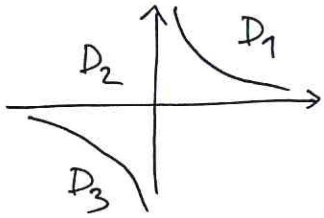
Exo 1

a) $\omega(x,y) = \dfrac{y^2-1}{(xy-1)^2}\,dx + \dfrac{1-x^2}{(xy-1)^2}\,dy$ domaine : $D = \{(x,y)\in\mathbb{R}^2 \mid xy \neq 1\}$

$= D_1 \cup D_2 \cup D_3$

union disjointe de trois domaines simplement connexes



$$d\omega(x,y) = \frac{\partial}{\partial x}\left(\frac{1-x^2}{(xy-1)^2}\right) - \frac{\partial}{\partial y}\left(\frac{y^2-1}{(xy-1)^2}\right) = \frac{-2x(xy-1)^{\cancel{2}} - (1-x^2)\cdot 2\cancel{(xy-1)}\cdot y}{(xy-1)^{\cancel{4}\,3}} - \frac{2y(xy-1)^{\cancel{2}} - (y^2-1)\cdot 2\cancel{(xy-1)}\cdot x}{(xy-1)^{\cancel{4}\,3}}$$

$$= \frac{-\cancel{2x^2y} + 2x - 2y + \cancel{2x^2y} - \cancel{2xy^2} + 2y - \cancel{2xy^2} + 2x}{(xy-1)^3} = 0$$ donc ω est fermée.

Par le lemme de Poincaré, ω est donc exacte sur chaque domaine D_i, $i=1,2,3$.

Cherchons sa primitive f sur chaque D_i :

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{y^2-1}{(xy-1)^2} & (1) \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{1-x^2}{(xy-1)^2} & (2) \end{cases} \implies f(x,y) \overset{(1)}{=} \int \frac{y^2-1}{(xy-1)^2}\,dx + g(y) \underset{\text{sur } D_1 \text{ et } D_3}{\overset{y\neq 0}{=}} \frac{y^2-1}{y}\int \frac{d(xy)}{(xy-1)^2} + g(y)$$

$$= -\frac{y^2-1}{y}\cdot\frac{1}{xy-1} + g(y) = \frac{1-y^2}{xy^2-y} + g(y)$$

$$\frac{\partial f}{\partial y} = \frac{-2y(xy^2-y) - (1-y^2)(2xy-1)}{(xy-1)^2y^2} + g'(y) = \frac{-2xy^3+2y^2-2xy+2xy^3+1-y^2}{(xy-1)^2y^2} + g'(y)$$

$$= \frac{y^2-2xy+1}{(xy-1)^2y^2} + g'(y) \overset{(2)}{=} \frac{1-x^2}{(xy-1)^2} \implies$$

$$g'(y) = \frac{y^2(1-x^2) - (y^2-2xy+1)}{(xy-1)^2y^2} = \frac{y^2-x^2y^2-y^2+2xy-1}{(xy-1)^2y^2} = \frac{-(xy-1)^2}{(xy-1)^2y^2} = -\frac{1}{y^2} \implies g(y) = \frac{1}{y} + c_i \quad i=1,3$$

$$\implies f(x,y) = \frac{1-y^2}{(xy-1)y} + \frac{1}{y} + c_i = \frac{\cancel{1}-y^2+xy\cancel{-1}}{(xy-1)y} + c_i = \frac{\cancel{y}(x-y)}{(xy-1)\cancel{y}} + c_i = \boxed{\frac{x-y}{xy-1} + c_i} \quad i=1,3$$

Après le calcul on voit que $f(x,y) = \dfrac{x-y}{xy-1} + c_i$ est aussi une primitive de ω sur D_2, donc $i=1,2,3$.

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b) $\eta(x,y) = (3x^2-2y^2)\,dx + 6xy\,dy$ domaine $D = \mathbb{R}^2$

$$d\eta(x,y) = \frac{\partial}{\partial x}(6xy) - \frac{\partial}{\partial y}(3x^2-2y^2) = 6y - (-4y) = 10y \neq 0$$

donc η n'est pas une 1-forme diff. fermée.

2) Exo 2 a) $\omega(x,y) = (y^2-3x^2)\,dx\wedge dy$ $D_\omega = \mathbb{R}^2$, $\omega \in \Omega^2(\mathbb{R}^2) \Rightarrow \omega$ fermée.

