

Perturbative Algebraic Quantum Field Theory

An Introduction for Mathematicians

Kasia Rejzner

This book is dedicated to the memory of Daniel Kastler, Uffe Haagerup, Raymond Stora and John Roberts, who passed away recently.

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Chapter 1

Introduction

Quantum field theory is an important theory in physics, with a wide range of applications and an impressive agreement with experiment. Despite this success, the mathematical foundations of this theory are still under investigation and many fundamental questions remain open. The rapid development of the field makes it difficult to find textbooks which are up to date with all the recent advances, especially if one looks for a mathematically rigorous approach. It is a common misconception that working in with QFT necessarily implies doing something “not-well defined”, while in fact most of the formal manipulations presented in the physics literature can be made completely rigorous.

For me quantum field theory is a beautiful bizarre world full of wonders suspended somewhere in between mathematics and physics. It charms physicists by providing results that agree with experiments with incredible precision. It lures mathematicians seeking to explore the land of QFT and get a closer look at beautiful mathematical structures that inhabit it. And yet, after more than 50 years of research, we do not fully understand what QFT really is and what wonders it is hiding from us deep in its conceptual roots.

As both a physicist and a mathematician I am fascinated by the richness of structures that one can encounter in the QFT land and from my first visit I have decided that I do not want to leave it ever again. So what is this book about? Well, maybe first I should explain what it isn't about. . . It is far from being a complete account of what has been done in the QFT research (this would have taken multiple volumes!). It also doesn't touch the problem of non-perturbative construction of models of interacting quantum field theories, which at the moment remains open.

You can think of this book as a mathematician's diary from a journey

into an exotic land. As opposed to some other textbooks on the subject I will not use the excuse that “physicists often do something which is not well defined”, so as mathematicians we don’t need to bother and just turn around for a while, until it’s over. Instead, I will jump straight into the lion’s den and will try to make mathematical sense of perturbative QFT all the way from the initial definition of the model to the interpretation of the results. This is not always easy and sometimes I will have to bring into the story results from several fields of mathematics at once. I hope this will not discourage you from exploration of the QFT wonderland. After all, its beauty lies in the fact that it is so diverse and full of surprises... So, come along! Our journey starts here.

Chapter 2

Algebraic approach to quantum theory

2.1 Algebraic quantum mechanics

Before entering the realm of quantum theory of fields, let's have a look at something simpler and better understood, namely *quantum mechanics* (QM). To prepare the ground for what follows, we will present an abstract formulation of QM and discuss how does it relate to the more standard Dirac-von Neumann axioms [Dir30, vN32]. The exposition presented in this chapter is based on [BF09b, Mor13, Fre13, Str08].

2.1.1 Observables and states

Let us start with recalling some basic definitions from functional analysis. For more information on operator algebras see [RS80, BR87, BR97, Kad83].

Definition 2.1. *An algebra \mathfrak{A} over the field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} is a vector a \mathbb{K} -vector space with an operation $\cdot : \mathfrak{A} \times \mathfrak{A} \rightarrow \mathfrak{A}$ called the **product** with the following properties:*

1. $(A \cdot B) \cdot C = A \cdot (B \cdot C)$, $\forall A, B, C \in \mathfrak{A}$ (*associativity*),
2. $A \cdot (B + C) = A \cdot B + A \cdot C$, $(B + C) \cdot A = B \cdot A + C \cdot A$,
 $\alpha(A \cdot B) = (\alpha A) \cdot B = A \cdot (\alpha B)$, for all $A, B, C \in \mathfrak{A}$, $\alpha \in \mathbb{K}$ (*distributivity*).

We will usually denote the algebra product \cdot simply by juxtaposition, i.e. $A \cdot B \equiv AB$.

Definition 2.2. An algebra \mathfrak{A} is said to have a unit (i.e. \mathfrak{A} is unital) if there exists an element $\mathbf{1} \in \mathfrak{A}$ such that $\mathbf{1}A = A\mathbf{1} = A$, for all $A \in \mathfrak{A}$.

Definition 2.3. An involutive complex algebra \mathfrak{A} is an algebra over the field of complex numbers, together with a map, $*$: $\mathfrak{A} \rightarrow \mathfrak{A}$, called an involution. The image of an element A of \mathfrak{A} under the involution is written A^* . Involution is required to have the following properties:

1. for all $A, B \in \mathfrak{A}$: $(A + B)^* = A^* + B^*$, $(AB)^* = B^*A^*$,
2. for every $\lambda \in \mathbb{C}$ and every $A \in \mathfrak{A}$: $(\lambda A)^* = \bar{\lambda}A^*$,
3. for all $A \in \mathfrak{A}$: $(A^*)^* = A$.

Up to now all the properties we have considered are purely algebraic. In order to quantify the notion of distance between the elements of the algebra we need some topology. This can be introduced for example by means of a norm.

Definition 2.4. A complex normed space is a vector space \mathcal{X} over \mathbb{C} , equipped with a map $\|\cdot\| : \mathcal{X} \rightarrow \mathbb{R}$, which satisfies:

1. $\|\lambda A\| = |\lambda|\|A\|$ (scaling),
2. $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality also called subadditivity),
3. If $\|A\| = 0$, then A is the zero vector (separates points).

If \mathfrak{A} is equipped with a norm, we can ask for the continuity of the algebraic relations with respect to the norm topology and for some notion of completeness. This leads to the following definitions.

Definition 2.5. A normed algebra \mathfrak{A} is a normed vector space whose norm $\|\cdot\|$ satisfies

$$\|AB\| \leq \|A\|\|B\|.$$

If \mathfrak{A} is unital, then it is a normed unital algebra if in addition $\|\mathbf{1}\| = 1$.

Definition 2.6. A Banach space is a normed vector space equipped with the norm-induced topology, which is complete with respect to this topology. A Banach (unital) algebra is a Banach space and a normed (unital) algebra with respect to the same norm.

A particularly important class of Banach algebras with involution is distinguished by the C^* -property. We will see in this chapter that such algebras can be used to describe spaces of observables in quantum systems.

Definition 2.7. A C^* -algebra is a Banach involutive algebra (Banach algebra with involution satisfying $\|A^*\| = \|A\|$), such that the norm has the C^* -property:

$$\|A^*A\| = \|A\|\|A^*\|, \quad \forall A \in \mathfrak{A}.$$

How come that such a mathematical structure is suitable for quantum physics? First note that in order to describe a physical system we need to specify a collection of physical quantities, which we want to measure (we call them *observables*) and a collection of *states* in which the system can be prepared. Now we want to deduce what kind of mathematical structure is suitable to describe observable and states. Operationally, each observable corresponds to some measurement apparatus, which measures given properties of the system. An example of such an apparatus is a particle detector localized in some region of space. Next, one considers operations that can be performed on observables. Scaling of the measurement apparatus corresponds to multiplying the corresponding observable A by a real number. One can also consider other functions of the observables, for example A^n is interpreted as measuring the observable A and taking the n -th power of the result.

Now we discuss the notion of states. We need to assume that we are able to repeat experiments, so that we can measure given observable repeatedly in the same state (i.e. for the same preparation of the system). This statistical interpretation presupposes that each experiment comes with a protocol that allows to obtain the same initial condition each time it is repeated. Under this assumption, a state ω associates to an observable A a real number $\omega(A)$ obtained by averaging the results of measurements of A for the system prepared to be in the state ω . It is natural to assume that $\omega(\lambda A) = \lambda\omega(A)$ for $\lambda \in \mathbb{R}_+$ (scaling), as well as $\omega(A^n) = (\omega(A))^n$. Let $\mathbb{1}$ be the observable, which always takes value 1. For such observable we require that $\omega(\mathbb{1}) = 1$. One can also deduce the positivity of states from the fact that the average of positive numbers is positive, so $\omega(A^2) \geq 0$.

If we assume that physical properties of observables can be measured only by looking at expectation values in various states of the system, it is natural to identify the observables, which give the same expectation values in all the states. Let now \mathcal{A} be the space of equivalence classes of observables, where $A \sim B$ if $\omega(A) = \omega(B)$ for all states ω of the system. A notion of a norm can be introduced by assigning to each observable $A \in \mathcal{A}$ a finite positive number defined by

$$\|A\| \doteq \sup_{\omega} |\omega(A)|$$

The operational properties of states imply that $\|\lambda A\| = |\lambda|\|A\|$ for $\lambda \in \mathbb{R}$ and $\|A\| = 0$ implies that $A = 0$ (states separate observables). It also follows that $\|A^2\| = \|A\|^2$, so we are getting close to the notion of a C^* -algebra. What is still missing is the linear structure on \mathcal{A} and the product. Let us start with the linear structure. Operationally “ $A + B$ ” has to satisfy

$$\omega(A + B) = \omega(A) + \omega(B),$$

for all states of the system. It is, however, not clear if an element “ $A + B$ ” exists in \mathcal{A} , so one needs to embed the initial space of observables in a larger structure in such a way that states will remain to be positive linear functionals on this enlarged space. This leads to the notion of Jordan algebras [Jor33, JvNW34] and finally C^* -algebras, introduced in [Gel43] and discussed in [Seg47a, Seg47b] in the context of quantum mechanics.

We can summarize the basic axioms in the algebraic approach to QM as follows:

1. A physical system is defined by its unital C^* -algebra \mathfrak{A} .
2. States are identified with positive, normalized linear functionals on \mathfrak{A} , i.e. $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$ and $\omega(\mathbb{1}) = 1$.

Note that on a unital C^* algebra a positive, normalized linear functional is automatically continuous with respect to the topology induced by the C^* -norm. More generally, we can define states also on involutive topological algebras.

Definition 2.8. *A state on an involutive algebra \mathfrak{A} is a linear functional ω , such that:*

$$\omega(A^*A) \geq 0, \quad \omega(\mathbb{1}) = 1.$$

Observables are self-adjoint elements of \mathfrak{A} and possible measurement results for an observable A are characterized by its spectrum $\sigma(A)$. We recall that an element A of a C^* -algebra is called self-adjoint if $A^* = A$.

Definition 2.9. *The spectrum $\text{spec}(A)$ of $A \in \mathfrak{A}$ is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda\mathbb{1}$ has no inverse in \mathfrak{A} .*

A standard result from functional analysis states that a spectrum of self-adjoint element is a subset of the real line and this agrees with the physical intuition, as outcomes of measurements have to be real.

2.1.2 Hilbert space representations

Having defined the abstract setup we can proceed to a more concrete description that provides a way to recover the Dirac-von Neumann axioms. The crucial observation is that abstract elements of an involutive algebra \mathfrak{A} can be realized as operators on some Hilbert space by a choice of a *representation*. First let us recall the definition of a Hilbert space.

Definition 2.10. *Let \mathcal{H} be a complex vector space. A map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ is a Hermitian inner product if*

1. $\langle u, v \rangle = \overline{\langle u, v \rangle}, \forall u, v \in \mathcal{H}$,
2. $\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle$ (linear in the second argument),
3. $\langle v, v \rangle \geq 0$ where the case of equality holds precisely when $v = 0$ (positive definite).

Properties 1 and 2 imply that $\langle \cdot, \cdot \rangle$ is antilinear in the first argument. One can define a norm on a Hilbert space \mathcal{H} by setting

$$\|v\| \doteq \sqrt{\langle v, v \rangle}.$$

Definition 2.11. *A Hilbert space \mathcal{H} is a complex vector space with a Hermitian inner product $\langle \cdot, \cdot \rangle$ such that the norm induced by this product makes \mathcal{H} into a Banach space.*

In physics *separable* Hilbert spaces play an important role.

Definition 2.12. *A Hilbert space is separable if it admits a countable dense subset.*

In fact a Hilbert space is separable if it is either finite dimensional or has a countable basis. We are now ready to define the notion of linear operators, which is important in the context of C^* -algebras and physical observables.

Definition 2.13. *An operator A on a Hilbert space \mathcal{H} is a linear map from a subspace $D \subset \mathcal{H}$ into \mathcal{H} . In particular, if $D = \mathcal{H}$ and A satisfies $\|A\| \doteq \sup_{\|x\|=1} \{\|Ax\|\} < \infty$, it is called bounded.*

An important class of bounded operators is provided by the unitary ones.

