche auf der allgemeinsten Fläche vierten Grades die sechs Tangenten bestimmt werden, die einen Berührungspunkt bestimmt werden, die einen Berührungspunkt gemein haben, und die Fläche außerdem jede noch einmal berühren, zerfällt für die Flächen vierten Grades mit 16 singulären Puntken in vier Faktoren ersten Grades und einen Faktor zweiten Grades, welche durch die Coordinaten des gemeinsamen Berührungspunktes rational ausgedrückt werden."

[33] [Kummer *Œuvres*, vol. 1, p. 101].

[34][Brill *et al.* 1873, pp. 30-37].

[35][Clebsch 1866]. In modern words, a representation of a surface on a plane is a birational map from the surface to the (projective) plane; Clebsch's representation of the cubic surface can be interpreted as the blowup of the projective plane along six points in general position. See [Lê 2013, pp. 59-60] for a description of Clebsch's proof of existence of the cubic surface representation on a plane.

[36][Brill *et al.* 1873, p. 33].

Accordingly, the equation of the sixteen lines is the one that appears in the paper. Although the quantitative room granted to it remains quite modest, the equation appears in the second paragraph, crowning the results on the groupings of the lines. Thus in contrast with Kummer's papers, the geometrical equation seems more to be an object of interest in itself, explicitly linked with the groupings of lines. As we shall see, the equation is the object of two other paragraphs, one devoted to the proof of the number of the lines lying upon the surface, the other to a refinement of its resolution process.[37]

These scattered appearances fit with the description of Clebsch's interest for the algebraic equations given in his obituary:

> The general theory of algebraic equations, as founded by Lagrange, further developed by Gauss and Abel and elevated by Galois to its present generality, interested Clebsch a great deal. However, he did no proper investigation in that direction; but he touched those questions indirectly by not letting any occasion pass, when a geometrical or algebraic problem led to equations of particular character, to point out these noteworthy equations. It was really the investigations of Hesse and furthermore of Abel that turned Clebsch's interest to the algebraic side of geometrical problems.[38] [Brill *et al.* 1873, p. 47]

We already mentioned the links between Clebsch and Hesse earlier. Moreover this quotation hints at points of an history of the reception of Galois theory that have not yet been studied by historians:[39] an actor, Clebsch, and particular equations, the geometrical equations—we will come back to this point later.

We now come to Clebsch's treatment of the sixteen lines equation in [Clebsch 1868]. The second paragraph of this article is entitled "Gruppierungen der Geraden." Clebsch uses the representation of the surface on a plane in order to establish the incidence relations between the 16 lines. At first, he proves that each line intersects five other lines that do not meet; the results are summed up in a table (for instance, the line 1 meets the lines 6, 7, 8, 9 and 16):

1)	6,	7,	8,	9,	16	9)	1,	5,	10,	11,	13
2)	6,	10,	11,	12,	16	10)	2,	3,	8,	9,	15
3)	7,	10,	13,	14,	16	11)	2,	4,	7,	9,	14
4)	8,	11,	13,	15,	16	12)	2,	5,	7,	8,	13
5)	9,	12,	14,	15,	16	13)	3,	4,	6,	9,	12
6)	1,	2,	13,	14,	15	14)	3,	5,	6,	8,	11
7)	1,	3,	11,	12,	15	15)	4,	5,	6,	7,	10
8)	1,	4,	10,	12,	14	16)	1,	2,	3,	4,	5

[37]Nevertheless, this last point uses long calculations and no special feature; since it would wander us off the subject of the premises of the *Erlanger Programm*, we will not deal with it

[38]"Die allgemeine Theorie der algebraischen Gleichungen, wie sie durch Lagrange begründet, durch Gauss und Abel weiter entwickelt, durch Galois zu ihrer jetzigen Allgemeinheit erhoben worden ist, hat Clebsch in hohem Masse interessirt. Er hat freilich in dieser Richtung nicht eigentlich einige Untersuchungen angestellt, aber er hat indirect diesen Fragen genützt, indem er keine Gelegenheit vorübergehen liess, wenn ein geometrisches oder algebraisches Problem zu Gleichungen besonderen Charakters hinleitete, auf eben diese Gleichungen als an und für sich beachtenswerth hinzuweisen. Es waren wohl die Untersuchungen von Hesse und weitherin von Abel gewesen, die Clebsch's Interesse für diese algebraische Seite der geometrischen Probleme rege gemacht hatten."

[39]See for instance [Kiernan 1971; Neumann 1996] and the more recent [Ehrhardt 2012].

From this table, Clebsch deduces another, representing the 40 couples of secant lines:

$1,6$	$2,6$	$3,7$	$4,8$	$5,9$	$6,13$	$7,15$	$9,11$
$1,7$	$2,10$	$3,10$	$4,11$	$5,12$	$6,14$	$8,10$	$9,13$
$1,8$	$2,11$	$3,13$	$4,13$	$5,14$	$6,15$	$8,12$	$10,15$
$1,9$	$2,12$	$3,14$	$4,15$	$5,15$	$7,11$	$8,14$	$11,14$
$1,16$	$2,16$	$3,16$	$4,16$	$5,16$	$7,12$	$9,10$	$12,13$

Now, to each couple correspond three other non-intersecting couples (for instance to the couple $(2,6)$ correspond the couples $(3,7)$, $(4,8)$ and $(5,9)$ because the lines 2, 6, 3, 7, 4, 8, 5 and 9 do not meet), so that the 40 couples split in ten groups of 4 formed by non secant lines. These ten groups split in five times two "conjugated" groups containing the 16 lines. Again, a table is used to represent these groups:

$2,6;$	$3,7;$	$4,8;$	$5,9.$	$1,16;$	$10,15;$	$11,14;$	$12,13.$
$1,6;$	$3,10;$	$4,11;$	$5,12.$	$2,16;$	$7,15;$	$8,14;$	$9,13.$
$1,7;$	$2,10;$	$4,13;$	$5,14.$	$3,16;$	$6,15;$	$8,12;$	$9,11.$
$1,8;$	$2,11;$	$3,13;$	$5,15.$	$4,16;$	$6,14;$	$7,12;$	$9,10.$
$1,9;$	$2,12;$	$3,14;$	$4,15.$	$5,16;$	$6,13;$	$7,11;$	$8,10.$

Clebsch writes directly after this last table:

> This table is of great importance, for it teaches that the equation of degree 16 of which the sixteen lines of the surface depend is solved with the help of an equation of degree 5. This equation, which delivers the five couples of groups, is no other than the one with the help of which Mr. Kummer obtained the five cones of order 2, the tangent planes of which doubly touch the surface, [Kummer 1863].[40] [Clebsch 1868, p. 145]

This quotation closes the second paragraph and the sixteen lines equation is not dealt with again until its next appearance many pages later. Here, the groupings of the lines seem to be an important matter for understanding the configuration of the sixteen lines. Represented in tables, the incidence relations imply directly the existence of a resolvent of degree 5.

