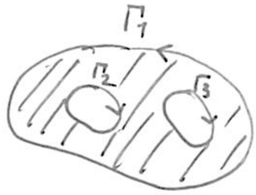
# Fiche TD2

**Exo 1.** $D \subseteq \mathbb{R}^2$, $\partial D = \bigcup_{j=1}^{N} \Gamma_j$




$$\mathrm{Aire}(D) := \iint_D dx\,dy = \iint_D F(x,y)\,d\lambda(x,y) \quad \text{pour} \quad F(x,y) = 1.$$

$$\text{Or } F(x,y) = \mathrm{rot}\, X(x,y) \text{ où } X = \begin{pmatrix} 0 \\ x \end{pmatrix} \text{ car } \mathrm{rot}\begin{pmatrix} 0 \\ x \end{pmatrix} = \frac{\partial}{\partial x}(x) - 0 = 1$$

$$= \mathrm{rot}\, Y(x,y) \text{ ou } Y = \begin{pmatrix} -y \\ 0 \end{pmatrix} \text{ car } \mathrm{rot}\begin{pmatrix} -y \\ 0 \end{pmatrix} = \frac{\partial 0}{\partial x} - \frac{\partial}{\partial y}(-y) = 1.$$

Donc

$$\mathrm{Aire}(D) = \iint_D dx\,dy = \iint_D F(x,y)\,d\lambda \overset{GR}{=\!=} \int_{\partial D} X\,dt = \sum_{j=1}^{N} \int_{\Gamma_j} X\,dt$$

Or, si $\Gamma_j$ est paramétrée par $\gamma_j(t) = (x(t), y(t))$, $t \in [a_j, b_j]$

$$\text{on a } \int_{\Gamma_j} X\,dt := \int_{a_j}^{b_j} \left\langle X(x(t),y(t)), \gamma_j'(t) \right\rangle dt = \int_{a_j}^{b_j} \left\langle \begin{pmatrix} 0 \\ x(t) \end{pmatrix}, \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} \right\rangle dt$$

$$= \int_{a_j}^{b_j} x(t) \underbrace{\dot{y}(t)\,dt}_{dy(t)} = \int_{a_j}^{b_j} x(t)\,dy(t)$$

$$\text{Donc } \mathrm{Aire}(D) = \int_D d\lambda = \int_{\partial D} X\,dt = \sum_{j=1}^{N} \int_{a_j}^{b_j} x(t)\,dy(t) = \int_{\partial D} x\,dy.$$

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**Exo 2** voir Fa1

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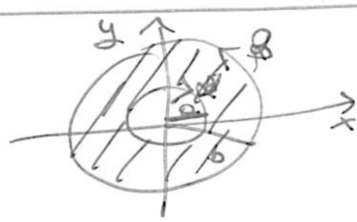
**Exo 3** $D = \{(x,y) \mid a^2 \le x^2 + y^2 \le b^2\}$ $0 < a < b$




(1) $$\mathrm{Aire}\, D = \iint_D dx\,dy = \int_{\partial D} x\,dy = \int_\alpha x\,dy + \int_\beta x\,dy$$

$\partial D = \alpha \sqcup \beta$, $\beta(t) = (b\cos t, b\sin t)$ $t \in [0, 2\pi]$

$\alpha(t) = (a\cos(-t), a\sin(-t)) = (a\cos t, -a\sin t)$ $t \in [0, 2\pi]$

$$= \int_0^{2\pi} a\cos t \cdot \underbrace{d(-a\sin t)}_{-a\cos t\,dt} + \int_0^{2\pi} b\cos t\, \underbrace{d(b\sin t)}_{b\cos t\,dt} = -a^2\int_0^{2\pi} \cos^2 t\,dt + b^2 \int_0^{2\pi} \cos^2 t\,dt$$

$$= (b^2 - a^2)\frac{1}{2}\left[\cos t \sin t + t\right]_0^{2\pi} = (b^2 - a^2)\frac{2\pi}{2}.$$

(2) $I = \iint_D 2xy\,dx\,dy$

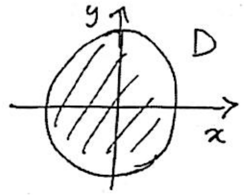
$$F(x,y) = 2xy = \mathrm{rot}\, X\ ? \quad X = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix} \quad \mathrm{rot}\, X = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \overset{!}{=} 2xy \quad \underline{\text{ex}} \begin{cases} P = 0 \\ Q = \int 2xy\,dx = x^2 y \end{cases}$$

$$I = \iint_D 2xy\,dx\,dy \overset{GR}{=\!=} \int_{\partial D} X\,dt = \int_{\partial D} \left\langle \begin{pmatrix} P \\ Q \end{pmatrix}, \begin{pmatrix} dx \\ dy \end{pmatrix} \right\rangle = \int_{\alpha \cup \beta} x^2 y\,dy = \int_\alpha x^2 y\,dy + \int_\beta x^2 y\,dy$$

$$= \int_0^{2\pi} -a^2\cos^2 t\, a\sin t \cdot -a\cos t\,dt + \int_0^{2\pi} b^2\cos^2 t\, b\sin t\, b\cos t\,dt = (a^4 + b^4)\int_0^{2\pi} \cos^3 t \underbrace{\sin t\,dt}_{d\cos t} = (b^2 - a^2)\left[\frac{\cos^4 t}{4}\right]_0^{2\pi} = 0.$$