Definition 2.14. *A bounded linear operator $U : \mathcal{H} \rightarrow \mathcal{H}$ on a Hilbert space \mathcal{H} is called a unitary operator if it satisfies $U^*U = UU^* = \mathbb{1}$.*

Note that the space $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space \mathcal{H} forms a C^* -algebra. We will see later on that one can argue the other way and realize any abstract C^* -algebra as the algebra of bounded operators on some \mathcal{H} . If A is a bounded operator on a Hilbert space then the self-adjointness is the same as hermiticity, i.e. is the condition that $A^* = A$. In general this is not sufficient.

Definition 2.15. *An operator A on a Hilbert space \mathcal{H} with a dense domain $D(A) \subset \mathcal{H}$ is called symmetric if for any vectors $u, v \in D(A)$ we have $\langle u, Av \rangle = \langle Au, v \rangle$. This implies that $D(A) \subseteq D(A^*)$. A symmetric operator A is self-adjoint if in addition $D(A^*) \subset D(A)$.*

Definition 2.16. *Let A be an operator on a Hilbert space \mathcal{H} with a dense domain $D(A) \subset \mathcal{H}$. A self-adjoint operator A' is called a self-adjoint extension of A if $D(A) \subseteq D(A')$ and if $A'v = Av$ for any $v \in D(A)$.*

A is called essentially self-adjoint if it admits a unique self-adjoint extension.

Abstract elements of an involutive algebra \mathfrak{A} are realized as operators on some Hilbert space by a choice of a *representation*.

Definition 2.17. *A representation of an involutive unital algebra \mathfrak{A} is a unital $*$ -homomorphism π into the algebra of linear operators on a dense subspace \mathcal{D} of a Hilbert space \mathcal{H} . In particular, a representation of a C^* -algebra \mathfrak{A} is a unital $*$ -homomorphism $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$.*

A representation π is called faithful if $\text{Ker } \pi = \{0\}$. It is called irreducible if there are no non-trivial subspaces invariant under $\pi(\mathfrak{A})$.

Definition 2.18. *Two representations (π_1, \mathcal{H}_1) and (π_2, \mathcal{H}_2) of a C^* -algebra \mathfrak{A} are called unitarily equivalent, if $U\pi_1(A) = \pi_2(A)U$ holds for all $A \in \mathfrak{A}$ with some unitary map $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$.*

In Dirac-von Neumann axioms, one postulates that physical observables are self-adjoint operators acting on a Hilbert space. The connection between the algebraic formulation and the Hilbert space picture is provided by means of the famous GNS (Gelfand-Naimark-Segal) theorem.

Theorem 2.1. *Let ω be a state on the involutive unital algebra \mathfrak{A} . Then there exists a representation π of the algebra by linear operators on a dense subspace \mathcal{D} of some Hilbert space \mathcal{H} and a unit vector $\Omega \in \mathcal{D}$, such that*

$$\omega(A) = (\Omega, \pi(A)\Omega),$$

and $\mathcal{D} = \{\pi(A)\Omega, A \in \mathfrak{A}\}$.

Proof. First we introduce a scalar product \mathfrak{A} on the algebra using the state ω :

$$\langle A, B \rangle \doteq \omega(A^*B).$$

Linearity for the right and antilinearity for the left argument are easy to prove. Hermiticity $\langle A, B \rangle = \langle B, A \rangle$ follows from the positivity of ω and the fact that we can write A^*B and B^*A as linear combinations of positive elements:

$$\begin{aligned} 2(A^*B + B^*A) &= (A + B)^*(A + B) - (A - B)^*(A - B), \\ 2(A^*B - B^*A) &= -i(A + iB)^*(A + iB) + i(A - iB)^*(A - iB). \end{aligned}$$

From the positivity of ω also follows that the scalar product is positive semidefinite, i.e. $\langle A, A \rangle \geq 0$ for all $A \in \mathfrak{A}$. We now study the set

$$\mathfrak{N} \doteq \{A \in \mathfrak{A} \mid \omega(A^*A) = 0\}.$$

We show that \mathfrak{N} is a left ideal of \mathfrak{A} . Because of the Cauchy-Schwarz inequality \mathfrak{N} is a subspace of \mathfrak{A} . The same inequality implies that for $A \in \mathfrak{N}$ and $B \in \mathfrak{A}$ we obtain

$$\omega((BA)^*BA) = \omega(A^*B^*BA) = \langle B^*BA, A \rangle \leq \sqrt{\langle B^*BA, B^*BA \rangle} \sqrt{\langle A, A \rangle} = 0,$$

hence $BA \in \mathfrak{N}$. Let us define \mathcal{D} as the quotient $\mathfrak{A}/\mathfrak{N}$. Clearly, the scalar product is positive definite on \mathcal{D} and we complete it to obtain a Hilbert space \mathcal{H} . The representation π is induced by the operation of left multiplication on \mathfrak{A} , i.e.

$$\pi(A)(B + \mathfrak{N}) \doteq AB + \mathfrak{N},$$

and we set $\Omega = \mathbb{1} + \mathfrak{N}$. If \mathfrak{A} is a C^* -algebra, one can show that the operators $\pi(A)$ are bounded, hence admitting unique continuous extensions to bounded operators on \mathcal{H} .

We now show that the construction is unique up to unitary equivalence. Let $(\pi', \mathcal{D}', \mathcal{H}', \Omega')$ be another quadruple satisfying the conditions of the theorem. Then we define an operator $U : \mathcal{D} \rightarrow \mathcal{D}'$ by

$$U\pi(A)\Omega \doteq \pi'(A)\Omega'.$$

U is well defined, since $\pi(A)\Omega = 0$ if and only if $\omega(A^*A) = 0$, but then also $\pi'(A)\Omega' = 0$. Moreover U preserves the scalar product and is invertible, hence it has a unique extension to a unitary operator from \mathcal{H} to \mathcal{H}' . It follows that π and π' are unitarily equivalent. \square

The representation π is in general not irreducible, i.e. there may exist a nontrivial closed invariant subspace. In this case, the state ω is not pure, which means that it is a convex combination of other states,

$$\omega = \lambda\omega_1 + (1 - \lambda)\omega_2, \quad 0 < \lambda < 1, \quad \omega_1 \neq \omega_2.$$

We have seen that the algebraic formulation of QM allows to characterize a physical system purely in terms of its observable C^* -algebra \mathfrak{A} and states on it. The Hilbert space representations can be then obtained from states by means of the GNS theorem. One can also obtain the probabilistic interpretation of QM as follows. Given an observable A and a state ω on a C^* -algebra \mathfrak{A} we reconstruct the full probability distribution $\mu_{A,\omega}$ of measured values of A in the state ω from its moments, i.e. the expectation values of powers of A ,

$$\int \lambda^n d\mu_{A,\omega}(\lambda) = \omega(A^n).$$

We can now apply these methods to some simple physical situations. The first example is related to the *canonical commutation relations*.

Example 2.1. *Let L be a real vector space with a symplectic form σ , i.e. a bilinear form σ on L which is antisymmetric,*

$$\sigma(x, y) = -\sigma(y, x),$$

and nondegenerate,

$$\sigma(x, y) = 0 \quad \forall y \in L \quad \text{implies} \quad x = 0.$$

We consider the unital $$ -algebra $\mathcal{W}(L, \sigma)$ over \mathbb{C} generated by abstract symbols $W(x)$ (the Weyl generators), satisfying the relation*

$$W(x)W(y) = e^{i\sigma(x,y)}W(x+y).$$

The involution is defined by

$$W(x)^* = W(-x)$$

and the unit is $\mathbb{1} = W(0)$.

We define the norm on $\mathcal{W}(L, \sigma)$ by

$$\left\| \sum_{i=1}^n \lambda_i W(x_i) \right\|_1 = \sum_{i=1}^n |\lambda_i|.$$

This norm satisfies the condition $\|AB\|_1 \leq \|A\|_1 \|B\|_1$ of an algebra norm. Moreover, the involution is isometric, $\|A^\|_1 = \|A\|_1$ and we obtain an involutive normed algebra $\mathcal{W}(L, \sigma)$.*

After [Mor13] we recall known facts about the existence of the unique C^* -norm on $\mathcal{W}(L, \sigma)$.

Proposition 2.1. *The following hold true:*

1. *There exists a norm $\|\cdot\|_0$ on $\mathcal{W}(L, \sigma)$ satisfying the C^* -property,*
2. *In a C^* -norm, Weyl generators are unitaries, i.e. satisfy*

$$W(x)^*W(x) = \mathbb{1} = W(x)W(x)^*$$

for all $x \in L$.

3. *If we set*

$$\|A\|_c \doteq \sup\{\|A\|_0, \text{ such that } \|\cdot\|_0 : \mathcal{W}(L, \sigma) \rightarrow [0, \infty) \text{ is a } C^*\text{-norm}\},$$

then $\|\cdot\|_c$ is a C^ -norm.*

4. *Let $\mathfrak{W}(L, \sigma)$ be the completion of $\mathcal{W}(L, \sigma)$ with respect to $\|\cdot\|_c$, then $\mathfrak{W}(L, \sigma)$ is a C^* -algebra, associated to (L, σ) uniquely up to isomorphism.*
5. *$\mathfrak{W}(L, \sigma)$ is simple, i.e. there are no non-trivial closed, $*$ -invariant two-sided ideals.*

Proof. For proof see [BGP07] as well as [Mor13]. To see that the supremum defining $\|\cdot\|_c$ is finite, note that generators $W(x)$ are unitaries with respect to every C^* -norm, so if $A = \sum_i a_i W(x_i)$, then $\|A\| \leq \sum_i |a_i| = \|A\|_1$, which provides the upper bound for the supremum. \square

Let's consider a particular example of a symplectic space (L, σ) , which realizes canonical commutation relations for a free quantum particle in d dimensional space. In this case $L = \mathbb{R}^{2d}$ and we write elements of L in the form $x = (\boldsymbol{\alpha}, \boldsymbol{\beta})$, where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d), \boldsymbol{\beta} = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$. We define

$$\sigma((\boldsymbol{\alpha}, \boldsymbol{\beta}), (\boldsymbol{\alpha}', \boldsymbol{\beta}')) = -\frac{1}{2}\hbar(\boldsymbol{\alpha} \cdot \boldsymbol{\beta}' - \boldsymbol{\alpha}' \cdot \boldsymbol{\beta}),$$

where \cdot is the scalar product on \mathbb{R}^d . If the generators of the resulting Weyl C^* -algebra $\mathfrak{W}(L, \sigma)$ are represented by operators on a Hilbert space in such a way that they depend strongly continuously¹ on the parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$, then

¹A net $\{T_\alpha\}$ of operators on a Hilbert space \mathcal{H} converges strongly to an operator T if and only if $\|T_\alpha x - Tx\| \rightarrow 0$ for all $x \in \mathcal{H}$.

such a representation is called regular. It was proven by von Neumann that all the regular reducible representations of the resulting Weyl algebra are unitary equivalent. Another theorem important in this context is due to Stone [Sto30]:

Theorem 2.2. *Let $(U_t)_{t \in \mathbb{R}}$ be a strongly continuous one-parameter unitary group. Then there exists a unique (not necessarily bounded) self-adjoint operator A such that*

$$U_t = e^{itA}, \quad \forall t \in \mathbb{R}.$$

Conversely, if A is a (not necessarily bounded) self-adjoint operator on a Hilbert space \mathcal{H} , then the one-parameter family $(U_t)_{t \in \mathbb{R}}$ of unitary operators defined by means of the Spectral Theorem for Self-Adjoint Operators (see for example chapter 9 [Mor13]) as

$$t \mapsto U_t := e^{itA}$$

is strongly continuous.