The use of this table recalls for instance Enrico Betti's treatment of the modular equation of order 5 in 1853. Indeed Betti represented in a table a partition of the group of the modular equation into 5 sets, and this proved the possibility of reducing the modular equation to an equation of degree 5.[41] One can also find such use of a table in [Galois 1846, p. 428] where Galois links the steps of the resolution of the equation of degree 4 with the decomposition of its group, the substitutions of which are represented in successive tables. Thus if a, b, c, d are the four roots of the general equation of degree 4, Galois describes in a table the group of the equation

[40]"Diese Tafel ist vorzugsweise von Wichtigkeit, weil sie lehrt, dass die Gleichung sechzehnten Grades, von welcher die sechzehn Geraden der Oberfläche abhängen, mit Hülfe einer Gleichung fünften Grades gelöst wird. Diese Gleichung, welche die fünf Paare von Gruppen liefert, ist keine andere, als diejenige, mit deren Hülfe Herr Kummer die fünf Kegel zweiter Ordnung erhalten hat, deren Seiten die fragliche Fläche doppelt berühren. (Sitzung der Berl. Acad. vom 16. Juli 1863.)"

[41]See [Goldstein 2011] for further explanations and in particular a copy of Betti's table.

reduced by the adjunction of a square root:

$$\begin{array}{lll} abcd, & acbd, & adbc, \\ badc, & cabd, & dacb, \\ cdab, & dbac, & bcad, \\ dcba, & dcba, & cbda. \end{array}$$

He goes on and writes that "this group divides into three groups [...]. Thus with the extraction of one radical of the third degree, there only remains the group

$$\begin{array}{l} abcd, \\ bacd, \\ cdab, \\ dcba.\text{"} \end{array}$$

Galois's process of selecting one of the three columns with the help of a radical of the third degree can be seen as analogous to Clebsch's process of finding a resolvent of degree 5 thanks to his table. Let us temporary leave Galois' line of thought and come back to Clebsch.

As stated earlier, the next appearance of this equation is linked with the proof that 16 is the actual number of lines upon the surface. In order to do that, Clebsch forms the equation of the lines as the result of an elimination and proves that its degree is 16. He then explains how the equation can be solved:

1. Search for a root of the equation of degree 5 of which the five cones depend.
2. Resolution of a quadratic equation [from which] the two families of conics [...] are deduced.
3. Complete resolution of two biquadratic equations [...] which deliver the 2.4 couples of lines of the two families of conics.
4. Resolution of eight quadratic equations giving the individual lines of the eight couples.[42] [Clebsch 1868, pp. 172-173]

One can read these steps and compare them to the last table that Clebsch gave. The search of one root of the equation of the cones corresponds to the selection of one line of the table; the resolution of the quadratic equation of the families of conics corresponds to the knowledge of both of the groups of 4 couples in the line; then the biquadratic equations split each group into their 4 couples; finally each couple is split with a quadratic equation.

Again, one can link these steps to what Galois did for the equation of degree 4. Indeed Galois continues from the group

$$\begin{array}{l} abcd, \\ bacd, \\ cdab, \\ dcba, \end{array}$$

[42]"1. Aufsuchung einer Wurzeln der Gleichung fünften Grades, von welcher die fünf Kegel abhängen. 2. Auflösung einer quadratischen Gleichung [...]. Durch diese quadratische Gleichung werden die beiden Kegelschnittschaaren [...] ermittelt. 3. Vollständige Auflösung zweier biquadratischen Gleichungen [...], welche die 2.4 Geradenpaare der beiden Kegelschnittschaaren liefern. 4. Auflösung der acht quadratischen Gleichungen, welche die einzelnen Geraden der acht Paare geben."

writing: "this group splits again in two groups:

$$\begin{array}{ll} abcd, & cdab, \\ badc, & dcba. \end{array}$$

Thus, after only one extraction of a square root, there will remain

$$\begin{array}{l} abcd, \\ badc; \end{array}$$

which will be eventually solved with only one extraction of a square root."

The comparison that we just made between Galois and Clebsch pleads in favor of an understanding of the equation theory arising from the geometry and somewhat analogous to the Galois theory. As explained earlier, Clebsch knew of substitution theory thanks to his links with Jordan; but in a letter to him, he wrote:

> I want to thank you for your friendly sending of the second part of your [*Traité des substitutions et des équations algébriques*]. I just wished you could have also sent the necessary understanding; for unfortunately I have to confess that these deep and important investigations go far beyond my competence. I stick to the first half, where geometry comes to my rescue [...].[43]

The difficulties of understanding the Galois theory[44] and the emphasis on the help provided by geometry confirm the idea that Clebsch went through equation theory thanks to a kind of geometrical reading of Galois' ideas. We shall encounter again this articulation between geometry and substitution theory in a paper of Klein just preceding the *Erlanger Programm*, so we will come back to it in the concluding remarks. Before that, let us return to the chronological progression of our corpus and take a look at Jordan's *Traité des substitutions et des équations algébriques* we just evoked.

