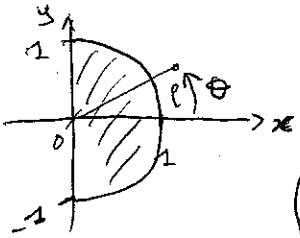
CC3 - UE géométrie et calcul différentiel - Printemps 2012 - Corrigé.

Exercice 1 :

1\. 

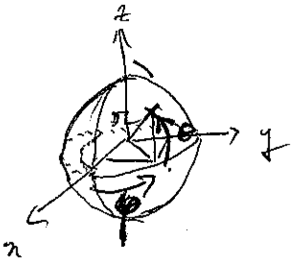
On effectue un changement de coordonnées en polaire :

$\mathcal{D} = \{(\rho, \theta) \in [0,1] \times [-\frac{\pi}{2}, \frac{\pi}{2}]\}$.

$$\iint_{\mathcal{D}} \frac{x}{1+x^2+y^2}\, dx\, dy = \int_{\rho=0}^{1} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho\cos\theta}{1+\rho^2}\, \rho\, d\rho\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta \int_{\rho=0}^{1} \frac{\rho^2}{1+\rho^2}\, d\rho = 2\int_0^1 \frac{1+\rho^2-1}{1+\rho^2}\, d\rho$$

$$= 2\int_0^1 \left(1 - \frac{1}{1+\rho^2}\right) d\rho = 2 - 2\left[\arctan(\rho)\right]_0^1 = \boxed{2 - \frac{\pi}{2}}$$

2\. 

$$\begin{cases} x = r\cos\theta\cos\varphi \\ y = r\cos\theta\sin\varphi \\ z = r\sin\theta \end{cases}$$

$\mathcal{D} = \{(r, \theta, \varphi) \in [0,1] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 2\pi]\}$.

$$\iiint_{\mathcal{D}} \sqrt{x^2+y^2+z^2}\, dx\, dy\, dz = \int_{r=0}^{1} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\varphi=0}^{2\pi} r \times r^2\cos\theta\, dr\, d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{2\pi} d\varphi \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta \int_{r=0}^{1} r^3\, dr = 2\pi \times 2 \times \frac{1}{4} = \boxed{\pi}.$$

Exercice 2 :

1\. $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \left(e^{x+y} + (x+y)e^{x+y}\right) dx + \left(e^{x+y} + (x+y)e^{x+y}\right) dy$

$$\boxed{df = (1+x+y)\, e^{x+y}\, (dx + dy)}.$$

$d\alpha = d(2xy) \wedge dx - d(x^2) \wedge dy = 2x\, dy \wedge dx - 2x\, dx \wedge dy$

$$\boxed{d\alpha = -4x\, dx \wedge dy}.$$

$d\beta = d\left(\frac{2x}{x^2+y^2}\right) \wedge dx + d\left(\frac{2y}{x^2+y^2} + 1\right) \wedge dy = -\frac{4xy}{(x^2+y^2)^2} dy \wedge dx - \frac{4xy}{(x^2+y^2)^2} dx \wedge dy$

$$\boxed{d\beta = 0}.$$

2\. Seule β est fermée. On cherche $g : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ telle que $\beta = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy$. Une primitive de $\frac{2x}{x^2+y^2}$ par rapport à x est $\ln(x^2+y^2) + K(y)$. En prenant $K(y) = y$, on obtient une primitive

$$\boxed{g(x,y) = \ln(x^2+y^2) + \ln y}.$$