**Exo 7** $D = \{(x,y) \mid x^2+y^2 \le 1\}$ $\quad K = \{(x,y) \mid x \ge 0,\ y \ge 0,\ x+y \le 1\}$

(1) Calculer $I = \int_{\partial D} \omega$ et $J = \int_{\partial K} \omega$ pour $\omega = xy^2 dx + 2xy\, dy \in \Omega^1(\mathbb{R}^2)$

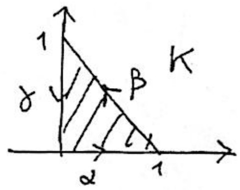
• 

$$\partial D = \{(x,y) = \gamma(t) = (\cos t, \sin t) \mid t \in [0, 2\pi]\}$$

$$\begin{cases} x(t) = \cos t \\ y(t) = \sin t \end{cases} \Rightarrow \begin{cases} dx = -\sin t\, dt \\ dy = \cos t\, dt \end{cases}$$

$$I = \int_0^{2\pi} (-\cos t \sin^3 t + 2\cos^2 t \sin t)\, dt = \left[-\frac{1}{4}\sin^4 t - \frac{2}{3}\cos^3 t\right]_0^{2\pi} = 0.$$

$u = \sin t$, $du = \cos t\, dt$ $\qquad$ $v = \cos t$, $dv = -\sin t\, dt$

• 

$$\partial K = \alpha \cup \beta \cup \gamma \quad \text{avec} \quad \begin{aligned} \alpha(t) &= (t, 0) & t \in [0,1] \\ \beta(t) &= (1-t, t) & t \in [0,1] \\ \gamma(t) &= (0, 1-t) & t \in [0,1] \end{aligned}$$

$$J = \int_\alpha \underbrace{xy^2 dx + 2xy\, dy}_{y=0 \Rightarrow\ =0} + \int_\beta xy^2 dx + 2xy\, dy + \int_\gamma \underbrace{xy^2 dx + 2xy\, dy}_{x=0 \Rightarrow\ =0}$$

$\beta$ : $x = 1-t$, $dx = -dt$ ; $y = t$, $dy = dt$

$$= 0 + \int_0^1 \left(-(1-t)t^2 + 2(1-t)t\right) dt + 0 = \left[\frac{1}{4}t^4 - t^3 + t^2\right]_0^1 = \frac{1}{4} - 1 + 1 = \frac{1}{4}.$$

$-t^2 + t^3 + 2t - 2t^2$

$t^3 - 3t^2 + 2t$

(2) Green-Riemann : $\int_{\partial D = \gamma} \cancel{\omega}\ X \cdot dr = \iint_D \cancel{d\omega}\ \mathrm{rot} X\, dxdy$ si $\cancel{\omega}$ X est déf sur tout D.

$\cancel{\omega}\, X = (xy^2 \cancel{dx} + 2xy \cancel{dy}) \Rightarrow \mathrm{rot} X = \frac{\partial(2xy)}{\partial x} - \frac{\partial}{\partial y}(xy^2) = 2y - 2xy$

$\cancel{d\omega = 2xy\, dy \wedge dx + 2x\, dx \wedge dy} = 2(1-x)y\ \cancel{dx \wedge dy}$

• $D = \{(r, \theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$ $\quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow dx\,dy = r\,dr\,d\theta$

~~convention : $\iint f(x,y)\, dx \wedge dy := \iint f(x,y)\, dx dy$ et $\iint f \cdot dy \wedge dx = -\iint f\, dx dy$.~~

$$I = \iint_D 2(1-x)y\, dx dy = \int_0^1 dr \int_0^{2\pi} d\theta\ 2(1 - r\cos\theta) r\sin\theta \cdot r\, dr\, d\theta$$

$$= 2\int_0^1 dr \int_0^{2\pi} r^2(1 - r\cos\theta)\sin\theta\, d\theta = 2\int_0^1 r^2 dr \int_0^{2\pi} (\sin\theta - r\cos\theta\sin\theta)\, d\theta$$

$u = \sin\theta$, $du = \cos\theta\, d\theta$

$$= 2\int_0^1 r^2 \left[-\cos\theta - \frac{1}{2} r \sin^2\theta\right]_0^{2\pi} dr = 0$$