6 Jordan: around the *Traité des substitutions et des équations algébriques*, 1870

As we saw in section 2, our corpus contains the *Traité des substitutions et des équations algébriques* and three short papers that announced the publication of the *Traité*. Accordingly we will focus on the *Traité* and more specifically on the chapter called "Applications géométriques." The three short publications are indeed about geometrical equations and one of them is devoted to a link between the twenty-seven lines equation and the equation of the trisection of hyperelliptic functions. This emphasis indicates that Jordan wanted to put in the foreground those special equations that consequently became the parts of the *Traité* that actually circulated and

[43]"Erst jetzt also kann ich Ihnen meinen Dank aussprechen für die freudliche Uebersendung der zweiten Abbildung Ihrer Bucher. Ich wollte nur, Sie hätten mir auch zugleich das nödrige Verständniss mitschicken können; denn leider, ich muss es gestehen, gehen diese liefen und wichtigen Untersuchungen bis jetztweit über meinn Fassungsgabe hinaus. Ich hate mich an die erste Hälfte, wo die Geometrie mir zu Hülfe kommt." Extract of a letter from Clebsch to Jordan, dated 5$^{\text{th}}$ March 1871. The letter is conserved in the Archives of the École Polytechnique.

[44]The authors of Clebsch's obituary do not mention any work that Clebsch would have done on substitution theory.

contributed to its fame.[45] Our corpus contains an additional note of Jordan in which he comes back to the link between the twenty-seven lines and the hyperelliptic functions; we will also discuss with this note in the present section.

In the introduction of the *Traité*, Jordan recalls some points on the development of equation theory, mentions the equations arising from the elliptic functions and then turns to the geometrical equations:

> Another fruitful research path has been opened up to analysts by Mr. Hesse's famous memoirs on the inflection points of the curves of third order. The problems of the analytic Geometry give indeed many other remarkable equations of which the properties, studied by the most illustrious geometers and especially by Mssrs. Cayley, Clebsch, Hesse, Kummer, Salmon, Steiner, are now well known and allow easy application to the methods of Galois. [...] We also want to thank here Mssrs. Clebsch and Kronecker for the precious hints they gave to us. Thanks to the liberal communications of Mr. Clebsch, we have been able to tackle the geometrical problems of the *Livre* III, *chapitre* III.[46] [Jordan 1870b, pp. VI-VIII]

We already mentioned Clebsch's influence on the elaboration of Jordan's chapter on the "Applications géométriques." It is divided into six parts, each dealing with a particular geometrical situation: the nine inflection points of a cubic curve, some curves with prescribed contact properties, the sixteen lines on a quartic surface with double conic section, the sixteen singular points of the Kummer surface, the twenty-seven lines on a cubic surface and eventually another kind of curves with given contact properties and which encompass the twenty-eight double tangents to a quartic curve. In each part, Jordan begins by recalling geometrical properties of the objects in hand, for instance the alignment three by three of the inflection points. For these geometrical properties, Jordan refers to the authors that appear in our previous quotation: Cayley, Clebsch, Hesse, Kummer, Salmon and Steiner.[47] Thus Jordan is probably assimilating the geometrical properties and the properties of the equations: this explains why Cayley, Salmon and Steiner do not appear in our corpus of the geometrical equations.

The application of "the methods of Galois" is one of the actual goals of the *Traité des substitutions et des équations algébriques*: in his book, Jordan develops the theory of substitutions and applies it to the theory of algebraic equations. With the help of groups of substitutions, he wants to place the problem of resolution with radicals as "the first ring of a long chain of questions concerning the transformation of the irrationals and their classification."[48] It must

[45]The letters that Jordan received after he sent the *Traité* to other mathematicians mainly contain comments about the link between the twenty-seven lines and the hyperelliptic functions. See [Brechenmacher 2011].

[46]"Une autre voie féconde de recherches a été ouverte aux analystes par les célèbres mémoires de M Hesse sur les points d'inflexion des courbes du troisième ordre. Les problèmes de la Géométrie analytique fournissent, en effet, une foule d'autres équations remarquables dont les propriétés, étudiées par les plus illustres géomètres, et principalement par MM. Cayley, Clebsch, Hesse, Kummer, Salmon, Steiner, sont aujourd'hui bien connues et permettent de leur appliquer sans difficulté les méthodes de Galois. [...] Nous tenons également à remercier ici MM. Clebsch et Kronecker des précieuses indications qu'ils nous ont fournies. C'est grâce aux libérales communications de M. Clebsch que nous avons pu aborder les problèmes géométriques du Livre III, Chapitre III."

[47]For the alignment of the inflection points, Jordan gives no reference but the corresponding section is called "Équation de M. Hesse.". For the twenty-seven lines, the property is that the lines meet three by three and form thus forty-five triangles. Jordan refers to Steiner, but adds in a note situated at the very end of the *Traité* that Cayley and Salmon had known this property before Steiner, [Jordan 1870b, p. 665].

[48][Jordan 1870b, p. VI]. See [Ehrhardt 2012, pp. 173-182] for an analysis of Jordan's works on Galois theory and [Brechenmacher 2011] for more general relations between Jordan and Galois.

be emphasized here that in the *Traité*, all of the groups are composed of substitutions; even in the geometrical applications—of which we will talk about right now— there is no group of transformations of space.[49]

For the case of the geometrical equations, Jordan has a specific method he applies to each of them: using the incidence relations existing between the objects, he creates an algebraic function having the same group of substitutions as the equation. Jordan then studies this group in order to get results on the resolution of the equation in particular. For instance, the nine roots of the inflection points equation are noted with the symbols (xy) where x, y are integers between 0 and 2 so that three points corresponding to the roots (xy), $(x'y')$, $(x''y'')$ are lined up if and only if $x + x' + x'' \equiv y + y' + y'' \equiv 0 \pmod 3$. The corresponding algebraic function is then $\varphi = \sum (xy)(x'y')(x''y'')$, the sum being taken for all $x, \ldots, y''$ satisfying the preceding condition. In other words, each term of φ corresponds to three inflection points that are lined up. Jordan then states that the group of the nine points equation is the same as the group leaving φ unaltered. He works on the latter—research of order, of chains of (normal) sub-groups—to prove eventually that the nine points equation is solvable with radicals.