ω fermée sur $\mathbb{R}^2$ contractile $\xRightarrow{L.P.}$ ω exacte : $\exists\, \eta \in \Omega^1(\mathbb{R}^2)$ t.q. $\omega = d\eta$.

Si $\eta(x,y) = a(x,y)\,dx + b(x,y)\,dy$, on a $\omega = d\eta \Leftrightarrow \dfrac{\partial b}{\partial x} - \dfrac{\partial a}{\partial y} = y^2 - 3x^2$ (*)

i) Supposons $b(x,y) \equiv 0$, alors (*) $\Leftrightarrow \dfrac{\partial a}{\partial y} = 3x^2 - y^2 \Leftrightarrow$

$a(x,y) = \int (3x^2-y^2)\,dy + f(x) = 3x^2y - \frac{1}{3}y^3 + f(x)$, $\forall f \in C^\infty(\mathbb{R})$.

Donc $\eta(x,y) = \left(3x^2y - \frac{1}{3}y^3 + f(x)\right)dx$ est une primitive de ω.

ii) Si $a(x,y) \equiv 0$, on a (*) $\Leftrightarrow \dfrac{\partial b}{\partial x} = y^2 - 3x^2 \Leftrightarrow b(x,y) = xy^2 - x^3 + g(y)$ $\forall g \in C^\infty(\mathbb{R})$

donc $\eta(x,y) = (xy^2 - x^3 + g(y))\,dy$ est une primitive de ω.

iii) Toute combinaison $\eta = \alpha\left(3x^2y - \frac{1}{3}y^3 + f(x)\right)dx + \beta\left(xy^2 - x^3 + g(y)\right)dy$,

avec $\alpha + \beta = 1$, est encore une primitive de ω, par. ex. $\alpha = \beta = \frac{1}{2}$.

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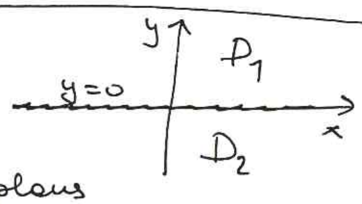
b) $\eta(x,y) = \left(y + \dfrac{x^2}{y^2}\right)dx\wedge dy$ $D_\eta = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$ $= D_1 \sqcup D_2$



union disjointe de deux demi-plans

$\eta \in \Omega^2(D_\eta)$ est fermée car 2-forme à $\mathbb{R}^2$, donc sur chaque composante connexe D_i $(i=1,2)$, qui est contractile, η est aussi exacte (L.P.).

Si $\xi = a\,dx + b\,dy$ est une primitive de η, on a : $\dfrac{\partial b}{\partial x} - \dfrac{\partial a}{\partial y} = y + \dfrac{x^2}{y^2}$ (*)

i) Si $b \equiv 0$, on a (*) $\Leftrightarrow \dfrac{\partial a}{\partial y} = -y - \dfrac{x^2}{y^2} \Leftrightarrow a(x,y) = \int\left(-y - \frac{x^2}{y^2}\right)dy + f(x) = -\frac{1}{2}y^2 + \frac{x^2}{y} + f(x)$,

donc $\xi = \left(\frac{x^2}{y} - \frac{1}{2}y^2 + f(x)\right)dx$ est une primitive de η.

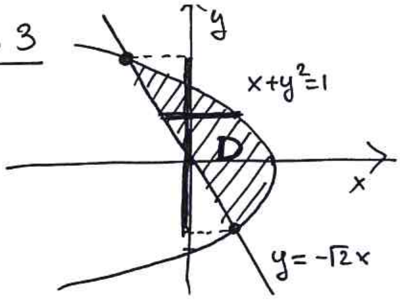
ii) Si $a \equiv 0$, on a (*) $\Leftrightarrow \dfrac{\partial b}{\partial x} = y + \dfrac{x^2}{y^2} \Leftrightarrow b(x,y) = \int\left(y + \frac{x^2}{y}\right)dx + g(y) = xy + \frac{x^3}{3y} + g(y)$

donc $\xi = \left(xy + \dfrac{x^3}{3y} + g(y)\right)dy$.

iii) Toute $\xi = \alpha\left(\frac{x^2}{y} - \frac{1}{2}y^2 + f(x)\right)dx + \beta\left(xy + \frac{x^3}{3y} + g(y)\right)dy$ avec $\alpha + \beta = 1$

est aussi une primitive de η.