For $\mathfrak{W}(L, \sigma)$ this implies that there exist self-adjoint generators $q^1, \dots, q^d, p^1, \dots, p^d$ of 1-parameter groups of unitary operators

$$W(0, \dots, \alpha_k, \dots, 0) = e^{i\alpha_k p^k}, \quad W(0, \dots, \beta_k, \dots, 0) = e^{i\beta_k q^k},$$

We denote $\mathbf{p} \doteq (p^1, \dots, p^d)$, $\mathbf{q} \doteq (q^1, \dots, q^d)$. Generators \mathbf{p} and \mathbf{q} satisfy the canonical commutation relations

$$[q^k, p^j] = \delta_{kj}, \quad [q^k, q^j] = 0, \quad [p^k, p^j] = 0$$

and one can write an arbitrary generator $W(\alpha, \beta)$ in the form

$$W(\alpha, \beta) = e^{-\frac{i\hbar\alpha \cdot \beta}{2}} e^{i\alpha \cdot \mathbf{p}} e^{i\beta \cdot \mathbf{q}} = e^{\frac{i\hbar\alpha \cdot \beta}{2}} e^{i\beta \cdot \mathbf{q}} e^{i\alpha \cdot \mathbf{p}}.$$

The Schrödinger representation of this Weyl algebra is defined on the Hilbert space of square integrable functions $\mathcal{L}^2(\mathbb{R}^d)$ with

$$(\pi(W(\alpha, \beta))\Phi)(\mathbf{X}) = e^{\frac{i\hbar\alpha \cdot \beta}{2}} e^{i\beta \cdot \mathbf{X}} \Phi(\mathbf{X} + \hbar\alpha), \quad (2.1)$$

for $\Phi \in \mathcal{L}^2(\mathbb{R}^d)$. As mentioned before, all the regular irreducible representations are unitary equivalent to this one. If one does not require continuity there are many more representations. In quantum field theory the uniqueness results do not apply, and one has to deal with a huge class of inequivalent representations. Note that the construction of the Weyl algebra makes sense also for L infinite dimensional, so can be applied in the context of field theory.

A particularly interesting class of states on $\mathfrak{W}(L, \sigma)$ is provided by *quasi-free states*.

Definition 2.19. A state ω on $\mathfrak{W}(L, \sigma)$ is called quasi-free if there exists $\eta : L \times L \rightarrow \mathbb{R}$, a symmetric form such that

$$\omega(W(x)) = e^{-\frac{1}{2}\eta(x,x)}.$$

The form η is then called covariance of the quasi-free state ω .

The following theorem provides a way to easily find quasi-free states.

Theorem 2.3. Let $\eta : L \times L \rightarrow \mathbb{R}$ be a symmetric form. The following are equivalent:

1. $\eta_{\mathbb{C}} + \frac{i}{2}\sigma_{\mathbb{C}} \geq 0$ on $L^{\mathbb{C}}$, the complexification of L , where $\eta_{\mathbb{C}}, \sigma_{\mathbb{C}} : L^{\mathbb{C}} \times L^{\mathbb{C}} \rightarrow \mathbb{C}$ are canonical sesquilinear extensions of η, σ .
2. $|\sigma(x_1, x_2)| \leq 2\sqrt{\eta(x_1, x_1)}\sqrt{\eta(x_2, x_2)}$, for all $x_1, x_2 \in L$.
3. There exists a quasi-free state ω_{μ} on $\mathfrak{W}(L, \sigma)$ with covariance η .

Proof. For proof see for example [AS71, DG13a] □

This result holds also if L is infinite dimensional and will be used later in section 5.3. We define a complex scalar product on the complex vector space $L_{\mathbb{C}}$ by

$$\langle x, y \rangle = \eta_{\mathbb{C}}(x, y) + \frac{i}{2}\sigma_{\mathbb{C}}(x, y). \quad (2.2)$$

The GNS Hilbert space representation corresponding to ω_{μ} turns out to be the bosonic Fock space:

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} (\mathcal{H}_1^{\otimes n})_{\text{symm}} ; \mathcal{H}_1 = \overline{L_{\mathbb{C}} / \text{Ker}(\langle \cdot, \cdot \rangle)}$$

The state ω_{μ} is pure (i.e. the associated GNS representation is irreducible) if and only if the map $L \rightarrow L_{\mathbb{C}} / \text{Ker}(\langle \cdot, \cdot \rangle)$ is surjective. The latter holds if and only if and only if the pair $(2\eta, \sigma)$ is Kähler.

Definition 2.20. A pair $(2\eta, \sigma)$ consisting of a symmetric form 2η and symplectic form σ on L is called Kähler if the ranges of the two coincide $\text{Ran}(2\eta) = \text{Ran}(2\sigma)$, 2η is positive definite and $J \doteq \sigma^{-1}2\eta$ satisfies $J^2 = -\mathbb{1}$ (i.e. J is an anti-involution).

If $(2\eta, \sigma)$ is Kähler, then the quadruple $(L, 2\eta, \sigma, J)$ is a Kähler space. We will come back to this structure in the context of QFT in section 5.3.

2.1.3 Dynamics and the interaction picture

If we want to model a physical system which evolves with time, we need to introduce the notion of dynamics. A very detailed discussion of quantum dynamics can be found in [BR87, BR97]. Here we only sketch the main ideas. Let \mathfrak{A} be a C^* -algebra of observables and let $A_t \in \mathfrak{A}$ be some observable corresponding to the measurement apparatus A at time t ². We postulate that the algebra of observables \mathfrak{A} doesn't change with time, so the assignment $t \mapsto A_t$ can be described by a 1-parameter group of automorphisms α_t , such that $A_t = \alpha_t(A)$ and we assume that α_t is strongly continuous.

For a given state ω we consider the family of states that are related to it by time-translations and it is natural to require some stability properties from the GNS-associated representation π_ω . If π_ω is irreducible, this stability requirement is realized as the condition that α_t has to be implemented by some unitary operator $U(t)$, i.e.

$$\pi_\omega(\alpha_t(A)) = U(t)^{-1}\pi_\omega(A)U(t), \quad \forall A \in \mathfrak{A}. \quad (2.3)$$

Now we apply Stone's theorem 2.2 to deduce the existence of a self-adjoint generator H , called the *Hamiltonian* and we write

$$U(t) = e^{-itH} \quad \forall t \in \mathbb{R}.$$

By differentiating (2.3) we obtain the known evolution equation in the Heisenberg picture,

$$\frac{d}{dt}A(t) = i[H, A(t)], \quad (2.4)$$

where we have put $A(t) = U(t)^*AU(t)$ and we have omitted the symbol π_ω . To get the Schrödinger picture, we consider $\psi \in D(H)$ a Hilbert space vector in the domain of essential selfadjointness (see definition 2.16) of H , and define $\psi_S(t) \doteq U(t)\psi$. We can now rewrite (2.4) in the form

$$i\frac{d}{dt}\psi_S(t) = H\psi_S(t). \quad (2.5)$$

This is the time-evolution in the Schrödinger picture. If we want to construct a model of a quantum dynamical system, we usually start with a Hamiltonian $H \in \mathcal{H}_\pi$ and solve (2.5) for some initial data $\psi_S(0)$, within the domain $D(H)$. A solution to the initial value problem defines then the *propagator* $U(t, 0)$, i.e.

$$\psi_S(t) = U(t, 0)\psi_S(0).$$

²As sharp localization is physically impossible, operationally we can think of A_t as the average over some interval $[t - \epsilon, t + \epsilon]$ centered at t , for a fixed value of $\epsilon > 0$.

Note that the main difference between (2.4) and (2.5) is that in the Heisenberg picture states remain stationary and operators evolve with time, while in the Schrödinger picture it's the other way round. Often solving the initial value problem of the form (2.5) is very difficult and it is convenient to split the Hamiltonian into two terms

$$H = H_0 + H_{int} ,$$

where the propagator for H_0 can be found relatively easily and we try to solve the problem perturbatively. This point of view is something in between the Heisenberg and the Schrödinger picture and we call it the *interaction picture*. H_{int} is then called the *interaction Hamiltonian*. Let $U_0 \doteq e^{-itH_0}$. In the interaction picture the states are represented by

$$\psi_I(t) = U_0^* \psi_S(t) = e^{itH_0} \psi_S(t) = e^{itH_0} e^{-itH} \psi ,$$

where ψ_S is a state in the Schrödinger picture and ψ is a state in the Heisenberg picture. Observables of the interaction picture evolve according to

$$A(t) = U_0(t)^* A_S U_0(t) ,$$

where A_S denotes the Schrödinger picture observable. In particular

$$H_{int} = U_0(t)^* H_{int} U_0(t)$$

for the interaction Hamiltonian H_{int} . Now the evolution equation (2.5) implies that

$$i \frac{d}{dt} \psi_I = H_{int} \psi_I . \quad (2.6)$$

Given initial data $\psi_I(t_0)$, we want to find the solution to this equations in terms of a propagator $U_I(t, t_0)$, so that

$$\psi_I(t) = U_I(t, t_0) \psi_I(t_0) .$$

By definition we have

$$U_I(t, t_0) = e^{itH_0} e^{-i(t-t_0)H} e^{-it_0H_0} ,$$

and from (2.6) follows that the propagator has to satisfy

$$i \frac{d}{dt} U_I(t, t_0) = H_{int}(t) U_I(t, t_0) , \quad U_I(t_0, t_0) = \mathbb{1} . \quad (2.7)$$

A formal solution to the above equation is then given by the Dyson series

$$\begin{aligned} U_I(t, t_0) &= \mathbb{1} - i \int_{t_0}^t H_{int}(t_1) dt_1 - \int_{t_0}^t \int_{t_0}^{t_2} H_{int}(t_2) H_{int}(t_1) dt_1 dt_2 + \cdots + = \\ &= \mathbb{1} + \sum_{n=1}^{\infty} (-i)^n \int_{t_0 < t_1 < \cdots < t_n < t} \cdots \int H_{int}(t_n) \cdots H_{int}(t_1) dt_1 \cdots dt_n. \end{aligned} \quad (2.8)$$

We can simplify the notation by introducing the *time-ordering operator* T defined on operators $A(t)$ and $B(t)$ by

$$T(A(t)B(t')) = \begin{cases} A(t)B(t'), & \text{if } t < t' \\ B(t')A(t), & \text{if } t' < t \end{cases}. \quad (2.9)$$

We can now rewrite the formula (2.8) as a time-ordered exponential, i.e.

$$U_I(t, t_0) = \mathbb{1} + T \left[\sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \left(\int_{t_0}^t H_{int}(t') dt' \right)^n \right] = T \left[\exp \left(-i \int_{t_0}^t H_{int}(t') dt' \right) \right]. \quad (2.10)$$

We define the Møller operators \mathcal{S}^{\pm} as the strong limits of $U_I(t, t_0)$ as t_0 approaches $\pm\infty$, as long as these limits exist.

$$\mathcal{S}_{\pm} \doteq \text{s-lim}_{t \rightarrow \pm\infty} U_I(0, t).$$

The scattering operator \mathcal{S} (the S-matrix) is then defined by

$$\mathcal{S} \doteq \mathcal{S}_+^* \mathcal{S}_-. \quad (2.11)$$

We will use these ideas later on, in section 6.1 to construct perturbatively QFT models.

2.2 Causality

After introducing basic notion from quantum mechanics, the next step towards quantum field theory leads through *spacetime geometry*. Historically, QFT was conceived as a framework, which allows to combine quantum mechanics with special relativity. The latter is based on concepts such as *Minkowski spacetime* and *causality*. In fact, the algebraic approach to QFT can be generalized beyond the Minkowski spacetime and one can apply it to construct models on a wide class of Lorentzian manifolds. In this section we will review some basic concepts from Lorentzian geometry that are relevant for our framework.

Definition 2.21. A spacetime is a pair $\mathcal{M} = (M, g)$, where M is a smooth (4-dimensional) manifold (we assume it to be Hausdorff, paracompact, connected) and g is a smooth Lorentzian metric, i.e. a smooth tensor field $g \in \Gamma(T^*M \otimes T^*M)$, s.t. for every $p \in M$, g_p is a symmetric non-degenerate bilinear form. We require the metric g to be of the Lorentz signature $(+, -, -, -)$.