Each geometrical equation is treated the same way. In the paragraph concerning the twenty-seven lines of a cubic surface though, Jordan wanders from his method for a moment and deduces some resolution properties directly from the existence of geometrical objects made from the twenty-seven lines. Those objects are the forty-five planes containing three by three the twenty-seven lines, the forty so-called "triples of double Steiner's trihedrons"[50] and the thirty-six double-sixes of Schläffli. For those double-sixes for instance, Jordan writes:

> One can determine in $27 \cdot 16/2$ different ways a couple of lines which do not intersect. Besides, one can group these couples by six (Schläffli's *double-sixes*) so that each line of a couple meets a line of every other couple of the double-six. The double-sixes depend thus on an equation of degree $27 \cdot 16/2 \cdot 6 = 36$ which will be equivalent to [the equation of the 27 lines].[51] [Jordan 1870b, p. 319]

This argument consisting of deducing algebraic results directly from the existence of geometrical objects encoding the incidence relations vividly recalls Clebsch and Klein putting the existence of the four triangles as the very reason of the resolution of the nine inflection points equation (see the end of section 3). This relation between the existence of certain superior geometrical objects and the understanding of a problem can also be found around the link between the twenty-seven lines and the hyperelliptic functions.

In the chapter "Applications à la théorie des transcendantes," Jordan deals with hyperelliptic functions, that is functions of two complex variables which are the inverse functions $x = \lambda_0(u, v)$, $y = \lambda_1(u, v)$ of hyperelliptic integrals

$$u = \int_0^x \frac{\mu + \nu x}{\Delta(x)}\,\mathrm{d}x + \int_0^y \frac{\mu + \nu y}{\Delta(y)}\,\mathrm{d}y \quad \text{and} \quad v = \int_0^x \frac{\mu' + \nu' x}{\Delta(x)}\,\mathrm{d}x + \int_0^y \frac{\mu' + \nu' y}{\Delta(y)}\,\mathrm{d}y,$$

[49]See [Lê 2013] for explanations about the mathematical content of the *Traité* and a discussion of the relations between the theory of substitutions and geometry around the twenty-seven lines equation.

[50]A "Steiner trihedron" is a trihedron made from some of 45 planes, having special incidence relations. These triherons can be grouped by pairs and then by triples of pairs, according to other incidence relations. The final objects are the "triples of double Steiner's trihedrons." See [Henderson 1911] for detailed explanations.

[51]"On peut déterminer de $27 \cdot 16/2$ manières différentes une paire de droites qui ne se coupent pas. On peut d'ailleurs grouper ces paires six à six (*doubles-six* de Schläfli), de sorte que les droites d'une paire rencontrent chacune une droite de chaque autre paire du double-six. Les doubles-six dépendent donc d'une équation de degré $27 \cdot 16/2 \cdot 6 = 36$, qui sera encore équivalente à la proposée."

where Δ^2 is a polynomial of degree 6. The problem of the trisection is to determine $\lambda_0(u/3, v/3)$ and $\lambda_1(u/3, v/3)$ with the help of $\lambda_0(u, v)$ and $\lambda_1(u, v)$, when u and v have particular values called the *periods*. Jordan tackles the problem studying the corresponding equation; he manages to prove that this equation has a reduced equation having the same group as the twenty-seven lines equation.

As explained, this result contributed to the immediate fame of the *Traité des substitutions et des équations algébriques*. In fact, in a letter he wrote to Jordan in December 1869, Luigi Cremona designated the link between the twenty-seven lines and the hyperelliptic functions as a "riddle to explain," to which Jordan answered that it would be interesting to find a "definitive demonstration" of it.[52] The note [Jordan 1870a] of our corpus is Jordan's answer to the problem. He explains:

> In order to facilitate the ulterior comparison between those two problems, apparently so different, it can be useful to reciprocally search for what is the combination of the 27 lines (or of the 45 triangles) which, taken as the unknown, will depend on an equation which is analogous to the equation of the division of an abelian function.[53] [Jordan 1870a, p. 326]

The combination that Jordan gives then is a system of nine planes with particular incidence properties. Going beyond the mathematical details, we can see again the importance given to the formation of geometrical objects encoding particular incidence relations and presented as keys to the understanding of an equation (or of links between equations). The groupings of lines, here represented in geometrical objects, are at the core of the resolution; as we saw, this feature is shared at least by Clebsch and Klein.

7 The theory of linear complexes: Klein, 1870

The paper we refer to is Klein's first publication, [Klein 1870]. It is a work taking elements from his thesis and deepening it, which he did under Plücker's supervision and which he presented in 1868.[54] In his *Gesammelte mathematische Abhandlungen*, Klein writes that a difference between his thesis and this paper is the influence that Göttingen, and especially Clebsch, had on him:

> Compared with the thesis, one recognizes the stimulating influence that the environment of Göttingen had on me. I chose this quite uncertain expression because besides Clebsch himself, the still little number of students he had gathered around him gained the most lively influence on me.[55] [Klein *Œuvres 1*, p. 50]

We will return to this difference between [Klein 1870] and his thesis a bit later. Let us note that if Klein's enthusiasm for Göttingen expressed in the last quotation is certainly genuine, one should

[52] The letter written by Cremona is conserved in the Archives of the École Polytechnique (ref. VI2A2(1855) 9). This letter and its answer have been edited by Simonetta Di Sieno and Paola Testi Saltini. Cremona's whole correspondence will be published soon, under the direction of Giorgio Israel.

[53] "Pour faciliter la comparaison ultérieure de ces deux problèmes, en apparence si différents, il peut être utile de rechercher réciproquement quelle est la combinaison des 27 droites (ou des 45 triangles) qui, prise pour inconnue, dépendra d'une équation analogue à celle qui dnne la division d'une fonction abélienne."

[54] [Klein 1868].