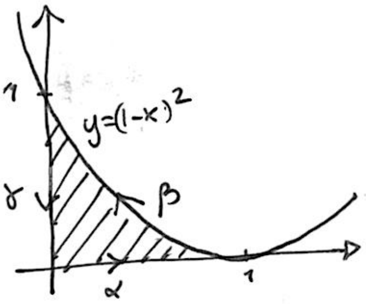
• $K = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1-x\}$

$$J = \iint_K 2(1-x)y\, dx dy = 2\int_0^1 (1-x)\, dx \int_0^{1-x} y\, dy = \cancel{2}\int_0^1 (1-x)\left[\frac{1}{\cancel{2}} y^2\right]_0^{1-x} dx$$

$$= \int_0^1 (1-x)^3 dx = -\int_1^0 u^3 du = \left[\frac{1}{4}u^4\right]_0^1 = \frac{1}{4}.$$

$u = 1-x$, $du = -dx$

Exo 4 $\quad \mathcal{D} = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le (1-x)^2\}$




(1) $\text{Aire}(D) := \iint_D dx\,dy = \int_0^1 dx \int_0^{(1-x)^2} dy$

$= \int_0^1 (1-x)^2\,dx \underset{u=1-x \Rightarrow du=-dx}{=} -\int_1^0 u^2\,du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}$

Avec Green-Riemann : $\text{rot}\,X = 1 \Rightarrow$ on choisit $X = \begin{pmatrix} 0 \\ x \end{pmatrix}$ et $\iint_D \text{rot}\,X\,dx\,dy = \int_{\partial D} X\,dt$

$\partial D = \alpha \cup \beta \cup \gamma$ avec
$$\left.\begin{aligned} \alpha(t) &= (t,0) \\ \beta(t) &= (1-t, (1-(1-t))^2) = (1-t, t^2) \\ \gamma(t) &= (0, 1-t) \end{aligned}\right\} t \in [0,1]$$

$\text{Aire}(D) = \int_\alpha \underbrace{x\,dy}_{0 \text{ sur } \alpha} + \int_\beta x\,dy + \int_\gamma \underbrace{x\,dy}_{0 \text{ sur } \gamma} = \int_0^1 2t(1-t)\,dt = \int_0^1 (2t-2t^2)\,dt = \left[t^2 - \frac{2}{3}t^3\right]_0^1$

$\beta \begin{cases} x = 1-t \\ y = t^2 \Rightarrow dy = 2t\,dt \end{cases}$ $\qquad = 1 - \frac{2}{3} = \frac{1}{3}$ Ok !

(2) $I = \iint_D (x^2+y^2)\,dx\,dy = \int_0^1 dx \int_0^{(1-x)^2} (x^2+y^2)\,dy = \int_0^1 \left[x^2y + \frac{1}{3}y^3\right]_0^{(1-x)^2} dx$

$= \int_0^1 \left(x^2(1-x)^2 + \frac{1}{3}(1-x)^6\right) dx = -\int_1^0 \left((1-u)^2u^2 + \frac{1}{3}u^6\right) du = \int_0^1 \left(u^2 - 2u^3 + u^4 + \frac{1}{3}u^6\right) du$

$u = 1-x$, $du = -dx$ $\qquad x = 1-u$

$= \left[\frac{1}{3}u^3 - \frac{2}{4}u^4 + \frac{1}{5}u^5 + \frac{1}{21}u^7\right]_0^1 = \frac{1}{3} + \frac{1}{2} + \frac{1}{5} + \frac{1}{3\times 7} = \frac{70-105+42+10}{2\times3\times5\times7} = \frac{17}{210}$

(3) $J = \int_{\partial D} (2y-y^3)\,dx + (2x+x^3)\,dy = \underbrace{\int_\alpha}_{\substack{dy=0 \\ y=0}} + \int_\beta + \underbrace{\int_\gamma}_{\substack{dx=0 \\ x=0}} = \int_\beta \Rightarrow \begin{matrix} x = 1-t & dx = -dt \\ y = t^2 & dy = 2t\,dt \end{matrix}$

$= \int_0^1 (2t^2-t^6)(-dt) + (2(1-t)+(1-t)^3)\,2t\,dt = \int_0^1 \Big(-2t^2+t^6+2t(\underbrace{2-2t+1-3t+3t^2-t^3}_{3-5t+3t^2-t^3})\Big)\,dt$

$= \int_0^1 (-2t^2+t^6+6t-10t^2+6t^3-2t^4)\,dt = \int_0^1 (6t-12t^2+6t^3-2t^4+t^6)\,dt$

$= \left[3t^2-4t^3+\frac{3}{2}t^4-\frac{2}{5}t^5+\frac{1}{7}t^7\right]_0^1 = \underbrace{3-4}_{-1}+\frac{3}{2}-\frac{2}{5}+\frac{1}{7} = \frac{-70+105-28+10}{2\times5\times7} = \frac{17}{70}$

(4) On remarque que $J = \frac{17}{70} = 3 \times \frac{17}{210} = 3I$ !