Remark 2.1. Let us make a few remarks concerning the above definition:

1. The assumption for a manifold to be *Hausdorff* means that points can be separated (for every pair of points x, y , there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint ($U \cap V = \emptyset$)). In general topology one can drop this assumption and an example of a non-Hausdorff manifold is a line with two origins, i.e. the quotient space of two copies of the real line $\mathbb{R} \times \{a\}$ and $\mathbb{R} \times \{b\}$, with the equivalence relation $(x, a) \sim (x, b)$ if $x \neq 0$.
2. The *paracompactness* is needed as a sufficient condition for the existence of partitions of unity. It means that for every open cover $(U_\alpha)_{\alpha \in A}$, there exists a refinement³ $(V_\beta)_{\beta \in B}$ that is locally finite, i.e. each $x \in M$ has a neighborhood that intersects only finitely many sets of $(V_\beta)_{\beta \in B}$.
3. We assumed also that M is connected, i.e. it cannot be represented as a disjoint union of two or more non-empty set. Later on we will see that in a more general context one has to drop this assumption and consider manifolds that are not connected.

Definition 2.22. A spacetime \mathcal{M} is said to be *orientable* if there exists a differential form of maximal degree (a volume form), which does not vanish anywhere. We say that \mathcal{M} is *time-orientable* if there exists a smooth vector field u on M such that for every $p \in M$, $g(u, u) > 0$ holds.

We will always assume that our spacetimes are orientable and time-orientable. We fix the orientation and choose the time-orientation by selecting a specific vector field u with the above property.

Example 2.2. A standard example is the 4 dimensional Minkowski spacetime \mathbb{M} , which is \mathbb{R}^4 with the diagonal metric $\eta = \text{diag}(1, -1, -1, -1)$.

An important feature of the Lorentzian signature, which distinguishes it from the Euclidean signature, is that it allows to introduce some important classes of smooth curves.

³An open cover $(V_\beta)_{\beta \in B}$ is a refinement of an open cover $(U_\alpha)_{\alpha \in A}$, if $\forall \beta \in B, \exists \alpha$ such that $V_\beta \subseteq U_\alpha$.

Definition 2.23. Let $\gamma : \mathbb{R} \supset I \rightarrow M$ be a smooth curve in M , for I an interval in \mathbb{R} and let $\dot{\gamma}$ be the vector tangent to the curve. We say that γ is

- *timelike*, if $g(\dot{\gamma}, \dot{\gamma}) > 0$,
- *spacelike*, if $g(\dot{\gamma}, \dot{\gamma}) < 0$,
- *lighlike (null)*, if $g(\dot{\gamma}, \dot{\gamma}) = 0$,
- *causal*, if $g(\dot{\gamma}, \dot{\gamma}) \geq 0$.

The classification of curves defined above is referred to as the *causal structure*. The presence of time orientation allows for a further refinement of this classification.

Definition 2.24. Given the global timelike vectorfield u (the time orientation) on M , a causal curve γ is called *future-directed* if $g(u, \dot{\gamma}) > 0$ all along γ . It is *past-directed* if $g(u, \dot{\gamma}) < 0$.

Using the causal structure one can distinguish points of spacetime which are in the future or in the past of a given point $p \in \mathcal{M}$.

Definition 2.25. Let $p \in \mathcal{M}$ be a point in a time-oriented spacetime.

- i) $J^\pm(p)$ is defined to be the set of all points in M which can be connected to x by a future(+)/past(-)-directed causal curve $\gamma : I \rightarrow M$ so that $x = \gamma(\inf I)$.
- ii) The set $J^+(p)$ is called the *causal future* and $J^-(p)$ the *causal past* of p . The boundaries $\partial J^\pm(p)$ of these regions are called respectively: the *future/past lightcone*.
- iii) The *future (past)* of a subset $B \subset M$ is defined by

$$J_\pm(B) = \bigcup_{p \in B} J_\pm(p) .$$

The physical importance of the structures presented above becomes clear in the context of general relativity (GR). One of the postulates of GR states that *massive particles can move only on time-like curves and light travels following null curves*, i.e. *nothing travels faster than light*. Consequently, one of the fundamental principles of physics, *the principle of causality*, states that an event happening at point p can be influenced only by events in $J^-(p)$ and that the consequences of this event can influence only the events in $J^+(p)$.

Definition 2.26. Two subsets O_1 and O_2 in M are called *causally separated* (or *spacetime separated*) if they cannot be connected by a causal curve, i.e. if for all $x \in \overline{O_1}$, $J^\pm(x)$ has empty intersection with $\overline{O_2}$.

Another important definition is that of the *causal complement* of a given region O .

Definition 2.27. O^\perp is defined as the causal complement of O , i.e. the largest open set in M which is causally separated from O .

It follows from the principle of causality that events happening at space-like separated points cannot influence each other. In classical physics this property is realized as a consequence of some properties of normally hyperbolic partial differential equations. In section 2.3 we will see how these ideas can be implemented into the framework of quantum theory.

Example 2.3. Consider the Minkowski spacetime $\mathbb{M} = (\mathbb{R}^4, \eta)$. Clearly this is an affine space. The set of points which are causally separated from the given point $P \in \mathbb{M}$ is called the *lightcone with apex P* . It is easy to verify that a point $Q \in \mathbb{M}$

- lies on the lightcone with apex P if and only if the vector \overrightarrow{PQ} is lightlike,
- is in the future (past) of P if and only if the vector \overrightarrow{PQ} is time-like and its 0-th component is positive (negative),
- is spacelike to P if and only if \overrightarrow{PQ} is spacelike.

These concepts are illustrated at the diagram 2.1.

Definition 2.28. Motivated by the example 2.3 we introduce the following notation:

- $\overline{V}_+ \subset \mathbb{R}^4$ is defined the subset of \mathbb{R}^4 consisting of vectors v that satisfy $\eta(v, v) \geq 0$ and $v_0 > 0$. We call \overline{V}_+ the closed future lightcone.
- $\overline{V}_- \doteq \{v \in \mathbb{R}^4 | \eta(v, v) \geq 0, v_0 \leq 0\}$ is called the closed past lightcone.

These definitions can also be applied to subsets of tangent and cotangent spaces $T_x M$ and $T_x^* M$, as these space can be mapped to \mathbb{R}^4 with the use of appropriate charts.

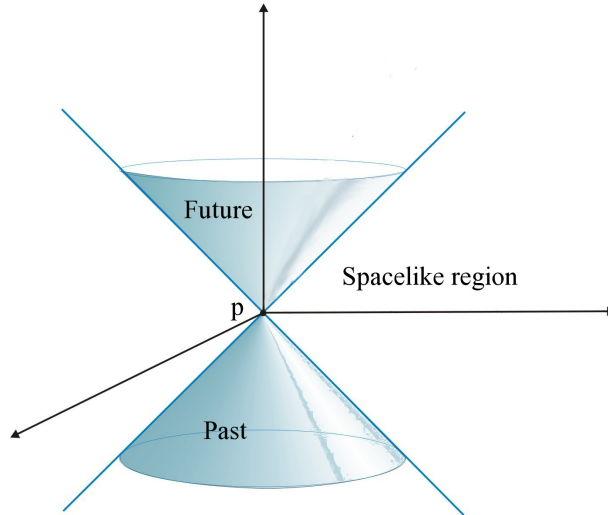


Figure 2.1: A lightcone in Minkowski spacetime.

Not all Lorentzian spacetimes are equally convenient for constructing quantum field theory models. For example, several conceptual and technical problems appear when we consider spacetimes with closed time-like curves. To exclude such situations, we will restrict ourselves to spacetimes that are *globally hyperbolic*.

Definition 2.29 (after [BS03]). *A spacetime is called globally hyperbolic if it does not contain closed causal curves and if for any two points x and y the set $J_+(x) \cap J_-(y)$ is compact.*

It was shown in [BS03] have many important properties and to understand them better it is important to introduce some further definitions.

Definition 2.30. *A causal curve is **future inextendible** if there is no $p \in M$ such that:*

$$\forall U \subset M \text{ open neighborhoods of } p, \exists t' \text{ s.t. } \gamma(t) \in U \forall t > t'.$$

Definition 2.31. *A **Cauchy hypersurface** in M is a smooth subspace of M such that every inextendible causal curve intersects it exactly once.*

The significance of Cauchy hypersurfaces lies in the fact that one can use them to formulate the initial value problem for partial differential equations

and for some classes of such equations this problem has a unique solution. The fundamental theorem relating different equivalent notions of global hyperbolicity has been proven in [BS03].

Theorem 2.4 (after [BS03]). *The following definitions of global hyperbolicity of a Lorentzian manifold \mathcal{M} are equivalent:*

- \mathcal{M} does not contain closed causal curves and for any two points x and y the set $J_+(x) \cap J_-(y)$ is compact.
- \mathcal{M} contain a Cauchy surface.
- \mathcal{M} admits a foliation by Cauchy surfaces.

2.3 Haag-Kastler axioms

In section 2.1 we have introduced some fundamental notions of quantum theory as states and observables. Now we want to make these compatible with the ideas of special and general relativity, reviewed in section 2.2, where the causal structure plays an important role. The main conceptual difficulty is to find a way to implement the idea that “nothing travels faster than light” in such a way that it doesn’t contradict the existence of quantum correlations the theory. The groundbreaking idea of Rudolf Haag was to combine these notion using the principle of *locality* (*Nahwirkungsprinzip*). In his framework, locality is the feature of observables, while states might exhibit correlations, i.e. they carry global information. One defines a QFT model by assigning to each bounded region $\mathcal{O} \subset \mathbb{M}$ of the Minkowski spacetimes the C^* -algebra of observables $\mathfrak{A}(\mathcal{O})$ that can be measured in this region. The notion of subsystem is realized by the requirement that if $\mathcal{O} \subset \mathcal{O}'$, then $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}')$. This condition is called *isotony* and it guarantees that one doesn’t lose observables when considering a larger region of spacetime. The complete set of axioms for algebraic quantum field theory (AQFT) can be found in [HK64, Haa93, Ara99]; we will recall them briefly in this section. We say that a net of C^* -algebras $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ satisfies the Haag-Kastler axioms (also called Araki-Haag-Kastler axioms) if the following hold:

- **Isotony.** For $\mathcal{O} \subset \mathcal{O}'$ we have $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}')$. The inductive limit of local algebras $\mathfrak{A}(\mathcal{O})$ defines the *quasilocal algebra* $\mathfrak{A} \doteq \overline{\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})}$ (the bar means taking the completion in the norm topology).
- **Locality (Einstein causality).** Algebras associated to spacelike separated regions commute: if \mathcal{O}_1 is spacelike separated from \mathcal{O}_2 , then

$[A, B] = 0, \forall A \in \mathfrak{A}(\mathcal{O}_1), B \in \mathfrak{A}(\mathcal{O}_2)$, where the commutator is taken in \mathfrak{A} . This expresses the “independence” of physical systems associated to regions \mathcal{O}_1 and \mathcal{O}_2 .

- **Covariance.** The Minkowski spacetime has a large group of isometries, namely the connected component of the Poincaré group. We require that there exists a family of isomorphisms $\alpha_L^\mathcal{O} : \mathfrak{A}(\mathcal{O}) \rightarrow \mathfrak{A}(L\mathcal{O})$ for Poincaré transformations L , such that for $\mathcal{O}_1 \subset \mathcal{O}_2$ the restriction of $\alpha_L^{\mathcal{O}_2}$ to $\mathfrak{A}(\mathcal{O}_1)$ coincides with $\alpha_L^{\mathcal{O}_1}$ and $\alpha_{L'}^{L\mathcal{O}} \circ \alpha_L^\mathcal{O} = \alpha_{L'L}^\mathcal{O}$.
- **Time slice axiom:** The algebra of a neighborhood of a Cauchy surface of a given region coincides with the algebra of the full region. Physically this correspond to a well-posedness of an initial value problem, i.e. we need to determine our observables in some small time interval $(t_0 - \epsilon, t_0 + \epsilon)$ to reconstruct the full algebra.
- **Spectrum condition.** Physically this condition is interpreted as the positivity of energy. One assumes that there exist a compatible family of faithful representations $\pi_\mathcal{O}$ of $\mathfrak{A}(\mathcal{O})$ on a fixed Hilbert space (i.e. the restriction of $\pi_{\mathcal{O}_2}$ to $\mathfrak{A}(\mathcal{O}_1)$ coincides with $\pi_{\mathcal{O}_1}$ for $\mathcal{O}_1 \subset \mathcal{O}_2$) such that translations are unitarily implemented, i.e. there is a unitary representation U of the translation group satisfying

$$U(a)\pi_\mathcal{O}(A)U(a)^{-1} = \pi_{\mathcal{O}+a}(\alpha_a(A)), \quad A \in \mathfrak{A}(\mathcal{O}),$$

and such that the joint spectrum of the generators P_μ of translations $e^{iaP} = U(a), aP = \sum_{\mu=0}^3 a^\mu P_\mu$, is contained in the closed future light-cone: $\sigma(P) \subset \overline{V}_+$.