[55] "Beim Vergleich mit der Dissertation wird man den anregenden Einfluß erkennen, den die Göttinger Umgebung auf mich ausgeübt hat. Ich wähle dieser etwas unbestimmten Ausdruck, weil neben Clebsch selbst die vorab noch kleine Zahl von Spezialschülern, die er um sich versammelt hatte, regsten Einfluß auf mich gewann."

bear in mind that in 1921, Klein had already managed to construct the "great Göttingen," erecting as heroes mathematicians from the past like Carl Friedrich Gauss and Bernhard Riemann. So even at a smaller scale, Clebsch's name could have been magnified as Klein was writing his history in 1921.[56]

The theme of [Klein 1870] is the study of the linear complexes of degree 2, that is, sets of lines of space satisfying some conditions. More precisely, a line of space can be described with six homogeneous coordinates satisfying a quadratic equation; a linear complex of degree 2 is the set of all lines, the coordinates of which satisfy an additional homogeneous quadratic equation. In this publication that Klein himself called "vague," special care is given to the link between the linear complexes and Kummer's quartic surface. Geometrical equations appear repeatedly at the conclusion of little paragraphs: they are not central objects of investigation. Let us now see this typical structure in an example.

Klein notes $a_1, a_2, \ldots, a_6$ the six homogeneous coordinates of the lines of space. He proves that one can suppose their quadratic identity to be $\sum a_i^2 = 0$, which happens to be invariant if one changes the signs of the a_i. Then Klein concludes that the lines of space can be grouped by 32: for a given 6-uple $a_1, \ldots, a_6$, the 32 lines are those of coordinates $\pm a_1, \pm a_2, \ldots, \pm a_6$. Now, Klein indicates that each group of 32 lines split into two groups of 16 lines according to the parity of the number of negative coordinates. Finally, if one line of a group of 16 is given, Klein proves that the lines of the other group of 16 split into two groups: those which are the conjugated polar lines of the given line with respect to the six fundamental complexes, respectively with respect with the ten fundamental surfaces.[57] He then writes immediately:

> The equation of the 32$^{\text{th}}$ degree, by which such a system of lines [...] is determined, requires, after the six fundamental complexes have been found by an equation of the 6$^{\text{th}}$ degree, only the resolution of equation of the second degree.[58] [Klein 1870, p. 210]

This result is not explained nor used or commented in the paper: it really seems to be a remark crowning a little paragraph. Other examples of such properties of equation resolution following geometrical groupings run throughout Klein's paper. In particular, Klein proves with this kind of reasoning a result that Jordan demonstrated with his substitutions techniques: that the 16 singular points equation depends on an equation of degree 6 and on other quadratic equations. All these algebraic results on geometrical equations following particular groupings of lines are precisely one obvious difference between [Klein 1870] and Klein's thesis. So we can see them as artifacts of the "stimulating atmosphere" of Göttingen that was mentioned in the first quotation of the present section.

With only this first publication, one can see Klein's early mathematical influences as well as some works related to his: Kummer and his investigation on quartic surfaces; Clebsch's emphasis on the groupings of lines; Jordan's result on the sixteen points equation. This knitting shows different types of relations between Klein and the cited authors, namely groundings of work,

[56]See [Rowe 1989a]. About Clebsch, a deeper investigation could be done to determine his precise relationship with Klein. Concerning a construction of Clebsch's posthumous fame (Clebsch died in November 1872), let us just recall here that the first lecture that Klein gave in the Evanston Colloquium is entirely devoted to Clebsch, [Klein 1894].

[57]The six fundamental complexes are the sets of lines defined each to be the vanishing locus of one of the six coordinates. The ten fundamental surfaces are the sets of lines defined by equations such that $a_1^2 + a_2^2 + a_3^2 = 0$.

[58]"Die Gleichung des 32$^{\text{ten}}$ Grades, durch welche ein derartiges System von geraden Linien [...] bestimmt wird, verlangt, nachdem die sechs Fundamentalcomplexe durch eine Gleichung des 6$^{\text{ten}}$ Grades gefunden worden sind, nur noch die Auflösung von Gleichungen des zweiten Grades."

influence and rereading of results. On this last point, we are about to see that Lie puts Klein's approach as the geometrical footing for the possibility to lower the sixteen points equation to an equation of degree 6. Thus we have again the idea that geometry is the way to understand links between equations.

This weaving of Klein's ideas between those of Clebsch, Kummer and Jordan shows a particular practice around the geometrical equations in the earliest works of Klein; moreover, the impression of closeness around this practice is about to be reinforced with some comments of Lie.

8 Interlude: Lie's comments, 1871

Lie's paper to which the *Encyklopädie* refers is an augmented and German-translated version of his doctoral thesis, [Lie 1872].[59] It is quite a long paper (more than 100 pages) but the *Encyklopädie* refers to a precise passage of it, where some comments on the geometrical equations appear. These comments are situated at the end of the paper, as Lie is about to tackle some quartic surfaces.

> Among the surfaces of the fourth order, there are two, firstly investigated by Mr. Kummer, that I shall consider here: the one with 16 nodes, f_4, and the one with a double conic section, F_4. Both give grounds to an equation of the sixteenth degree; one by its nodes, the other by the lines lying upon it. The latter led Mr. Clebsch to an equation of the fifth degree already considered by Mr. Kummer (Borchardt's Journal vol. 67[60]). On the other hand, Mr. Jordan found that the former comes back to an equation of the sixth degree [Jordan 1869c]. The geometrical grounds of it lie in Mr. Klein's investigations relative to that surface [Klein 1870].[61] [Lie 1872, pp. 250-251]

This quotation does not contain anything new mathematically speaking, but it is interesting to see that Lie refers to almost everything we have been dealing with until now. Not only Lie is acquainted with the geometrical research concerning the quartic surfaces, but he also knows of the corresponding geometrical equations.

It is true that Lie does not here develop anything about geometrical equations. Yet, his comments emphasize the fact that these equations are really part of the mathematical bath in which he and Klein are immersed at those times just preceding the elaboration of the *Erlanger Programm*.

[59][Engel 1900, p. 36].

[60]This volume contain no article written by Kummer, and those written by Clebsch have nothing to do with quartic surfaces with double conic section. Lie is probably referring to [Kummer 1863] or [Clebsch 1868].