L'explication est que si on pose $X = \begin{pmatrix} 2y-y^3 \\ 2x+x^3 \end{pmatrix}$, alors

$\text{rot}\,X = \frac{\partial}{\partial x}(2x+x^3) - \frac{\partial}{\partial y}(2y-y^3) = \cancel{2}+3x^4-\cancel{2}+3y^4 = 3(x^4+y^4) = 3F$

où $F(x,y) = x^4+y^4$, et le thm de Green-Riemann dit exactement que

$I = \iint_D F\,dx\,dy = \frac{1}{3}\int_{\partial D} X\,dt = \frac{1}{3}J$.

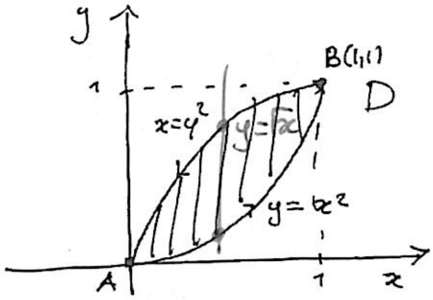
[Fiche TD4 - suite] (1

~~Exo 9~~ Exo 5  $D$ = domaine de $\mathbb{R}^2$ délimité par

$P_1 = \{(x,y) \mid y = x^2\}$

$P_2 = \{(x,y) \mid x = y^2\}$

$\cancel{\omega}(x,y) = \Big((2x+3y^2)\cancel{dx}, (6xy+\cos y)\cancel{dy}\Big)$




(1) $\text{Aire } D = \iint_D dx\,dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy = \int_0^1 (\sqrt{x} - x^2)\,dx = \left[\frac{1}{\frac{1}{2}+1} x^{3/2} - \frac{1}{3}x^3\right]_0^1$ $x^{\frac12}$

$D = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}$ $= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

$1 = \text{rot}\begin{pmatrix} 0 \\ x \end{pmatrix}$

GR: $\text{Aire } D = \iint_D \overset{dx\,dy}{\cancel{d(x\,dy)}} \overset{GR}{=} \int_{\partial D} x\,dy = \int_{P_1} x\,dy + \int_{P_2} x\,dy =$

$P_1 : \begin{cases} \alpha(t) = (t, t^2) & t \in [0,1] \\ x = t,\ y = t^2 \Rightarrow dy = 2t\,dt \end{cases}$  $P_2 : \begin{cases} \beta(t) = ((1-t)^2, 1-t) & t \in [0,1] \\ x = (1-t)^2,\ y = 1-t \Rightarrow dy = -dt \end{cases}$

$= \int_0^1 2t^2 dt + \int_0^1 -(1-t)^2 dt = \frac{2}{3}\left[t^3\right]_0^1 + \int_1^0 u^2 du = \frac{2}{3} - \left[\frac{1}{3}u^3\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$ ok.

$1-t = u$
$du = -dt$

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(2) ~~M.q. $\omega$ est fermée~~ : $d\omega = 6y\,\underbrace{dy \wedge dx}_{-dx \wedge dy} + 6y\,dx \wedge dy = 0 \Rightarrow \omega$ fermée.

(3) M.q. $\omega$ est exacte : $\omega$ fermée déf sur $\mathbb{R}^2$ simplement convexe $\overset{\text{Poincaré}}{\Longrightarrow}$ $\omega$ exacte.

Primitive $f$ tq $df = \omega$ ; i.e. $\begin{cases} \dfrac{\partial f}{\partial x} = 2x + 3y^2 & (1) \\ \dfrac{\partial f}{\partial y} = 6xy + \cos y & (2) \end{cases}$ $(1) \Rightarrow f(x,y) = \int (2x+3y^2)\,dx + g(y)$
$= x^2 + 3xy^2 + g(y)$

$\frac{\partial f}{\partial y} = 6xy + g'(y) \overset{(2)}{=} 6xy + \cos y \Rightarrow g'(y) = \cos y \Rightarrow g(y) = \sin y + c$

Une primitive est $f(x,y) = x^2 + 3xy^2 + \sin y$ (suffit $c = 0$).