All these axioms, apart from the spectrum condition, can be generalized to QFT's on general globally hyperbolic spacetimes. We will discuss this in more detail in the next section. There are many important conceptual results that have been proven in the AQFT framework. The first major success was the development of the Haag-Ruelle scattering theory [Haa58, Rue62], which provided an explanation why quantum field theory yields a theory of interacting particles. It is, however, an open question, whether all states in the vacuum representation admit a particle interpretation (the problem of asymptotic completeness). For recent works on that topic see [DT11, DG14b, DG14a] Another remarkable result of AQFT is the Reeh-Schlieder Theorem [Haa93, RS61], see [BS14] for a recent discussion. Another known result achieved with the AQFT methods is the analysis of the superselection structure of QFT models [DHR71, DHR74]. Despite all this insight into

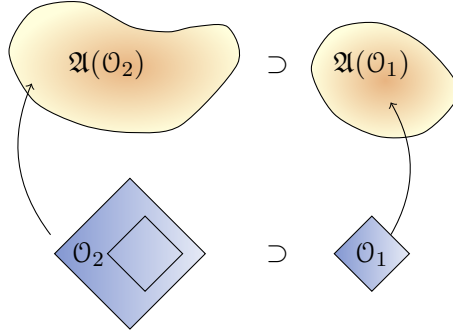


Figure 2.2: Diagram representing inclusion of spacetime regions and corresponding C^* -algebras.

the general structure of QFT, there remains the difficulty with constructing 4 dimensional interacting models that fulfil the Haag-Kastler axioms. For models in 2 dimensions see [Lec08, Tan12, BT13, Ala13, BC13] and references therein.

2.4 pAQFT axioms

In this book we explore the possibility to drop some of the assumptions of the Haag-Kastler framework, in order to allow for models that exist only in the formal, perturbative sense. The resulting framework is called *perturbative Algebraic Quantum Field Theory* (pAQFT). The generalization of the HK axioms to the perturbative context has been developed in a series of papers [DF01a, DF02, DF04, DF07, DF01b, BD08, Boa00, DB01, BDF09, Rej11b]. The generalization of the HK framework to curved spacetime has been for a long time, an independent development. Some important early contributions include [Kay78, Dim80, KW91, Dim92]. Later these two generalizations met as the pAQFT on curved spacetimes after a seminal series of papers [BFK96, BF97, BF00, BFV03, HW01, HW02a, HW02b, HW05]. Abelian gauge theories were later treated in [DF98], while the Yang-Mills theories are the subject of [Hol08]. At the same time the mathematical foundations of pAQFT became better understood, mainly with the use of the functional approach, which is also the approach we take in this book. In [FR12b, FR12a, Rej11a] this framework has been used to add the Batalin-Vilkovisky (BV) formalism in the pAQFT toolbox, which allows to treat very general theories possessing local symmetries, including the bosonic string

[BRZ14] and effective quantum gravity [BFR13].

2.4.1 Functional analytic preliminaries

On the functional analytic side, we leave the realm of Banach spaces and allow for structures that have more complicated topologies. This involves some technical complications, but gives more flexibility in terms of model building. The most general class of topological vector spaces, which we will use is the class of locally convex ones.

Definition 2.32. *A topological vector space X is called a locally convex vector space (LCVS) if there is a local base \mathcal{B} whose members are convex.*

Here by a local base we mean a collection \mathcal{B} , of neighborhoods of 0 such that every neighborhood of 0 contains a member of \mathcal{B} . The open sets of X are then precisely those that are unions of translates of members of \mathcal{B} .

There is another way to characterize locally convex vector spaces, which allows us to make a connection with normed spaces, introduced in definition 2.4. Instead of having one norm that characterizes the topology, we have a family of *seminorms*. A seminorm differs from a norm by not fulfilling the property 3 in definition 2.4. More precisely:

Definition 2.33. *A seminorm on a vector space X is a real-valued function p on X such that:*

1. $p(x + y) < p(x) + p(y)$ for all $x, y \in X$.
2. $p(\lambda x) = |\lambda|p(x)$ for all $x \in X$ and all scalars $\lambda \in \mathbb{K}$.

We see that a seminorm already provides us with a lot of information, but it doesn't separate points. However, it is possible that a certain family of seminorms is already separating.

Definition 2.34. *A family \mathcal{P} of seminorms on X is said to be separating if to each $x \neq 0$ corresponds at least one $p \in \mathcal{P}$ with $p(x) \neq 0$.*

We can see that a separating family of seminorms already allows us to distinguish two elements of X . From the following theorem it becomes clear why a locally convex vector space is a useful concept.

Theorem 2.5. *With each separating family of seminorms on X we can associate a locally convex topology τ on X and vice versa: every locally convex topology is generated by some family of separating seminorms.*

Proof. See [Rud91]. □

In the pAQFT framework a LCVS is usually the best one can expect. Unfortunately it doesn't share many nice properties of a Banach space, but there are some distinguish classes of LCVS, which are relatively well behaving and good for defining calculus on them. The “nicest” ones are *Fréchet* spaces. They are distinguished by the fact that their topology can be described in terms of a *metric*.

Theorem 2.6. *A locally convex vector space $\mathcal{X} = (X, \tau)$ is metrizable if and only if τ can be defined by $\mathcal{P} = \{p_n : n \in \mathbb{N}\}$ a countable separating family of seminorms on X . One can equip X with a metric which is compatible with τ and which provides a family of convex balls.*

Proof. See [Köt69, Rud91]. □

A LCVS from theorem 2.6 can be equipped with the metric:

$$d(x, y) := \sum_{n \in \mathbb{N}} 2^{-n} \frac{p_n(x - y)}{1 + p_n(x - y)} \quad (2.12)$$

This metric is compatible with τ but in general it doesn't provide convex balls (see the discussion in [Rud91] after theorem 1.24 and exercise 18). Nevertheless it is good to know that you have a metric that can actually be written down in a closed form.

Definition 2.35. *If X is complete with respect to the metric from theorem 2.6 it is a *Fréchet space*.*

Usually a Fréchet space topology is defined explicitly by providing a countable separating family of seminorms.

To end this section we remark on one more important aspect of LCVS, namely the definition of tensor products. In quantum theory tensor products are used to model systems that constitute of independent subsystems. This is closely related to the notion causality and we will come back to this issue in section 2.5.

Definition 2.36. *Let E and F be locally convex vector spaces and let $\otimes : E \times F \rightarrow E \otimes F$ be the canonical map into the corresponding tensor product. The finest topology on $E \otimes F$ which makes \otimes continuous is called **the projective tensor topology** or the π -topology. The space $E \otimes F$ equipped with this topology is denoted by $E \otimes_{\pi} F$ and its completion by $E \hat{\otimes}_{\pi} F$.*

It can be shown that the topology π is locally convex. Another possible topology on $E \otimes F$ is the so called *injective tensor topology*. Its definition is a little bit more involved. In some sense it is the weakest well behaving topology one can put on $E \otimes F$, while the projective tensor topology is the strongest.

The idea behind the injective topology is to define it via the topology on the space of continuous linear mappings $L(E'_\gamma, F)$.

Definition 2.37. *We equip E' with the finest locally convex topology γ that coincides with the weak one on equicontinuous⁴ sets. One can identify $E \otimes F$ with a subspace of $L(E'_\gamma, F)$. Next we equip $L(E'_\gamma, F)$ with a topology of uniform convergence on equicontinuous compact sets in E' . We denote the resulting topological space by $E \varepsilon F$. It is called the ε -product of E and F . The corresponding topology induced on $E \otimes F$ is called the ε -topology and $E \otimes F$ equipped with it is the injective tensor product $E \otimes_\varepsilon F$. The corresponding completion is denoted by $E \hat{\otimes}_\varepsilon F$*

This topology is better behaving if we want to consider for example vector valued distributions and was used (in a slightly modified version) by L. Schwartz in [Sch57, Sch58]. Inequivalent notions of tensor products on LCVS can possibly create a problem, but there is a large class of spaces, where these notions coincide. These are the *nuclear* locally convex topological vector spaces, studied by A. Grothendieck in [Gro55].

2.4.2 Axioms

In this section we introduce the generalization of the Haag-Kastler axioms which is the foundation of pAQFT. It is in fact convenient to extend the pAQFT framework also to classical field theory, to keep a uniform language.

Definition 2.38. *A classical field theory model on a spacetime \mathcal{M} is a net of locally convex topological Poisson $*$ -algebras with sequentially continuous operations*

$$\mathcal{O} \mapsto \mathfrak{P}(\mathcal{O}),$$

⁴A set A of continuous functions between two topological spaces E and F is equicontinuous at the points $x_0 \in E$ and $y_0 \in F$ if for any open set \mathcal{O} around y , there are neighborhoods U of x_0 and V of y_0 such that for every $f \in A$, if the intersection of $f(U)$ and V is nonempty, then $f(U) \subseteq \mathcal{O}$. One says that A is equicontinuous if it is equicontinuous for all points $x_0 \in E$, $y_0 \in F$. The notion of equicontinuity becomes more intuitive, if we choose E and F to be metric spaces. The family A is equicontinuous at a point x if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_0), f(x)) < \epsilon$ for all $f \in A$ and all x such that $d(x_0, x) < \delta$. In other words we require all member of the family A to be continuous and to have equal variation over a given neighbourhood.

where $\mathcal{O} \subset \mathcal{M}$ are bounded, simply-connected regions. The global algebra is obtained as the inductive limit

$$\mathfrak{P}(\mathcal{M}) \doteq \lim_{\mathcal{O} \subset \mathcal{M}} \mathfrak{P}(\mathcal{O}).$$

We require that the **Locality** holds, i.e. if \mathcal{O}_1 is spacelike separated from \mathcal{O}_2 , then

$$[A, B] = 0,$$

$\forall A \in \mathcal{P}(\mathcal{O}_1), B \in \mathcal{P}(\mathcal{O}_2)$, where the Poisson bracket $[\cdot, \cdot]$ is taken in \mathcal{P} .

In chapter 4 we show how to construct models of classical field theories in agreement with the above definition. In chapters 5-8 we will show how to quantize such classical models perturbatively. The resulting structure is not a net of C^* algebras, due to the perturbative character of the construction. Nevertheless, many of the features of the Haag-Kastler net are still present.

Definition 2.39. A *perturbative algebraic quantum field theory (pAQFT) model on a spacetime \mathcal{M}* is a net of topological $*$ -algebras with sequentially continuous operations

$$\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}),$$

where $\mathcal{O} \subset \mathcal{M}$ are bounded, simply-connected regions. The global algebra is obtained as the inductive limit

$$\mathfrak{A}(\mathcal{M}) \doteq \lim_{\mathcal{O} \subset \mathcal{M}} \mathfrak{A}(\mathcal{O}),$$

and we require **Locality**.

The remaining Haag-Kastler axioms, apart from the Spectrum Condition, can be easily translated to a pAQFT context.

Definition 2.40. *Further axioms:*

1. A classical/quantum field theory model on a globally hyperbolic spacetime \mathcal{M} satisfies the **Time-slice axiom** if the algebra of a neighborhood of a Cauchy surface of a given region coincides with the algebra of the full region.
2. If the underlying spacetime \mathcal{M} has a non-trivial group of symmetries \mathcal{G} , we say that a model is covariant on \mathcal{M} , if there exists a family of isomorphisms $\alpha_g^\mathcal{O} : \mathfrak{A}(\mathcal{O}) \rightarrow \mathfrak{A}(g\mathcal{O})$ for $g \in \mathcal{G}$, such that for $\mathcal{O}_1 \subset \mathcal{O}_2$ the restriction of $\alpha_g^{\mathcal{O}_2}$ to $\mathfrak{A}(\mathcal{O}_1)$ coincides with $\alpha_g^{\mathcal{O}_1}$ and $\alpha_{g'}^{g\mathcal{O}} \circ \alpha_g^\mathcal{O} = \alpha_{g'g}^\mathcal{O}$.