[61]"Unter den Flächen vierter Ordnung, giebt es zwei, zuerst von Herrn Kummer untersuchte, welche ich hier betrachten will: die mit 16 Knotenpunkte, f_4, und die mit einem Doppelkegelschnitt, F_4. Beide Flächen geben Anlass zu einer Gleichung sechzehnten Grades; die eine durch ihre Knotenpunkte, die andere durch die auf ihr gelegenen geraden Linien. Die letzere fürhte Herr Clebsch auf eine schon von Herr Kummer aufgestellte Gleichung fünften Grades zurück. (Borchardt's Journal Bd. 67.) Andererseits fand Herr Jordan, dass die ertere Gleichung auf eine solche vom sechsten Grade zurückkommt (Borchardt's Journal Bd. 70.). Es fand dies seine geometrische Begründung in den auf diese Fläche bezüglichen Untersuchungen des Herrn Klein (Math. Ann. Bd. 2.)."

9 Clebsch's geometrical interpretation of the quintic, 1871

We now turn to [Clebsch 1871] which is not a direct reference of the *Encyklopädie*, but which is quoted in Klein's paper *Ueber eine geometrische Repräsentation der Resolventen algebraischer Gleichungen* we evoked in the introduction. Clebsch's goal is to give a geometrical interpretation of the theory of the equation of the fifth degree:[62]

> The object of present article is mainly the different forms that an equation of the fifth degree can take by a superior transformation. [...] So one obtain [...] a complete overview of the relations existing between the equations of degree 5 and their resolvents, and in particular of the relations with the Jerrard form and the modular equation.[63] [Clebsch 1871, pp. 284-285]

The actual core of this geometrical interpretation is to consider the coefficients of the intervening transformations as coordinates of space. For instance, a transformation $\xi = \frac{y_1+y_2\lambda+y_3\lambda^2}{x_1+x_2\lambda+x_3\lambda^2}$ meant to act on an algebraic equation $f(\lambda) = 0$ is represented as the line of the plane joining the points of homogeneous coordinates x_1, x_2, x_3 and y_1, y_2, y_3. One of the main points of Clebsch's method is to find transformations such that prescribed invariants of the transformed equation vanish. Such transformations are geometrically represented by lines that are tangent to certain curves, created from the prescribed invariants.

Clebsch also interprets the Tschirnhaus transformation $\xi = a\lambda^4 + b\lambda^3 + c\lambda^2 + d\lambda + e$. If one wants the transformed equation to be of the form $\xi^5 + A\xi + B = 0$, the coefficients $a, b, \dots, e$ must satisfy three conditions $\Phi(a, b, c, d, e) = 0$, $\Psi(a, b, c, d, e) = 0$ and $\mathrm{X}(a, b, c, d, e) = 0$, where Φ, Ψ and X are homogeneous polynomials of degree 1, 2 and 3 (resp.). The linear identity $\Phi(a, b, c, d, e) = 0$ allows Clebsch to interpret a, b, c, d, e as pentaedric coordinates of space, that is to eliminate one of the five coefficients $a, \dots, e$ with the help of $\Phi = 0$ and then consider the four remaining as homogeneous coordinates of space. Then $\Psi = 0$ becomes the equation of a quadratic surface and $\mathrm{X} = 0$ the equation of a cubic surface. So, in order to determine an appropriate Tschirnhaus transformation, one has to find a point on the curve (of order 6) which is the intersection of these surfaces.

Geometrical equations appear incidentally as ways to control the irrationalities in the process of finding a point on the curve of order 6—since the Tschirnhaus transformation does not involve radicals superior to the fourth order, so must the geometrical process. Clebsch summarizes his method as follows:

> One has just to determine any point of the space curve of the sixth order, which is carried out by intersecting a generator of the surface $\Psi = 0$ with the [cubic surface $\mathrm{X} = 0$]. For this, a quadratic equation and a cubic equation are to be solved; the former to find a generator of the surface of the second order; the latter to determine the intersection of this surface with the [cubic] surface.[64] [Clebsch 1871, p. 341]

[62] On the general equation of degree 5, see the chapter VII of [Houzel 2002] and the chapter IV of [Gray 2000].

[63] "Der vorliegende Aufsatz hat vorzugsweise die verschiedenen Formen zum Gegenstande, welche einer Gleichung 5^{ten} Grades durch eine höhere Transformation gegeben werden können. [...] So erhält man [...] eine vollständige geometrische Uebersicht über die Zusammenhänge, welche zwischen den Gleichungen 5^{ten} Grades und ihren Resolventen bestehen, insbesondere über den Zusammenhan mit der Jerrard'schen Form und der Modulargleichung."

[64] "Man hat dann nur einen beliebigen Punkt der Raumcurve 6^{ter} Ordnung zu ermitteln, was geschieht, indem man eine Erzeugende der Fläche $\Psi = 0$ mit der Diagonalfläche schneidet. Dazu ist eine quadratische und eine

Other passages of Clebsch's paper contain geometrical equations. For instance, Clebsch interprets Kronecker's method of resolution of the quintic, which consists of resolving the modular equation of degree 6 in order to transform the quintic into a pure equation.[65] In the geometrical interpretation, the modular equation is proved to be the same as the equation of degree 6, which "separates" six double tangents to a certain curve.

This mixing of equation theory, invariant theory and geometry is very characteristic of this paper.[66] These mathematical domains are precisely those which are at the core of Klein's paper on the geometrical representation of equations.

10 Geometrical representation of resolvents: Klein 1871

We finally come to [Klein 1871], that we mentioned at the beginning of the present paper. In the introduction, Klein emphasizes the intuitive character of the geometrical equations:

> The large advantage of these examples is that they intuitively present to the eye the in themselves so abstract ideas of the theory of substitutions. They refer most of the time to equations of very particular character, of which the roots present particular groupings and thus give a view on why such particular equations can occur."[67] [Klein 1871, p. 346]

He goes on and announces his general and fundamental principle: to conceive every algebraic equation as a geometrical equation, embodying the roots of an equation in geometrical objects and replacing the substitutions of the roots by transformations of the space. Together with the previous quotation, this articulation between algebra and geometry reveals two main mottoes of Klein: to bring geometry to the fore because of the intuition it allows and to stress the importance of transformation groups.