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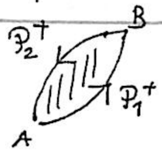
(4) $A(0,0)$, $B(1,1)$ calculer $\int_{A, P_1^+}^{B} \cancel{\omega} + \int_{B, P_2^+}^{A} \cancel{\omega} =$




$X = \begin{pmatrix} 2x+3y^2 \\ 6xy+\cos y \end{pmatrix}$

On a $P_1^+ \cup P_2^+ = \partial D^+$, donc :

$\int_{A, P_1^+}^{B} \cancel{\omega}\,dl + \int_{B, P_2^+}^{A} \cancel{\omega}\,dl = \oint_{\partial D^+} \cancel{\omega}\,dl \overset{GR}{=} \iint_D \text{rot} X\, d\lambda \quad \cancel{\oint_{\partial D} df} = 0$ car $\text{rot} X = \frac{\partial}{\partial x}(6xy+\cos y) - \frac{\partial}{\partial y}(2x+3y^2) = 6y - 6y = 0$

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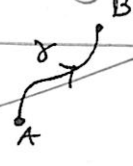
En effet, si $\omega = df$, on a $\int_A^B \omega = \int_A^B df = f(B) - f(A)$




quelconque soit la courbe $\gamma : A \to B$.

Donc si $\partial D^+ =$ est une courbe fermée, quelconque soit $A \in \partial D^+$ on a

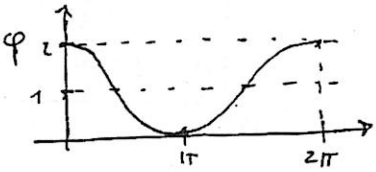
$\oint_{\partial D^+} \omega = \int_A^A df = f(A) - f(A) = 0.$

4) Exo6

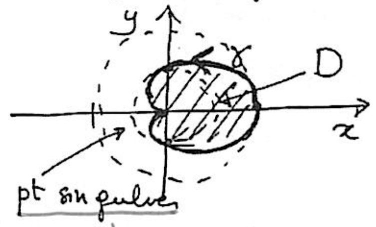
**Exo 10** <u>Cardioïde</u> $\gamma(\varphi) = \rho(\varphi)e^{i\varphi}$ avec $\rho(\varphi) = 1+\cos\varphi$, $\varphi \in [0, 2\pi]$.

· Calculer $L_0^{2\pi}(\gamma)$, Aire $(D)$ où $\partial D = \gamma$ et vérifier l'inégalité isopérimétrique.

· $\rho(\varphi) = 1+\cos\varphi$




$\Longrightarrow$




car $\rho'(\varphi) = -\sin\varphi$
en $\varphi = \pi$ on a $\begin{cases} \rho(\pi) = 1-1=0 \\ \rho'(\pi) = 0 \end{cases}$

· $L_0^{2\pi}(\gamma) = \left| \int_0^{2\pi} \|\gamma'(\varphi)\| \, d\varphi \right|$

$\gamma(\varphi) = (1+\cos\varphi)(\cos\varphi, \sin\varphi) = \left((1+\cos\varphi)\cos\varphi, (1+\cos\varphi)\sin\varphi\right)$

$\gamma'(\varphi) = \left(-\sin\varphi - 2\cos\varphi\sin\varphi, \cos\varphi - \sin^2\varphi + \cos^2\varphi\right) = \left(-\sin\varphi - \sin(2\varphi), \cos\varphi + \cos(2\varphi)\right)$

$\|\gamma'(\varphi)\|^2 = \sin^2\varphi + 2\sin\varphi\sin(2\varphi) + \sin^2(2\varphi) + \cos^2\varphi + 2\cos\varphi\cos(2\varphi) + \cos^2(2\varphi)$

$= 1 + 2\cos(2\varphi - \varphi) + 1 = 2 + 2\cos\varphi = 2(1+\cos\varphi)$

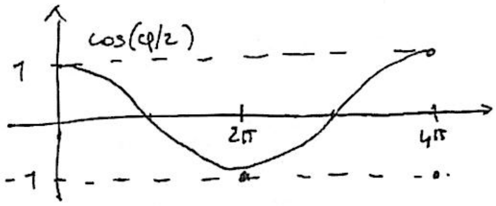
Puisque $\cos\varphi = \cos(2\cdot\varphi/2) = 2\cos^2(\varphi/2) - 1$, on a $\cos\varphi + 1 = 2\cos^2(\varphi/2)$

donc $\|\gamma'(\varphi)\|^2 = 2(1+\cos\varphi) = 4\cos^2(\varphi/2)$, i.e. $\|\gamma'(\varphi)\| = 2|\cos(\varphi/2)|$

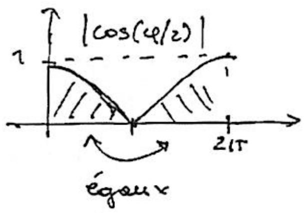
Alors $L_0^{2\pi}(\gamma) = \left| \int_0^{2\pi} 2|\cos(\varphi/2)| \, d\varphi \right| = \left| 4\int_0^{\pi} \cos(\varphi/2)\, d\varphi \right| = 4\times 2 \int_0^{\pi} \cos(\varphi/2)\, d(\varphi/2)$