The spectrum condition cannot be meaningfully defined on an arbitrary globally hyperbolic spacetime, as it relies on the action of translations, which is special feature of \mathbb{M} . We will replace this condition with a requirement we impose on preferred states on our net of algebras. These preferred states are called *Hadamard states* and they realize the idea of a positivity of energy. We discuss them in detail in section 5.1.

2.5 Locally covariant quantum field theory

In the previous section we have recalled the Haag-Kastler axioms and reviewed the generalization of these axioms to the situation where we drop some of the regularity conditions on the topology of local algebras and we drop the restriction to Minkowski spacetime, allowing for general globally hyperbolic backgrounds. We can go a step further and see what happens if we replace the embeddings of bounded regions \mathcal{O} into a fixed spacetime \mathcal{M} with arbitrary embeddings between pairs of globally hyperbolic spacetimes \mathcal{N} and \mathcal{M} . We formalize this idea by introducing the notion of an *admissible embedding*.

Definition 2.41. *We call an embedding $\chi : \mathcal{M} \rightarrow \mathcal{N}$ of a globally hyperbolic manifold \mathcal{M} into another one \mathcal{N} admissible if it is an isometry, it preserves the metric, orientations, the causal structure. The property of preserving the causal structure is defined as follows: let $\chi : \mathcal{M} \rightarrow \mathcal{N}$, for any causal curve $\gamma : [a, b] \rightarrow \mathcal{M}$, if $\gamma(a), \gamma(b) \in \chi(\mathcal{M})$ then for all $t \in]a, b[$ we have: $\gamma(t) \in \chi(\mathcal{M})$.*

The generalization of AQFT which we discuss in this section is called *Locally Covariant Quantum Field Theory* (LCQFT). For a recent extensive review of the area, see [FV15].

As in the original AQFT framework, we assign algebras of observables to globally hyperbolic spacetimes and we also want to require that for each such admissible embedding there exists an injective homomorphism

$$\alpha_\chi : \mathfrak{A}(\mathcal{M}) \rightarrow \mathfrak{A}(\mathcal{N}) \tag{2.13}$$

of the corresponding algebras of observables assigned to them, moreover if $\chi_1 : \mathcal{M} \rightarrow \mathcal{N}$ and $\chi_2 : \mathcal{N} \rightarrow \mathcal{L}$ are embeddings as above then we require the covariance relation

$$\alpha_{\chi_2 \circ \chi_1} = \alpha_{\chi_2} \circ \alpha_{\chi_1} . \tag{2.14}$$

Such an assignment \mathfrak{A} of algebras to spacetimes and algebra-morphisms to embeddings can be interpreted in the language of category theory as a *covariant functor* between two categories: the category **Loc** which is an appropriate sub category of the category whose objects are globally hyperbolic spacetimes and arrows are the admissible embeddings; and the category **Obs** of topological $*$ -algebras. The precise choice of the category **Loc** depends on the kind of objects we want to study. If the physical theory we consider is sensitive to some topological (hence non-local) features of the underlying manifold, one first restricts the class of objects considered and then studies possible extensions. The detailed analysis of such topological effects has been provided in [BSS14]. In this section we will present the framework in the simplest version, suitable for the study of scalar fields, as introduced in [BFV03]. First we recall some basic notions of category theory, which are relevant for LCQFT.

Definition 2.42. *A category \mathbf{C} consists of:*

- a class of objects $\text{Obj}(\mathbf{C})$,
- a class of morphisms (arrows) $\text{hom}(\mathbf{C})$, such that each $f \in \text{hom}(\mathbf{C})$ has a unique **source object** and **target object** (both are elements of $\text{Obj}(\mathbf{C})$). For a fixed $a, b \in \text{Obj}(\mathbf{C})$, we denote by $\text{hom}(a, b)$ the set of morphisms with a as a source and b as a target,
- a binary associative operation $\circ : \text{hom}(a, b) \times \text{hom}(b, c) \rightarrow \text{hom}(a, c)$, $f, g \mapsto f \circ g$, called *composition of morphisms*,
- the identity morphism id_c for each $c \in \text{Obj}(\mathbf{C})$.

Definition 2.43. *Let \mathbf{C}, \mathbf{D} be categories. A covariant functor \mathfrak{F} assigns to each object $c \in \mathbf{C}$ and object $\mathfrak{F}(c)$ of \mathbf{D} and to each morphism $f \in \text{hom}(\mathbf{C})$, a morphism $\mathfrak{F}(f) \in \text{hom}(\mathbf{D})$ in such a way that the following two conditions hold:*

- $\mathfrak{F}(\text{id}_c) = \text{id}_{\mathfrak{F}(c)}$ for every object $c \in \mathbf{C}$.
- $\mathfrak{F}(g \circ f) = \mathfrak{F}(g) \circ \mathfrak{F}(f)$ for all morphisms $f : a \rightarrow b$ and $g : b \rightarrow c$.

Definition 2.44. *Let \mathfrak{F} and \mathfrak{G} be functors between categories \mathbf{C} and \mathbf{D} , then a natural transformation η from \mathfrak{F} to \mathfrak{G} associates to every object $a \in \mathbf{C}$ a morphism $\text{Hom}(\mathbf{D}) \ni \eta_a : \mathfrak{F}(a) \rightarrow \mathfrak{G}(a)$, such that for every morphism $\mathbf{C} \ni f : a \rightarrow b$ we have:*

$$\eta_b \circ \mathfrak{F}(f) = \mathfrak{G}(f) \circ \eta_a.$$

We denote the family of natural transformation between \mathfrak{F} and \mathfrak{G} by $\text{Nat}(\mathfrak{F}, \mathfrak{G})$.

For more details on categories and functors, see [ML78]. In LCQFT applied to scalar fields we adopt the following definitions of categories **Loc** and **Obs**.

Definition 2.45. *The category **Loc** is a category where objects are globally hyperbolic, oriented time-oriented spacetimes and morphisms are admissible embeddings (see definition 2.41).*

Remark 2.2. Note that **Loc** is a large category, i.e. its class of objects $\text{Obj}(\mathbf{Loc})$ is not a small set. It was shown in [Few07] that one can improve the situation with the use of Whitney embedding theorem, which states that every smooth manifold of dimension d may be embedded as a smooth submanifold of \mathbb{R}^{2d+1} . Hence the collection of isomorphism equivalence classes in $\text{Obj}(\mathbf{Loc})$ may be identified with a subset of the power set of \mathbb{R}^{2d+1} , so it is a small set. This makes **Loc** essentially small.

Definition 2.46. *Depending on the context, we have the following choices for the category of observables.*

- i) In the non-perturbative setting: **Obs** is the category with unital C^* -algebras as objects and injective unit-preserving $*$ -homomorphisms as arrows.*
- ii) In classical theory: **Obs_c** is the category with locally convex topological Poisson algebras as objects and injective Poisson homomorphism as arrows.*
- iii) In the perturbative setting: **Obs_p** is the category with locally convex topological unital $*$ -algebras as objects and injective unit-preserving $*$ -homomorphisms as arrows.*

We are now ready to give a definition of a classical/quantum field theory model in the LCQFT setting.

Definition 2.47. *In the LCQFT framework, a model is a functor \mathfrak{A} from **Loc** to ...*

- i) ... **Obs** for a non-perturbative locally covariant QFT model,*
- ii) ... **Obs_c** for a locally covariant classical field theory model,*
- iii) ... **Obs_p** for a perturbative locally covariant QFT model.*

*If we don't want to specify the context, we write **Loc_{*}**. Moreover, we often use the notation $\alpha_\chi \equiv \mathfrak{A}_\chi$, where $\chi \in \text{Hom}(\mathbf{Loc})$.*

Another useful category is the category of locally convex topological vector spaces.

Definition 2.48. *Define \mathbf{Vec} to be the category whose objects are locally convex topological vector spaces (LCVS) and whose morphisms are injective homomorphisms of LCVS.*

The requirement that \mathfrak{A} is a covariant functor already generalizes the Haag-Kastler axioms of *Isotony* and *Covariance*. We can impose further requirements:

- **Einstein causality:** let $\chi_i : \mathcal{M}_i \rightarrow \mathcal{M}$, $i = 1, 2$ be morphisms of \mathbf{Loc} such that $\chi_1(M_1)$ is causally disjoint from $\chi_2(M_2)$, then we require that:

$$[\alpha_{\chi_1}(\mathfrak{A}(\mathcal{M}_1)), \alpha_{\chi_2}(\mathfrak{A}(\mathcal{M}_2))] = \{0\},$$

- **Time-slice axiom:** let $\chi : \mathcal{N} \rightarrow \mathcal{M}$, if $\chi(\mathcal{N})$ contains a neighborhood of a Cauchy surface $\Sigma \subset M$, then α_χ is an isomorphism.

The **Einstein causality** requirement reflects the commutativity of observables localized in spacelike separated regions. From the point of view of category theory, this property is encoded in the tensor structure of the functor \mathfrak{A} . In order to make this statement precise, we need to equip our categories \mathbf{Loc} and \mathbf{Obs}_* with tensor structures (for a precise definition of a tensor category, see [ML78]).

Definition 2.49. *We call a category \mathbf{C} strictly monoidal (tensor category) if there exists a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$ which is associative, i.e. $\otimes(\otimes \times 1) = \otimes(1 \times \otimes)$ and there exists an object e which is a left and right unit for \otimes .*

The category of globally hyperbolic manifolds \mathbf{Loc} can be extended to a monoidal category \mathbf{Loc}^\otimes , if we extend the class of objects with finite disjoint unions of elements of $\mathbf{Obj}(\mathbf{Loc})$,

$$\mathcal{M} = \mathcal{M}_1 \sqcup \dots \sqcup \mathcal{M}_k,$$

where $\mathcal{M}_i \in \mathbf{Obj}(\mathbf{Loc})$. Morphisms of \mathbf{Loc}^\otimes are isometric embeddings, preserving orientations and causality. More precisely, they are maps $\chi : \mathcal{M}_1 \sqcup \dots \sqcup \mathcal{M}_k \rightarrow \mathcal{M}$ such that each component satisfies the requirements for a morphism of \mathbf{Loc} and additionally all images are spacelike to each other, i.e., $\chi(M_i) \perp \chi(M_j)$, for $i \neq j$. \mathbf{Loc}^\otimes has the disjoint union as a tensor product, and the empty set as unit object. It is a monoidal category and, using

the results of [JS93], it is tensor equivalent to a strict monoidal category, which we will denote by the same symbol \mathbf{Loc}^\otimes .