At the end of the introduction, Klein reveals that both Clebsch and Lie helped shape the ideas of the paper. The former with the "geometrical considerations" that he used in the interpretation of the quintic in [Clebsch 1871] and that he "had the kindness to share with [Klein] during repeated personal conversations."[68] The latter because of his common investigations on linear transformations of geometrical objects published in [Klein and Lie 1871].

This last reference is one of those which usually appears in the historiography of the *Erlanger Programm*. On the one hand, this confirms a link between Klein's early research on algebraic

cubische Gleichung zu lösen; erstere, um eine Erzeugende der Fläche 2^{ter} Ordnung zu finden; die andere, um die Durchschnitte derselben mit der Diagonalfläche zu bestimmen."

[65] On the modular equation of degree 6 and its links with the quintic (without geometry) treated in particular by Charles Hermite, see [Goldstein 2011].

[66] However, let us recall that the geometrical interpretation of the theory of invariants had been developed by Clebsch himself years before. See [Brill *et al.* 1873, p. 37 ff.]. Besides, the application of the invariant theory to geometry had begun with the works of Cayley, Sylvester and Salmon. About the history of invariant theory, see [Fisher 1966; Hunger Parshall 1989]

[67] "Der hohe Nutzen dieser Beispiele liegt darin, daß sie die an und für sich so eigenartige abstrakten Vorstellungen der Substitutionstheorie in anschaulicher Weise dem Auge vorführen. Sir beziehen sich zumeist auf Gleichungen von sehr partikulärem Character, zwischen deren Wurzeln besondere Gruppierungen statthaben, und lassen also übersehen, wieso derartige besondere Gleichungen auftreten können."

[68] "Die nächste Veranlassung zu den hiermit angedeuteten Dingen sind mir die geometrische Betrachtungen gewesen, die Herr Clebsch in dem vorstehenden Aufsatze behufs Discussion der Gleichungen 5^{ten} Grades angewandt hat, und welche mir derselbe in wiederholten persönlichen Unterhaltungen mitzutheilen die Güte hatte." [Klein 1871, p. 347].

equations and his common work with Lie; on the other hand, the reference to Clebsch indicates an influence that has been, in our view, not emphasized or studied enough in the historiography. We will come back to these points in the concluding remarks. We will now roughly describe each of the four parts in which Klein's paper is divided.

In the first part, Klein gives more accurate explanations on his general principle of geometrical representation. Thus if the equation is general and of degree n, an element of space takes in general $n!$ different places when the $n!$ transformations corresponding to the $n!$ substitutions act on it. According to Klein, this system of $n!$ elements of space is the image of the Galois resolvent of the equation. Furthermore, Klein points out that there are special elements in space such that some of the $n!$ transformed elements are coincident. In that case, "the Galois resolvent becomes a power of an expression which is designated as a particular resolvent."[69]

In the second part of the paper, Klein links the theory of equations with the theory of covariants: the groups of $n!$ elements obtained in general are covariants of the system made of the n given elements corresponding to the n roots.

The third part deals briefly with particular equations, like the nine inflection points equation but also the cyclotomic equations. Klein emphasizes again the importance of the transformations of space:

> As a geometrical image for the equation of the 9th degree, we do not consider the curve of the third order which contains the inflection points, but rather *the inflection points themselves and the transformation cycles by which they are permuted among themselves.*[70] [Klein 1871, p. 354]

Finally, in the fourth and last part, Klein comes to the general equation of degree 6. He applies his theory of linear complexes and refers to [Klein 1870] for geometrical results. Substitutions of the roots correspond to transformations of linear coordinates and just like in the other references of our corpus, groupings of lines lead directly to algebraic results. For instance, Klein finds 15 pairs of directrices, each being associated to a pair of fundamental complexes, and he deduces immediately that:

> *The* 15 *pairs of directrices are the image of a resolvent of degree* 15.
>
> The 15 pairs of directrices build the edges of 15 tetrahedrons (in the sense that one can, in 15 ways, divide six elements into 3 groups of 2).
>
> *These* 15 *tetrahedrons represent a second resolvent of degree* 15.
>
> From these 15 tetrahedrons, one can now choose five that have for edges the 30 directrices, in 6 ways.
>
> *These groups of five tetrahedrons represent a resolvent of degree* 6.[71] [Klein 1871, p. 357]

[69] "Die Galoissche Resolvente wird dann eine Potenz eines Ausdrucks, der als eine besondere Resolvente bezeichnet wird." [Klein 1871, p. 348].

[70] "Als geometrisches Bild für die Gleichung 9ten Grades betrachten wir nun nicht die Curve 3ter Ordnung, welche die Wendepunkte besitzt, sondern *die Wendepunkte selbst und den Transformationscyclus, durch welche diese untereinander vertauscht werden.*" (Klein's emphasis).

[71] "*Die* 15 *Directricenpaaren sind das Bild einer Resolvente* 15*ten* *Grades.* Die 15 Directricenpaaren bilden nun die Kanten von 15 Tetraedern (dem entsprechend, dass man 6 Elemente auf 15 Weisen in 3 Gruppen von 2 theilen kann). *Diese* 15 *Tetraedern stellen eine zweite Resolvente* 15*ten* *Grades dar.* Aus den 15 Tetraedern nun kann man auf 6 Weisen solche 5 aussuchen, die zusammen alle 30 Directricen zu Kanten haben. *Diese Gruppen von* 5 *Tetraedern repräsentieren eine Resolvente des* 6*ten* *Grades.*" (Klein's emphasis).

We recognize here the formation of resolvents directly from the existence of geometrical objects, as it was presented for instance in Jordan's *Traité des substitutions et des équations algébriques* (cf. *supra*).

At the very end of the paper, Klein explains very quickly (and very vaguely) how, in his geometrical representation, one can find the rational transformation of an equation of degree 6, the effect of which is to vanish a given invariant. In particular, he refers at Clebsch's paper on the geometrical interpretation of the quintic, [Clebsch 1871], where that kind of transformations was at the core.