$= 8\left[\sin(\varphi/2)\right]_0^{\pi}$

$= 8\sin\pi/2 - 0 = 8$




$\Longrightarrow$




· Aire $D = \iint_D dx\,dy \overset{GR}{=} \oint_\gamma x\,dy = \int_0^{2\pi} (1+\cos\varphi)\cos\varphi\,(\cos\varphi + 2\cos^2\varphi - 1)\,d\varphi =$

$\gamma: \begin{cases} x = (1+\cos\varphi)\cos\varphi \\ y = (1+\cos\varphi)\sin\varphi \Rightarrow dy = (\cos\varphi + \cos(2\varphi))\,d\varphi \end{cases}$ $\quad \cos(2\varphi) = 2\cos^2\varphi - 1$

$= \int_0^{2\pi} (\cos\varphi + \cos^2\varphi)(2\cos^2\varphi + \cos\varphi - 1)\,d\varphi = \int_0^{2\pi} \left(2\cos^4\varphi + \cos^3\varphi - \cancel{\cos^2\varphi} + 2\cos^3\varphi + \cancel{\cos^2\varphi} - \cos\varphi\right) d\varphi$

$= \int_0^{2\pi} \left(2\cos^4\varphi + 3\underset{\cos\varphi(1-\sin^2\varphi)}{\cos^3\varphi} - \cos\varphi\right) d\varphi = \int_0^{2\pi} 2\cos^4\varphi\,d\varphi - \int_0^{2\pi} 3\sin^2\varphi\cos\varphi\,d\varphi$

$\int \cos^4\varphi\,d\varphi = \cos^3\varphi\sin\varphi + 3\int\cos^2\varphi\underbrace{\sin^2\varphi}_{1-\cos^2\varphi}\,d\varphi = \cos^3\varphi\sin\varphi + 3\int\cos^2\varphi\,d\varphi - 3\int\cos^4\varphi\,d\varphi$

$u = \cos^3\varphi$, $v' = \cos\varphi$
$u' = -3\cos^2\varphi\cdot\sin\varphi$, $v = \sin\varphi$ $\quad \left| \quad \int\cos^2\varphi\,d\varphi = \frac{1}{2}(\cos\varphi\sin\varphi + \varphi) \right. \Longrightarrow$

$\int\cos^4\varphi\,d\varphi = \frac{1}{4}\left[\cos^3\varphi\sin\varphi + \frac{3}{2}\cos\varphi\sin\varphi + \frac{3}{2}\varphi\right]$

Aire $D = \frac{1}{2}\cancel{\frac{2}{4}}\left[\cos^3\varphi\sin\varphi + \frac{3}{2}\cos\varphi\sin\varphi + \frac{3}{2}\varphi\right]_0^{2\pi} - \left[\sin^3\varphi\right]_0^{2\pi} = \frac{1}{\cancel{2}}\cdot\frac{3}{2}\cancel{2}\pi = \frac{3\pi}{2}$

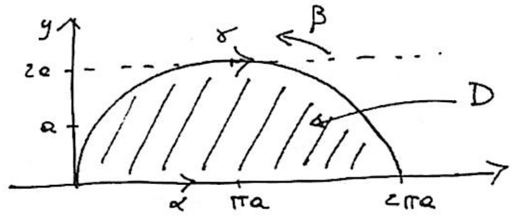
· <u>Inégalité isopérimétrique</u>: $\boxed{(L(\gamma))^2 \geq 4\pi \text{ Aire } D}$ En effet $L(\gamma)^2 = 64 > 4\pi\cdot\frac{3\pi}{2} \approx 6\times 9{,}86$
$= 59{,}16$

**Exo 11** ~~**Exo 7**~~ Cycloïde $\gamma(t) = a(t-\sin t,\ 1-\cos t)$ $\quad t\in[0,2\pi]$ $\quad a>0$ $\quad \alpha$ = segment $[0,1]\times\{0\}$

Calculer $L_0^{2\pi}(\gamma\cup\alpha)$, Aire$(D)$ où $\partial D = \gamma\cup\alpha$ et vérifier l'inégalité isopérimétrique.

- Courbe déjà vue et étudiée :

  $2\pi$-périodique




$\partial D = \alpha\cup\beta$ où $\alpha(t) = (at, 0)$, $t\in[0,2\pi]$

$\beta(t) = \gamma(2\pi - t) = a(2\pi - t - \sin(2\pi - t),\ 1-\cos(2\pi-t))$

$= a(2\pi - t + \sin t,\ 1-\cos t)$ $\quad t\in[0,2\pi]$

- $L(\beta\cup\alpha) = L(\partial D^+) = L_0^{2\pi}(\alpha) + L_0^{2\pi}(\beta)$

$\alpha'(t) = (a,0) \Rightarrow \|\alpha'(t)\| = a \Rightarrow L_0^{2\pi}(\alpha) = a\int_0^{2\pi} dt = 2\pi a$