On the level of C^* -algebras the choice of a tensor structure is less obvious, since, in general, the algebraic tensor product $\mathfrak{A}_1 \odot \mathfrak{A}_2$ of two C^* -algebras can be completed to a C^* -algebra with respect to many non-equivalent tensor norms. The choice of an appropriate norm has to be based on some further physical indications. This problem was discussed in [BFIR14], where it is shown that a physically justified tensor norm is the minimal C^* -norm $\|\cdot\|_{\min}$ defined by

$$\|A\|_{\min} \doteq \sup\{\|(\pi_1 \otimes \pi_2)(A)\|_{\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)}\}, \quad A \in \mathfrak{A}_1 \otimes \mathfrak{A}_2,$$

where π_1 and π_2 run through all representations of \mathfrak{A}_1 and of \mathfrak{A}_2 on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ respectively. \mathcal{B} denotes the algebra of bounded operators. If we choose π_1 and π_2 to be faithful, then the supremum is achieved, i.e. $\|A\|_{\min} = \|(\pi_1 \otimes \pi_2)(A)\|_{\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)}$. The completion of the algebraic tensor product $\mathfrak{A}_1 \odot \mathfrak{A}_2$ with respect to the minimal norm $\|A\|_{\min}$ is denoted by $\mathfrak{A}_1 \otimes_{\min} \mathfrak{A}_2$. It was shown in [BFIR14] that, under some technical assumptions, a functor $\mathfrak{A} : \mathbf{Loc} \rightarrow \mathbf{Obs}$ satisfies the axiom of Einstein causality if and only if it can be extended to a tensor functor $\mathfrak{A}^\otimes : \mathbf{Loc}^\otimes \rightarrow \mathbf{Obs}^\otimes$, which means that

$$\mathfrak{A}^\otimes(\mathcal{M}_1 \sqcup \mathcal{M}_2) = \mathfrak{A}^\otimes(\mathcal{M}_1) \otimes_{\min} \mathfrak{A}^\otimes(\mathcal{M}_2), \quad (2.15)$$

$$\mathfrak{A}^\otimes(\chi \otimes \chi') = \mathfrak{A}^\otimes(\chi) \otimes \mathfrak{A}^\otimes(\chi'), \quad (2.16)$$

$$\mathfrak{A}^\otimes(\emptyset) = \mathbb{C}. \quad (2.17)$$

In the perturbative setting, we also face the same problem with extending \mathbf{Obs}_p to a tensor category, as there are many possibilities to chose a tensor product. The most natural choices are the injective tensor product (definition 2.37) and the projective tensor product (definition 2.36). A way out it to restrict \mathbf{Obs} to the category of *nuclear* topological algebras, where these two notions coincide.

Let us now discuss the **time slice axiom**. We use it to describe the evolution between different Cauchy surfaces. Firstly, we associate to each Cauchy surface Σ the inverse limit

$$\mathfrak{A}(\Sigma) = \lim_{N \supset \Sigma}^{\leftarrow} \mathfrak{A}(\mathcal{N}). \quad (2.18)$$

We denote by ι_{MN} the inclusion of a subset N in M and by $\alpha_{MN} \doteq \mathfrak{A}\iota_{MN}$, the corresponding morphism in $\text{hom}(\mathbf{Obs})$. Elements of the inverse limit

(2.18) are sequences $A = (A_N)_{L_A \supset N \supset \Sigma}$ with $\alpha_{KN}(A_N) = A_K$, $K \subset L_A$, with the equivalence relation

$$A \sim B \text{ if } \exists L \subset L_A \cap L_B \text{ such that } A_N = B_N \text{ for all } N \subset L . \quad (2.19)$$

We define the embedding of the algebra $\mathfrak{A}(\Sigma)$ into $\mathfrak{A}(\mathcal{M})$ by

$$\alpha_{M\Sigma}(A) = \alpha_{MN}(A_N) \text{ for some (and hence all) } \Sigma \subset N \subset L_A . \quad (2.20)$$

From the time slice axiom follows that each homomorphism α_{MN} is an isomorphism. Hence $\alpha_{M\Sigma}$ is also an isomorphism, and we obtain the propagator between two Cauchy surfaces Σ_1 and Σ_2 by

$$\alpha_{\Sigma_1\Sigma_2}^M = \alpha_{M\Sigma_1}^{-1} \circ \alpha_{M\Sigma_2} . \quad (2.21)$$

This construction resembles constructions in topological field theory [Seg].

The next important notion in LCQFT is that of a *local quantum field*. In the Haag-Kastler framework on Minkowski space an essential ingredient was the translation symmetry. This symmetry allowed the comparison of observables in different regions of spacetime. This is not possible in the general covariant framework we describe here, because on generic spacetime the isometry group might be trivial. It follows that there is a priori no natural way to say what does it mean to have the same observable in a different region. We need to introduce some extra labels for the observables, which make such a comparison possible. This is where locally covariant quantum field come into the game. We can think of them as operator-valued distributions assigned to all the objects of **Loc** in a coherent way. Before we give the precise definition, we need to make clear what we mean by test function spaces

Definition 2.50. *Let \mathfrak{D} denote the functor from **Loc** to **Vec** which associates to every spacetime M its space of compactly supported \mathcal{C}^∞ -functions,*

$$\mathfrak{D}(\mathcal{M}) = \mathcal{D}(M) \doteq \mathcal{C}_c^\infty(M, \mathbb{R}) , \quad (2.22)$$

and to every embedding $\chi : M \rightarrow N$ of spacetimes the pushforward of test functions $f \in \mathfrak{D}(\mathcal{M})$

$$\mathfrak{D}\chi \equiv \chi_* , \quad \chi_* f(x) = \begin{cases} f(\chi^{-1}(x)) & , \quad x \in \chi(M) \\ 0 & , \quad \text{else} \end{cases} . \quad (2.23)$$

Note that \mathfrak{D} is a covariant functor and its target category contains also the category of topological algebras which is the target category for \mathfrak{A} . We are now ready to state the definition of a locally covariant quantum field.

Definition 2.51. *A locally covariant quantum/classical field Φ is defined as a natural transformation from the functor \mathfrak{D} of test function spaces to the functor \mathfrak{A} of field theory composed with the forgetful functor from \mathbf{Obs}_* to \mathbf{Vec} .*

More concretely, Φ is defined by a family of morphisms $\Phi_{\mathcal{M}} : \mathfrak{D}(\mathcal{M}) \rightarrow \mathfrak{A}(\mathcal{M})$, $\mathcal{M} \in \mathbf{Obj}(\mathbf{Loc})$ such that

$$\mathfrak{A}_{\chi} \circ \Phi_{\mathcal{M}} = \Phi_{\mathcal{N}} \circ \mathfrak{D}_{\chi} \quad (2.24)$$

The category theory language, which is used to formulate the axioms of LCQFT in not only a convenient way to phrase known results, but also leads to new insights. For example, one can use it to formulate what does it mean to have the same physics in all spacetimes. This property, called SPASS, is a property of the QFT functor and it has been extensively studied in [FV11a, FV11b]. Further study of structures appearing in LCQFT lead recently to construction of new theories by using symmetries of the QFT functor [Few13].

The next step in the LCQFT research is the proper understanding of the structures of gauge theories, where the topological features lead to new difficulties [DL12, SDH14, BSS14]. It would be desirable to obtain for local symmetries a framework similar to the DHR analysis done for global symmetries [DHR71, DHR74]. Another possible extension of the LCQFT framework is to replace \mathbf{Loc} with the category of framed manifolds. This idea has been used in [FV15] to prove the locally covariant version of the PCT theorem.

Chapter 3

Kinematical structure

In the framework of perturbative algebraic quantum field theory (pAQFT) we start with the classical theory, which is subsequently quantized. We work in the Lagrangian framework, but there are some modifications that we need to make to deal with the infinite dimensional character of field theory. In this chapter we give an overview of mathematical structures that will be needed later on to construct models of classical and quantum field theories. Since we do not fix the dynamics yet, the content of this chapter describes the kinematical structure of our model. Readers familiar with some of the concepts we introduce here can skip corresponding sections.

3.1 The space of field configurations

We start with a globally hyperbolic spacetime $\mathcal{M} = (M, g)$ (see definition 2.21). The next step is to define the space of field configurations. This specifies what kind of objects our model describes (e.g. scalar fields, Dirac fields, gauge fields, etc.). In the simplest situation the configuration space \mathcal{E} is a vector space.

Definition 3.1. *The configuration space \mathcal{E} on the fixed spacetime $\mathcal{M} = (M, g)$ is realized as the space of smooth sections $\Gamma(E)$ of some vector bundle $E \xrightarrow{\pi} M$ over M . Let V be the finite dimensional vector space which constitutes the fibre of E . We assume that there exists a bilinear pairing $\langle \cdot, \cdot \rangle_E : V \times V \rightarrow \mathbb{R}$. This pairing also defines an isomorphism between V and its dual V' .*

Example 3.1. *Examples of configuration spaces for commonly used theories:*

- for the real scalar field: $\mathcal{E} = \mathcal{C}^\infty(M, \mathbb{R})$,

- for the Yang-Mills theories (see chapter 7) with the trivial principal bundle: $\mathcal{E} = \Omega_1(M, g)$, where g is a Lie algebra of a compact Lie group,
- for effective quantum gravity (see chapter 8): $\mathcal{E} = \Gamma((T^*M)^{2\otimes})$.

Let us now comment on the differentiable structure on \mathcal{E} . A natural way to introduce a smooth structure on \mathcal{E} is to equip it first with the standard Fréchet topology (as defined below) and use this topology to define functional derivatives on \mathcal{E} .

Definition 3.2. Let $\Omega \subset \mathbb{R}^n$ be an open subset and $\mathcal{C}^\infty(\Omega, \mathbb{R})$ the space of smooth functions on it. We equip this space with a Fréchet topology generated by the family of seminorms:

$$p_{K,m}(\varphi) = \sup_{\substack{x \in K \\ |\alpha| \leq m}} |\partial^\alpha \varphi(x)|, \quad (3.1)$$

where $\alpha \in \mathbb{N}^N$ is a multiindex and $K \subset \Omega$ is a compact set. This is just the topology of uniform convergence on compact sets, of all the derivatives.

The definition above can be applied to define a Fréchet topology on $\mathcal{C}^\infty(M, \mathbb{R})$ with the use of coordinate charts, as M is locally \mathbb{R}^4 . It also generalizes easily to the vector-valued case $\mathcal{E} = \Gamma(E \xrightarrow{\pi} M)$. We will always assume that \mathcal{E} is equipped with this Fréchet topology.

Next we introduce a natural topology on the space of compactly supported functions $\mathcal{D}(M) \doteq \mathcal{C}_c^\infty(M, \mathbb{R})$. This topology is locally convex, but is not Fréchet.

Definition 3.3. Let $\mathcal{D}(\Omega) \doteq \mathcal{C}_c^\infty(\Omega, \mathbb{R})$, $\Omega \subset \mathbb{R}^n$. The fundamental system of seminorms on $\mathcal{D}(\Omega)$ is given by:

$$p_{\{m\}, \{\epsilon\}, a}(\varphi) = \sup_{\nu} \left(\sup_{\substack{|x| \geq \nu, \\ |p| \leq m_\nu}} |D^p \varphi^a(x)| / \epsilon_\nu \right), \quad (3.2)$$

where $\{m\}$ is an increasing sequence of positive numbers going to $+\infty$ and $\{\epsilon\}$ is a decreasing one tending to 0.

The generalization of the above topology to the vector-valued case $\mathcal{E}_c \doteq \Gamma_c(E \xrightarrow{\pi} M)$ is straightforward.

Remark 3.1. We can also consider situations in which the configuration space \mathcal{E} is not a vector space, but still can be made into an infinite dimensional affine manifold in the sense of [Mic84]. It happens for example with the space of all Lorentzian metrics or the space of gauge connections, but we this is beyond the scope of this book.

3.2 Functionals on the configuration space

We model classical and quantum observables as smooth functions on \mathcal{E} . Intuitively, a classical measurement assigns a real number to a field configuration. The smoothness condition is a regularity requirement which we need in our formalism in order to introduce various algebraic structures on the space of functionals. The main feature of the functional approach, which we advocate in this book, is that both the classical and quantum theory are defined in terms of the *same set* of functionals and differ by algebraic structures on this set. The classical theory is defined in term of a Poisson bracket, while the quantum theory in terms of a non-commutative product. We will construct these structures for the scalar field in chapters 4-6, but first we need to define the underlying space of functionals, which is the main subject of the present section. Having the quantization in mind, we work from the start with complex-valued functionals.

For functions on \mathcal{E} , the smoothness is understood in the sense of Bastiani calculus [Bas64, Ham82, Mil84, Nee06], i.e.