Exo 3



points d'intersection des deux courbes :

$$\begin{cases} x+y^2=1 \\ y=-\sqrt{2}\,x \end{cases} \quad \begin{cases} x+2x^2=1 \\ y=-\sqrt{2}\,x \end{cases} \quad 2x^2+x-1=0 \quad x=\frac{-1\pm\sqrt{1+8}}{4}=\frac{-1\pm3}{4}=\begin{cases}-1\\ 1/2\end{cases}$$

$$\begin{cases} y=+\sqrt{2} \\ y=-\frac{\sqrt{2}}{2}\end{cases}$$

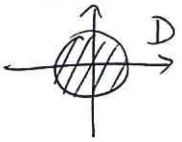
intersections : $(-1,+\sqrt{2})$ et $(1/2,-\sqrt{2}/2)$.

$$D=\left\{(x,y)\in\mathbb{R}^2 \;\middle|\; -\frac{\sqrt{2}}{2}\le y\le +\sqrt{2},\quad -\frac{y}{\sqrt{2}}\le x\le 1-y^2\right\}$$

$$\text{Aire } D=\iint_D dx\,dy=\int_{-\sqrt{2}/2}^{\sqrt{2}}dy\int_{-y/\sqrt{2}}^{1-y^2}dx=\int_{-\sqrt{2}/2}^{\sqrt{2}}\left(1-y^2+\frac{y}{\sqrt{2}}\right)dy=\left[y+\frac{y^2}{2\sqrt{2}}-\frac{1}{3}y^3\right]_{-\frac{\sqrt{2}}{2}}^{\sqrt{2}}$$

$$=\sqrt{2}+\frac{\cancel{2}}{2\sqrt{2}}-\frac{1}{3}\cdot2\sqrt{2}-\left(-\frac{\sqrt{2}}{2}\right)-\frac{1}{2\sqrt{2}}\cdot\frac{\cancel{2}}{4}+\frac{1}{3}\left(-\frac{2\sqrt{2}}{\cancel{8}\,4}\right)=2\sqrt{2}-\frac{2\sqrt{2}}{3}-\frac{\sqrt{2}}{8}-\frac{\sqrt{2}}{12}=\boxed{\frac{9\sqrt{2}}{8}}$$

Exo 4



$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le1\}$ ∂D = cercle unitaire : $x^2+y^2=1$ orienté ↺

a) $\omega(x,y)=xy^2\,dx\wedge dy\in\Omega^2(\mathbb{R}^2)$ est fermée sur domaine contractile, donc exacte.

Cherchons une primitive $\eta(x,y)=f(x,y)\,dy$:

$$d\eta=\omega\iff\frac{\partial f}{\partial x}=xy^2\iff f(x,y)=\int xy^2dx+g(y)=\frac{1}{2}x^2y^2+g(y).$$

Choix : $\eta(x,y)=\frac{1}{2}x^2y^2\,dy$. Alors, par le thm. de Stokes, on a

$$\iint_D xy^2\,dx\,dy=\oint_{\partial D}\frac{1}{2}x^2y^2\,dy\underset{\substack{x=\cos t\\ y=\sin t\quad dy=\cos t\,dt}}{=}\frac{1}{2}\int_0^{2\pi}\underbrace{\cos^3t}_{\cos t(1-\sin^2t)}\sin^2t\,dt=\frac{1}{2}\int_0^{2\pi}(\sin^2t-\sin^4t)\underbrace{\cos t\,dt}_{d(\sin t)}$$

$$=\frac{1}{2}\left[\frac{1}{3}\sin^3t-\frac{1}{5}\sin^5t\right]_0^{2\pi}=\boxed{0}.$$

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b) $\omega(x,y)=x^2y^2\,dx\wedge dy\in\Omega^2(\mathbb{R}^2)$ fermée sur contractile, donc exacte : primitive $\eta(x,y)=f(x,y)\,dy$ t.q. $\frac{\partial f}{\partial x}=x^2y^2$, i.e. $f(x,y)=\int x^2y^2dx=\frac{1}{3}x^3y^2$.