All of the main points developed by Klein in the paper to which this section was devoted form one of the bases of Klein's theory of the quintic and the icosahedron.[72] However, as explained in the introduction, equation theory is part of the mathematical background of the *Erlanger Programm*. Our point here is thus to emphasize that what is usually linked to the research on the icosahedron must also be understood as premises of the *Erlanger Programm*.

11 Conclusion

The present investigation put the geometrical equations to the fore, filling up a hole in the historiography well fitted to these particular equations. Following these particular equations, one can indeed track some important points of the history of algebraic equations: the formation of resolvents in the tradition of Gauss and Abel; Galois theory as explained and used by Jordan; works on the quintic crowned with Klein's approach linked to the icosahedron. What also appears, although less directly, is a proximity with the modular equation on the one hand, and the cyclotomic equation on the other hand. But like those special equations, one should not consider the geometrical equations just as mere examples illustrating the development of equation theory, for they did contribute to this development.

The cyclotomic and modular equations have been extensively studied by historians, and their importance as paradigmatic models has already been emphasized. Moreover, a link between the groupings of the roots of the cyclotomic equation and geometry has been commented on in [Boucard 2011]. To the contrary, the connections between geometry and the modular equations as they appear for instance in Clebsch's paper on a geometrical interpretation of the quintic theory have not yet been investigated.

In fact, Clebsch himself remains historically too little studied. Although he is usually connected to the domains of invariant theory, abelian functions and surface representations, his interest for the algebraic equations have been emphasized in the present paper. A central actor in our corpus because of his relations with Hesse, Jordan and Klein, Clebsch is the one giving for the first time a kind of geometrical understanding of Galois theory. As we noticed earlier, this shared understanding goes through a careful study of the groupings of geometrical objects (and not of roots), sometimes expressed in tables or incarnated in superior objects. We touch here on an important point of the development of the substitution theory, already stressed by Max Noether in 1875:

> With the research on the inflection points of the cubic curve [...] an insight into the particular nature of the [inflection points equation] has been given through the proof of the four triangles [...]. A geometrical image for all the relations concerning

[72]In the book on the icosahedron, Klein explicitly refers to this paper. See [Klein 1884, p. 160].

the groupings of the roots had been gained. Such intuitive [*anschauliche*][73] special examples really participated to an easier perception and to the development of the so abstruse substitution theory.[74] [Noether 1875, p. 86]

Yet the geometrical equations are not only interesting for the history of substitution theory.

Indeed, nourished by his common research with Lie, Klein connected substitutions and linear transformations of space through a geometrical interpretation of the roots of any algebraic equation based on the examples of the geometrical equations. But the transformations of space are at the core of the *Erlanger Programm*; more specifically, the central pillar of the *Programm* is the very notion of *groups* of transformations. As mentioned in the introduction, the importance of groups of transformation was stressed in the *Programm* with an analogy with the Galois theory. The emphasis on invariant elements of space comes also from this analogy:

> In the theory of equations the first subjects to engage the attention are the symmetric functions of the coefficients, and in the next place those expressions which remain unaltered, if not under all, yet under a considerable number of permutations of the roots. In treating a manifoldness on the basis of a group our first inquiry is similarly with regard to the bodies [...], viz., the configurations which remain unaltered under all the transformations of the group. But there are configurations admitting not all but some of the transformations of the group, and they are next of particular interest from the point of view of the treatment based on the group. [Klein 1893, p. 242]

Thus Klein's wish for a theory of transformation groups similar to the theory of substitutions indicates how important was the background of equation theory.

What happened next is better known. The "few readers and listeners [of the *Erlanger Progrmam*] must have realized that they were being exposed to new and fundamental perspectives, even if they barely understood them."[75] It would be of great interest to know the reactions to the *Programm* of all the authors we studied in this paper; but this is another story. At least we can take for granted that Clebsch could not have reacted, for he suddenly died in November 1872.

> "During the elaboration of my program, I naturally always thought of what my dear professor Clebsch (whom I undoubtedly had to thank for the early nomination to Erlangen) would have to say about my explanations, which differ so much from his systematic-projective way of thinking. In vain! For Clebsch suddenly succumbed from a case of diphtheria on the November 7$^{\text{th}}$ 1872, at the age of just 39. His opinion, which I was awaiting with a mixing of hope and fear, thus never came.[76] [Klein *Œuvres 1*, p. 412]

[73]A discussion about the nature of this intuition, its manifestations and its uses in the texts is worth a detailed investigation. Such an investigation will be done later.

[74]"Bei der Untersuchung der Wendepunkte der Curve dritter Ordnung [...] ergab sich durch den Nachweis der vier Dreiseite, in die sich die Geraden gruppiren, ein Einblick in die besondere Natur der Gleichung neunten Grades, welche die neun Punkte bestimmt. [...] [Es] war auch ein geometrisches Bild für alle auf die Gruppirungen der Wurzeln bezüglichen Verhältnisse gewonnen. Solche anschauliche speciellere Beispiele haben wesentlich auf die leichtere Auffassung und auch auf die Ausbildung der an sich so abstrusen Substitutionentheorie gewirkt."

[75] [Birkhoff and Bennett 1988, p. 152].

[76]"Bei der Ausarbeitung meines Programms habe ich selbsverständlich immer daran gedacht, was wohl mein verehrter Lehrer Clebsch (dem ich zweifellos auch die frühzeitige Berufung nach Erlangenzu verdanken hatte) zu meinen Darlegungen, die so vielfach von seiner systematische-projective Denkweise abwischen, sagen würde. Vergeblich! Denn Clebsch ist am 7. November 1872, im Alter von nur 39 Jahren, einem Anfall von Diphteritis

This mixed feeling that Klein recalled serves to tone down the image of the dazzling 23-year-old *Ordinatus* and shows the zealous mathematician at the beginning of his "Faustian quest,"[77] both aware of the boldness of his ideas and yet still caring for his elders' approval.

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[77] [Rowe 1989a, p. 187].

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