Around the History of the Twenty-seven Lines upon Cubic Surfaces: Uses and Non-Uses of models

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October 2015

In 1849, Arthur Cayley and George Salmon proved a theorem which can be phrased in modern terms as follows: "Every non-singular cubic surface of $\mathbf{P}_3(\mathbf{C})$ contains exactly 27 lines". In the corresponding papers, Cayley 1849; Salmon 1849, Cayley and Salmon also showed that the 27 lines of a cubic surface are coplanar by threes, thus forming 45 triangles. Later on, in 1858, Ludwig Schläfli defined the "double-sixes" of a cubic surface, which are sets of 12 lines (among the 27) with prescribed incidence relations,[1] and he proved that there are exactly 36 double-sixes, Schläfli 1858.

The issue of constructing models of cubic surfaces, of the 27 lines without the surface to which they belong, or of a double-six, has been tackled since the beginning of the 1860s. In my talk, I mainly focused on the models of the 27 lines and of a double-six, thus exploring texts of James Joseph Sylvester (1861), Cayley (1870), Percival Frost (1882), and Henry Martyn Taylor (1900).[2] I also aimed attention at two models (including one of the so-called "diagonal" cubic surface) presented in 1872 by Alfred Clebsch. Taking a look at a paper of Clebsch linked to these two models, I finally addressed the question of the actual uses of models in the second half of the 19th century.

1 Models of the 27 lines and of a double-six

Sylvester's text is a note to the *Comptes rendus des séances de l'Académie des sciences* of 1861, Sylvester 1861. In this paper, Sylvester touched upon the idea of creating a model of the 27 lines:

[1]To be more precise, the lines $a_1, \dots, a_6, b_1, \dots, b_6$ form a double-six if, in the table

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{pmatrix},$$

two lines intersect if and only if they do not belong to the same row and the same column.

[2]The corresponding references have been gathered from the "historical summary" of Henderson 1911.

> Je me propose de faire construire en fil de fer ou d'archal un système de 27 droites [...], de sorte qu'on pourra éprouver le plaisir inattendu de voir avec les yeux du corps toutes les droites (le squelette pour ainsi dire) d'une surface du 3^{e} degré avec leurs 135 points d'intersection, les 45 triangles, [...] et les autres merveilles de cette involution si compliquée, mais en même temps si symétrique. Sylvester 1861, 979-980

This intention of building a model does not seem to have been realized by Sylvester. However, the quotation shows the latter's interest for the incidence relations linking the 27 lines, incarnated in the 45 triangles for instance. It also hints to the importance of the "skeleton" made of the 27 lines as the key to understand a cubic surface, an idea that has been used by other mathematicians.[3]

The 1870 article of Cayley deals with a model of a double-six, Cayley 1870. In this paper, Cayley started computing equations of the lines of a double-six, as well as their "Plücker coordinates." He then turned to "the numerical computations for enabling the creation of a drawing or model" of the double-six, Cayley 1870, p. 68, and calculated the coordinates of the points of intersection of the lines forming the double-six. He concluded: "I find however, on laying down the figure, that the lines 3 and 4, $3'$ and $4'$ come so close together, that the figure cannot be obtained with any accuracy." Cayley 1870, p. 71. Cayley did not comment about the creation of a model of a double-six, but the case illustrates the difficulties of finding "good" equations and "good" numerical values when trying to make a satisfying model.

In Frost's paper Frost 1882, the emphasis was also put on the issue of finding simple equations for the lines as well as adequate numerical values, so that the points of intersection of the 27 lines do not appear too close. Frost encouraged his readers "to spend a few minutes on the subject, and possibly to amuse themselves, as [he has] done, by constructing a model" Frost 1882, p. 89, which points to a recreational aspect of the building of a model. However, he admitted to have failed creating a complete model:

> I shall be most happy to show what I have done to anybody who may like to see what to avoid and what to adopt. My model is anything but perfect, two or three of the lines are too far off to appear, and with them their ten points of intersection are out of sight. The only satisfaction I have is that I know where they all are, and 9 or 10 lines all pointing at the deserters tell where they ought to be. Frost 1882, p. 96

This statement hints at the comings and goings of the process of creating a model; it also emphasizes that trying to build a model requires investigations which allow one to gain knowledge of the situation even though the final model is not complete.

Taylor's 1900 article Taylor 1900 essentially involves the same questions and ingredients as the preceding (finding simple equations and adequate numerical values) but ends up with

[3] For example, when Hieronymous Zeuthen investigated the possible shapes of cubic surfaces in Zeuthen 1875, he used the 27 lines to define their "sides," their "triangles", and their "openings".

a success. For the sake of brevity, I only display here two pictures of Taylor's 27 lines model, without commenting it any further (see figure 1).