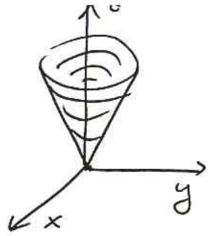
Donc

$$\iint_D x^2y^2\,dx\,dy=\oint_{\partial D}\frac{1}{3}x^3y^2\,dy\underset{\substack{x=\cos t\\ y=\sin t}}{=}\frac{1}{3}\int_0^{2\pi}\cos^4t\underbrace{\sin^2t}_{1-\cos^2t}\,dt=\frac{1}{3}\int_0^{2\pi}(\cos^4t-\cos^6t)\,dt$$

$$=\frac{1}{3}\int_0^{2\pi}\cos^4t\,dt-\frac{1}{3}\underbrace{\left[\frac{1}{6}\sin t\cos^5t\right]_0^{2\pi}}_{=0}-\frac{1}{3}\cdot\frac{5}{6}\int_0^{2\pi}\cos^4t\,dt=\frac{1}{3}\left(1-\frac{5}{6}\right)\int_0^{2\pi}\cos^4t\,dt$$

$$=\frac{1}{\cancel{18}\,6}\cdot\frac{3}{8}\cdot\big[t\big]_0^{2\pi}=\frac{2\pi}{48}=\boxed{\frac{\pi}{24}}.$$

4/ **Exo 5** a)



$C = \{(x,y,z) \mid x^2+y^2 \leq z^2,\ 0 \leq z \leq 3\}$

coord. cylindriques : $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$ éq. $\rho = z \in [0,3]$, $\theta \in [0,2\pi]$

$\Rightarrow$ paramétrisation de C :

$f(\rho,\theta) = (\rho\cos\theta,\ \rho\sin\theta,\ \rho)$, $\rho \in [0,3]$, $\theta \in [0,2\pi]$.

$$\begin{cases} \dfrac{\partial f}{\partial \rho} = (\cos\theta,\ \sin\theta,\ 1) \\ \dfrac{\partial f}{\partial \theta} = (-\rho\sin\theta,\ \rho\cos\theta,\ 0) \end{cases} \qquad \frac{\partial f}{\partial \rho} \wedge \frac{\partial f}{\partial \theta} = (-\rho\cos\theta, -\rho\sin\theta, \rho)$$

$$\left\| \frac{\partial f}{\partial \rho} \wedge \frac{\partial f}{\partial \theta} \right\| = \rho \cdot \sqrt{\cos^2\theta + \sin^2\theta + 1} = \sqrt{2}\,\rho.$$

$$\text{Aire } C = \iint_{[0,3]\times[0,2\pi]} \left\| \frac{\partial f}{\partial \rho} \wedge \frac{\partial f}{\partial \theta} \right\| d\rho\, d\theta = \sqrt{2} \int_0^3 \rho\, d\rho \int_0^{2\pi} d\theta = \sqrt{2}\left[\frac{1}{2}\rho^2\right]_0^3 \cdot [\theta]_0^{2\pi} = \frac{\sqrt{2}}{\cancel{2}} \cdot 9 \cdot \cancel{2}\pi = \boxed{9\sqrt{2}\pi}.$$

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b)

$\Omega = \{(x,y,z) \mid x^2+y^2 \leq z,\ 0 \leq z \leq 3\}$

$\partial\Omega = C \cup D$ où $C =$ cône $x^2+y^2 = z^2$, $0 \leq z \leq 3$

$D =$ disque $x^2+y^2 \leq 9$, $z = 3$

Puisque $dx\wedge dy \wedge dz = d(x\, dy\wedge dz)$, en utilisant le thm. de Gauss-Ostrogradski on a :

$$\text{Vol } \Omega = \iiint_\Omega dx\,dy\,dz = \left| \iint_{\partial\Omega = C\cup D} x\, dy\wedge dz \right| = \left| \iint_C x\,dy\,dz + \iint_D x\,dy\,dz \right|$$

Sur D on a $z = 3$, donc $dz = 0$ $\Rightarrow \iint_D x\,dy\,dz = 0$. Alors

$$\text{Vol } \Omega = \left| \iint_C x\,dy\,dz \right| = \left| \iint_{[0,2\pi]\times[0,3]} \rho\cos\theta \cdot \rho\cos\theta\, d\theta\wedge d\rho \right| = \left| \int_0^3 \rho^2\, d\rho \int_0^{2\pi} \cos^2\theta\, d\theta \right|$$

Sur C : $x = \rho\cos\theta$

$y = \rho\sin\theta \quad dy = \sin\theta\, d\rho + \rho\cos\theta\, d\theta$

$z = \rho \quad dz = d\rho$

$\Big\} \Rightarrow dy\wedge dz = \rho\cos\theta\, d\theta\wedge d\rho$

$$= \left[\frac{1}{3}\rho^3\right]_0^3 \cdot \frac{1}{2}[\theta]_0^{2\pi} = \frac{\cancel{3}\cdot 9}{\cancel{3}} \cdot \frac{1}{\cancel{2}} \cdot \cancel{2}\pi = \boxed{9\pi}.$$