$\beta'(t) = a(-1+\cos t,\ \sin t) \Rightarrow \|\beta'(t)\| = a\sqrt{1-2\cos t + \cos^2 t + \sin^2 t} = a\sqrt{2(1-\cos t)} = 2a\sin(t/2)$ sur $[0,2\pi]$

$\Rightarrow L(\partial D^+) = 2\pi a + \int_0^{2\pi} 2a\sin(t/2)\,dt = 2\pi a + 4a\left[-\cos(t/2)\right]_0^{2\pi} = 2\pi a + 4a(+1+1) = 2\pi a + 8a$

$= 2a(\pi+4)$

- Aire$(D) = \left|\iint_D dx\,dy\right| \overset{GR}{=} \left|\oint_{\partial D^+} x\,dy\right| = \left|\int_\alpha x\,dy + \int_\beta x\,dy\right|$

le long de $\alpha$ : $x = at$, $y = 0 \Rightarrow dy = 0\,dt$

le long de $\beta$ : $x = a(2\pi - t + \sin t)$, $y = a(1-\cos t) \Rightarrow dy = a\sin t\,dt$

Aire$(D) = \left|\int_0^{2\pi} 0 \cdot + \int_0^{2\pi} a^2(2\pi - t + \sin t)\sin t\,dt\right| = a^2\left|2\pi\underbrace{\int_0^{2\pi}\sin t\,dt}_{=0} - \int_0^{2\pi} t\sin t\,dt + \int_0^{2\pi}\sin^2 t\,dt\right|$

$\int_0^{2\pi} t\sin t\,dt = \left[-t\cos t\right]_0^{2\pi} + \underbrace{\int_0^{2\pi}\cos t\,dt}_{=0} = -2\pi + 0 = -2\pi$

$u = t \quad v' = \sin t$
$u' = 1 \quad v = -\cos t$

$\int_0^{2\pi}\sin^2 t\,dt = \frac{1}{2}\left[t - \sin t\cos t\right]_0^{2\pi} = \frac{1}{2}(2\pi - 0) = \pi$

$\Rightarrow$ Aire$(D) = a^2\left|-(-2\pi) + \pi\right| = 3\pi a^2$

- Inégalité isopérimétrique : $(L(\partial D))^2 \geq 4\pi$ Aire$(D)$ $\quad (*)$

$(L(\partial D))^2 = (2a(\pi+4))^2 = 4a^2(\pi^2 + 8\pi + 16)$

$4\pi$ Aire$(D) = 12a^2\pi^2$

Donc $(*) \Leftrightarrow \cancel{4a^2}(\pi^2 + 8\pi + 16) \geq \underset{3}{\cancel{12}}\cancel{a^2}\pi^2 \Leftrightarrow \pi^2 + 8\pi + 16 \geq 3\pi^2 \Leftrightarrow 2\pi^2 - 8\pi - 16 \leq 0$

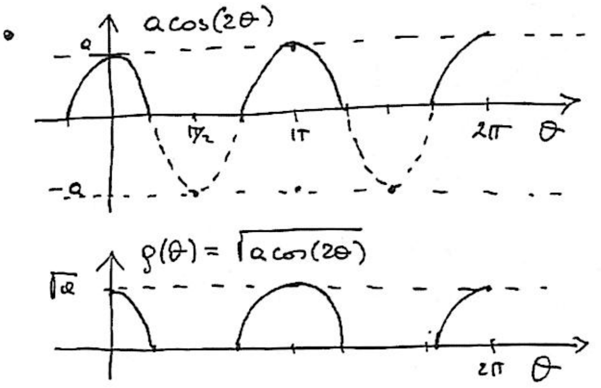
$\Leftrightarrow \pi^2 - 4\pi - 8 \leq 0$

Or $3 \leq \pi \leq 4 \Rightarrow \left.\begin{matrix}\pi^2 \leq 16\\ -4\pi \leq -12\end{matrix}\right\} \Rightarrow \pi^2 - 4\pi - 8 \leq 16 - 12 - 8 = -4 < 0$ donc $(*)$ est vérifiée.

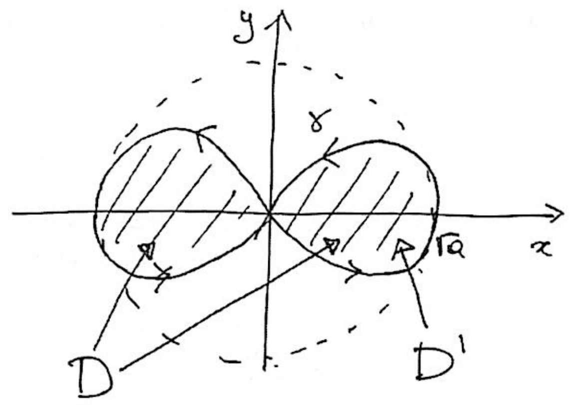
16) **Exo 12** ~~Exo 8~~ Lemniscate de Bernoulli : $\gamma(\theta) = \rho(\theta) e^{i\theta}$ avec $\rho(\theta) = \sqrt{a\cos(2\theta)}$, $a>0$

bien déf pour $\theta \in [-\pi/4, \pi/4] \cup [3\pi/4, 5\pi/4]$.