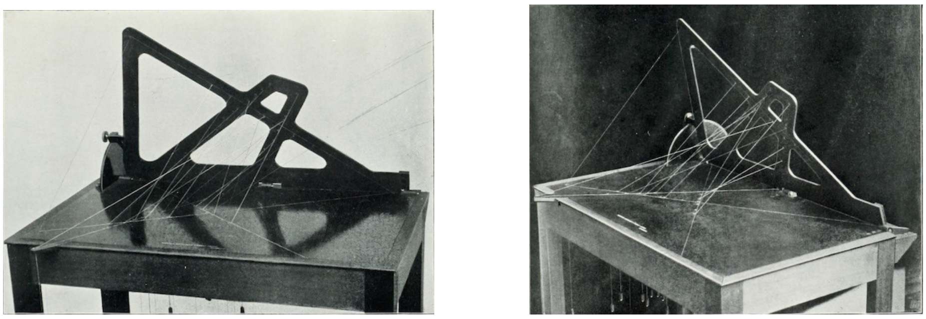

Figure 1 – Model of the 27 lines made of chord, Taylor 1900.

2 From the diagonal surface to geometrical equations

The *Nachrichten der Königlichen Gesellschaft der Wissenschaften und der G.A. Universität zu Göttingen* of 1872 contains a account (p. 402) of when Clebsch presented two models:

> Hr. Clebsch legte zwei Modelle vor, welche Hr. stud. Weiler hierselbst dargestellt hatte, und welche auf eine besondere Classe von Flächen dritter Ordnung beziehen. [...] Das eine der beiden Modelle stellte die 27 Geraden dar, das andere die Fläche selbst, ein Gypsmodell, auf welchen die 27 Geraden gezeichnet waren.

The rest of the account does not mention the first model anymore; it describes the second one, with an insistence on the shape of the "diagonal surface."

This surface had appeared in a 1871 paper of Clebsch where he intended to geometrically interpret the theory of the quintic equation, Clebsch 1871. One of his aim was to interpret the "Tschirnhaus transformation" $\xi = a + \lambda b + \lambda^2 c + \lambda^3 d + \lambda^4 e$, which was supposed to act on a quintic $f(\lambda) = 0$ so that the transformed equation would have the form $\xi^5 + A\xi + B = 0$. The roots of the transformed equation being noted $\xi_1, \ldots, \xi_5$, the transformation would have the desired effect when

$$\sum_{i=1}^{5} \xi_i = 0, \qquad \sum_{i=1}^{5} \xi_i^2 = 0, \qquad \sum_{i=1}^{5} \xi_i^3 = 0.$$

These conditions led Clebsch to study the two surfaces defined by $\sum \xi_i = \sum \xi_i^2 = 0$ and $\sum \xi_i = \sum \xi_i^3 = 0$ respectively. The second surface is the one that Clebsch called "diagonal surface." The associated "27 lines equation" and "36 double-sixes equation" then played an important role in Clebsch's final interpretation of Kronecker's approach of the quintic.

In fact, these equations are part of a larger family, that of "geometrical equations," which are algebraic equations (in one unknown) associated to diverse geometrical configurations like the 27 lines or the 36 double-sixes, but also the 9 inflection points of the cubic curves, the 16 nodes of Kummer's surface, etc. Geometrical equations played a crucial role (ca. 1870) for a geometrical, intuitive understanding of the theory of equations and the theory of substitutions for people like Clebsch and Klein among others:[4]

> Der hohe Nutzen dieser Beispiele liegt darin, daß sie die an und für sich so eigenartig abstrakten Vorstellungen der Substitutionstheorie in anschaulicher Weise dem Auge vorführen. Klein 1871, p. 346

A central feature of this "intuitive" approach consisted in replacing the search of resolvents of a geometrical equation by the search of configurations made from the objects linked to the main equation. For instance, the very existence of the 36 double-sixes meant the existence of a resolvent of degree 36 (the double-sixes equation) of the 27 lines equation.

Now, I found no evidence in favor of any kind of use of models for issues relative to geometrical equations[5]—for instance, one could have expect that the inspection of a model of the 27 lines would have make someone discover a new configuration made from these lines, and therefore a new resolvent of the 27 lines equation. So, even if models were used to find new results in some cases, this was only true for certain situations. As for geometrical equations, even though the objects involved were the same as those which were modeled, and even though the mathematicians were essentially the ones implied in the production of models, these models were not used. Therefore, this observation helps us delimit the realm of uses of models in the mathematical research of the 19th century. Additionally, it proves that the "intuition" mobilized for geometrical equations was not necessarily related to a concrete kind of visualization supported by material objects like models.

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[4]The mathematical activities linked to geometrical equations are analyzed in Lê 2015a and Lê 2015b, where the notion of "cultural system" is discussed as descriptive category of these activities.

[5]Either in the published texts, the letters, or the manuscripts I studied.

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