Dessiner la lemniscate et calculer $Aire(D)$ où $\partial D = \gamma$.

- 


$\Longrightarrow$




- Soit D' la portion du plan entourée par $\gamma\big|_{[-\pi/4,\pi/4]}$, alors

$$Aire(D) = 2\,Aire(D') = 2\left|\iint_{D'} dx\,dy\right| \overset{GR}{=} 2\left|\oint_{\partial D'} x\,dy\right| = 2\left|\int_{-\pi/4}^{\pi/4} x(\theta)\,y'(\theta)\,d\theta\right|$$

$$\gamma : \begin{cases} x(\theta) = \sqrt{a\cos(2\theta)}\cos\theta \\ y(\theta) = \sqrt{a\cos(2\theta)}\sin\theta \end{cases} \Longrightarrow y'(\theta) = \frac{-\cancel{2}a\sin(2\theta)}{\cancel{2}\sqrt{a\cos(2\theta)}}\sin\theta + \sqrt{a\cos(2\theta)}\cos\theta$$

$$= \frac{a}{\sqrt{a\cos(2\theta)}}\left(-\sin(2\theta)\sin\theta + \cos(2\theta)\cos\theta\right) = \frac{a\cos(3\theta)}{\sqrt{a\cos(2\theta)}}$$

$$Aire(D) = 2\left|\int_{-\pi/4}^{\pi/4} \cancel{\sqrt{a\cos(2\theta)}}\cos\theta \cdot \frac{a\cos(3\theta)}{\cancel{\sqrt{a\cos(2\theta)}}}\,d\theta\right| = 2a\left|\int_{-\pi/4}^{\pi/4}\cos\theta\cos(3\theta)\,d\theta\right|$$

primitive : $\int \cos\theta\cos(3\theta)\,d\theta = \frac{1}{3}\cos\theta\sin(3\theta) + \frac{1}{3}\int\sin\theta\sin(3\theta)\,d\theta =$

$\begin{vmatrix} u = \cos\theta & v' = \cos(3\theta) \\ u' = -\sin\theta & v = \frac{1}{3}\sin(3\theta)\end{vmatrix}$ $\qquad$ $\begin{vmatrix} u = \sin\theta & v' = \sin(3\theta) \\ u' = \cos\theta & v = -\frac{1}{3}\cos(3\theta)\end{vmatrix}$

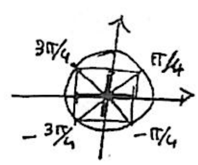
$$= \frac{1}{3}\cos\theta\sin(3\theta) + \frac{1}{3}\left[-\frac{1}{3}\sin\theta\cos(3\theta) + \frac{1}{3}\int\cos\theta\cos(3\theta)\,d\theta\right]$$

$$= \frac{1}{3}\cos\theta\sin(3\theta) - \frac{1}{9}\sin\theta\sin(3\theta) + \frac{1}{9}\int\cos\theta\cos(3\theta)\,d\theta$$

$$\Longrightarrow \underbrace{\left(1-\frac{1}{9}\right)}_{8/9}\int\cos\theta\cos(3\theta)\,d\theta = \frac{1}{3}\cos\theta\sin(3\theta) - \frac{1}{9}\sin\theta\sin(3\theta)$$

$$\Longrightarrow \int\cos\theta\cos(3\theta)\,d\theta = \frac{\overset{3}{\cancel{9}}}{8}\cdot\frac{1}{\cancel{3}}\cos\theta\sin(3\theta) - \frac{\cancel{9}}{8}\cdot\frac{1}{\cancel{9}}\sin\theta\sin(3\theta)$$

$$Aire(D) = \frac{\cancel{2}a}{\cancel{8}_4}\left|\Big[3\cos\theta\sin(3\theta) - \sin\theta\cos(3\theta)\Big]_{-\pi/4}^{\pi/4}\right|$$




$$= \frac{a}{4}\left|3\frac{\sqrt2}{2}\frac{\sqrt2}{2} - \frac{\sqrt2}{2}\cdot\left(-\frac{\sqrt2}{2}\right) - 3\cdot\frac{\sqrt2}{2}\left(-\frac{\sqrt2}{2}\right) + \left(-\frac{\sqrt2}{2}\right)\left(-\frac{\sqrt2}{2}\right)\right|$$

$$= \frac{a}{4}\left|3\cdot\frac{1}{2} + \frac{1}{2} + 3\cdot\frac{1}{2} + \frac{1}{2}\right| = \frac{a}{4}\cdot 8\cdot\frac{1}{2} = a.$$