Definition 3.4 (after [Nee06]). *Let \mathcal{X} and \mathcal{Y} be topological vector spaces, $U \subseteq \mathcal{X}$ an open set and $f : U \rightarrow \mathcal{Y}$ a map. The derivative of f at $x \in U$ in the direction of $h \in \mathcal{X}$ is defined as*

$$\left\langle f^{(1)}(x), h \right\rangle \doteq \lim_{t \rightarrow 0} \frac{1}{t} (f(x + th) - f(x)) \quad (3.3)$$

whenever the limit exists. The function f is called differentiable at x if $\langle f^{(1)}(x), h \rangle$ exists for all $h \in \mathcal{X}$. It is called continuously differentiable if it is differentiable at all points of U and $f^{(1)} : U \times \mathcal{X} \rightarrow \mathcal{Y}$, $(x, h) \mapsto df(x)(h)$ is a continuous map. It is called a \mathcal{C}^1 -map if it is continuous and continuously differentiable. Higher derivatives are defined by

$$\left\langle f^{(k)}(x), v_1 \otimes \cdots \otimes v_k \right\rangle \doteq \frac{\partial^k}{\partial t_1 \dots \partial t_k} f(x + t_1 v_1 + \cdots + t_k v_k) \Big|_{t_1 = \dots = t_k = 0}, \quad (3.4)$$

and f is \mathcal{C}^k if $f^{(k)}$ is jointly continuous as a map $U \times \mathcal{X}^k$.

This notion of differentiability is also referred to as Michal-Bastiani differentiability, since the definition of differentiability at a point is equivalent to the one proposed by Michal [Mic38, Mic40]. However, Bastiani differentiability on an open set is a stronger notion.

We apply this to define derivatives of \mathbb{C} -valued functionals F on \mathcal{E} . By definition $F^{(1)}(\varphi)$, if it exists, is an element of the complexified dual space $\mathcal{E}'^{\mathbb{C}} \doteq \mathcal{E}' \otimes \mathbb{C}$. More generally, we have the following result.

Proposition 3.1. *Let $F : \mathcal{E} \rightarrow \mathbb{C}$ be a Bastiani smooth functional, then*

- i) $F^{(n)}(\varphi)$ is a linear continuous map from \mathcal{E}^n to \mathbb{C} ,
- ii) $F^{(n)}(\varphi)$ induces a continuous map on the completed projective tensor product $\mathcal{E}^{\hat{\otimes}_{\pi^k}} \cong \Gamma(E^{\boxtimes n} \rightarrow M^n)$, where \boxtimes is the exterior tensor products of vector bundles, defined below in 3.6. Here we denote the map on $\mathcal{E}^{\hat{\otimes}_{\pi^k}}$ by the same symbol as the original differential, i.e $F^{(n)}(\varphi)$.

Proof. Property i) follows directly from the definition of continuous differentiability. For ii) it is crucial that \mathcal{E} is a Fréchet space. For the proof of the claim, see for example [Trè06]. \square

Definition 3.5. *Let $E_1 \xrightarrow{\pi_1} M_1$, $E_2 \xrightarrow{\pi_2} M_2$ be two vector bundles over M_1 and M_2 with fibers V_1 , V_2 respectively. The exterior tensor product $E_1 \boxtimes E_2$ is defined as the vector bundle over $M_1 \times M_2$, whose fiber over $(x, y) \in M_1 \times M_2$ is $V_{1x} \otimes V_{2y}$.*

This definition has to be contrasted with the definition of the ordinary tensor product of vector bundles.

Definition 3.6. *The tensor product of vector bundles $E_1 \xrightarrow{\pi_1} M$, $E_2 \xrightarrow{\pi_2} M$ is a vector bundle over M , denoted by $E_1 \otimes E_2$ whose fiber over a point $x \in M$ is the tensor product of vector spaces $E_{1x} \otimes E_{2x}$.*

It is clear from the discussion above that in order to understand better the behavior of functional derivatives of smooth functionals on \mathcal{E} we need to bring some notions from the theory of distributions into our framework.

Definition 3.7. *The space of distributions on $\Omega \subset \mathbb{R}$ is defined to be the dual $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$ with respect to the topology given in the definition 3.3.*

Equivalently, given a linear map L on $\mathcal{D}(\Omega)$ we can decide if it is a distribution by checking one of the equivalent conditions given in the theorem below [Trè06, Rud91, Hör03].

Theorem 3.1. *A linear map u on $\mathcal{D}(\Omega)$ is a distribution if it satisfies the following equivalent conditions:*

1. *To every compact subset K of Ω there exists an integer m and a constant $C > 0$ such that for all $\varphi \in \mathcal{D}$ with support contained in K it holds:*

$$|u(\varphi)| \leq C \max_{p \leq k} \sup_{x \in \Omega} |\partial^p \varphi(x)|.$$

We call $\|u\|_{\mathcal{C}^k(\Omega)} \doteq \max_{p \leq k} \sup_{x \in \Omega} |\partial^p \varphi(x)|$ the \mathcal{C}^k -norm and if the same integer k can be used in all K for a given distribution u , then we say that u is of order k .

2. If a sequence of test functions $\{\varphi_k\}$, as well as all their derivatives converge uniformly to 0 and if all the test functions φ_k have their supports contained in a compact subset $K \subset \Omega$ independent of the index k , then $u(\varphi_k) \rightarrow 0$.

Proof. See for example [Hör03]. □

The regularity of a distribution can be characterized in terms of the falloff conditions for its Fourier transform.

Theorem 3.2. *A distribution $u \in \mathcal{E}'(\Omega)$ is smooth if and only if for every N there is a constant C_N such that:*

$$|\hat{u}(k)| \leq C_N(1 + |k|)^{-N},$$

where \hat{u} denotes the Fourier transform of u .

Proof. See for example [Hör03]. □

Definition 3.8. *The singular support of a distribution u is the complement of the largest open set on which u is smooth.*

If a distribution has a nonempty singular support we can give a further characterization of its singularity structure by specifying the direction in which it is singular. This is exactly the purpose of the definition of a wave front set.

Definition 3.9. *For a distribution $u \in \mathcal{D}'(\Omega)$ the wavefront set $\text{WF}(u)$ is the complement in $\Omega \times \mathbb{R}^n \setminus \{0\}$ of the set of points $(x, k_0) \in \Omega \times \mathbb{R}^n \setminus \{0\}$ such that there exist*

- a function $f \in \mathcal{D}(\Omega)$ with $f(x) = 1$,
- an open conic neighborhood C of k_0 , with

$$\sup_{k \in C} (1 + |k|)^N |\widehat{f \cdot u}(k)| < \infty \quad \forall N \in \mathbb{N}_0.$$

On a manifold M the definition of the Fourier transform depends on the choice of a chart, but the property of strong decay in some direction (characterized now by a point (x, k) , $k \neq 0$ of the cotangent bundle T^*M)

turns out to be independent of this choice. Therefore the wave front (WF) set of a distribution on a manifold M is a well defined closed conical subset of the cotangent bundle (with the zero section removed).

The wavefront sets provide a simple criterion for the existence of pointwise products of distributions. Before we give it, we prove a more general result concerning the pullback. Here we follow closely [BF09a, Hör03]. Let $\alpha : \Omega \rightarrow \tilde{\Omega}$ be a smooth map between $\Omega \subset \mathbb{R}^m$ and $\tilde{\Omega} \subset \mathbb{R}^n$. We define the normal set N_α of the map α as:

$$N_\alpha \doteq \{(\alpha(x), \eta) \in \tilde{\Omega} \times \mathbb{R}^n \mid (d\alpha_x)^T(\eta) = 0\},$$

where $(d\alpha_x)^T$ is the transposition of the differential of α at x .

Theorem 3.3. *Let Γ be a closed cone in $\tilde{\Omega} \times (\mathbb{R}^n \setminus \{0\})$ and $\alpha : \Omega \rightarrow \tilde{\Omega}$ as above, such that $N_\alpha \cap \Gamma = \emptyset$. Then the pullback of functions $\alpha^* : \mathcal{E}(\tilde{\Omega}) \rightarrow \mathcal{E}(\Omega)$ has a unique, sequentially continuous extension to a sequentially continuous map $\mathcal{D}'_\Gamma(\tilde{\Omega}) \rightarrow \mathcal{D}'(\Omega)$, where $\mathcal{D}'_\Gamma(\tilde{\Omega})$ denotes the space of distributions with WF sets contained in Γ .*

Proof. For proof see [BF09a, Hör03]. □

Using this theorem we can define the pointwise product of two distributions t, s on an n -dimensional manifold M as a pullback by the diagonal map $D : M \rightarrow M \times M$ if the pointwise sum of their wave front sets

$$\text{WF}(t) + \text{WF}(s) = \{(x, k + k') \mid (x, k) \in \text{WF}(t), (x, k') \in \text{WF}(s)\},$$

does not intersect the zero section of \dot{T}^*M (see theorem 8.2.10 of [?]). To see that this is the right criterion, note that the set of normals of the diagonal map $D : x \mapsto (x, x)$ is given by $N_D = \{(x, x, k, -k) \mid x \in M, k \in T^*M\}$. The product ts is then defined by: $ts = D^*(t \otimes s)$ and if one of t, s is compactly supported, then so is ts and we define the contraction by $\langle t, s \rangle \doteq \widehat{ts}(0)$. Another way of seeing that the construction works is to look at the Fourier transformed version. For $t, s \in \mathcal{E}(\Omega)$ we have

$$\langle ts, fg \rangle = \frac{1}{(2\pi)^n} \int \widehat{tf}(k) \widehat{sg}(-k) dk, \quad (3.5)$$

where $f, g \in \mathcal{D}(\Omega)$ are chosen with sufficiently small support. We will now give a brief argument for why the integral above converges. Note that if $k \neq 0$, then either \widehat{tf} is fast-decaying in a conical neighborhood around k or \widehat{sg} is fast-decaying in a conical neighborhood around $-k$, while the other factor is polynomially bounded.

Motivated by the criterion above we can distinguish certain important classes of functionals by analyzing the WF set properties of their derivatives. Before we move to that task, there is one more concept we need to introduce first. Following the spirit of AQFT we would like to define some notion of localization (on spacetime) of functionals we consider. More precisely, later on we will construct algebraic structures associated to bounded regions of M , so we need to be able to decide if a given observable (modeled as a smooth functional) belongs to a given region or not. We achieve this by introducing the notion of the *spacetime support*.

Definition 3.10. *Let F be a map from $\mathcal{X} = \Gamma(E \rightarrow M)$ to \mathcal{Y} , where E is a vector bundle over M and \mathcal{Y} is a set. The spacetime support of F is defined by*

$$\text{supp } F \doteq \{x \in M \mid \forall \text{ neighborhoods } U \text{ of } x \exists \varphi, \psi \in \mathcal{X}, \text{supp } \psi \subset U, \quad (3.6)$$

$$\text{such that } F(\varphi + \psi) \neq F(\varphi)\} .$$

Here we rely on the fact that $\Gamma(E \rightarrow M)$ is equipped with a linear structure, but the concept of spacetime support generalizes to the case where \mathcal{X} is a space of sections of an arbitrary bundle over M [BFR13]. Note that if F is linear and $\mathcal{X} = \mathcal{C}_c^\infty(\Omega, \mathbb{R})$, where $\Omega \subset \mathbb{R}^n$, then definition 3.10 reduces to the standard definition of the support of a distribution on Ω .

Definition 3.11. *Let $u \in \mathcal{D}'(\Omega)$. The support $\text{supp } u$ of a distribution $u \in \mathcal{D}'(\Omega)$ is the smallest closed set \mathcal{O} such that $u|_{\Omega \setminus \mathcal{O}} = 0$. In other words:*

$$\text{supp } u \doteq \{x \in \Omega \mid \forall U \text{ open neigh. of } x, U \subset \Omega \exists \varphi \in \mathcal{D}(\Omega), \text{supp } \varphi \subset U, \text{ s.t. } \langle u, \varphi \rangle \neq 0\} .$$

An even closer relation between the distributional support and the spacetime support of a functional can be seen in the following result [BDLGR15]:

$$\text{supp } F = \overline{\bigcup_{\varphi \in \mathcal{E}} \text{supp}(F^{(1)}(\varphi))} .$$

The concept of spacetime support applies also to functions of several arguments.

Definition 3.12. *Let F be a map on the product $\mathcal{X} \equiv \Gamma(E_1 \rightarrow M) \times \cdots \times \Gamma(E_k \rightarrow M)$ taking values in a set \mathcal{Y} . Then the spacetime support of F is defined as*

$$\text{supp } F \doteq \{x \in M \mid \forall \text{ neighborhoods } U \text{ of } x \exists \varphi = (\varphi_1, \dots, \varphi_k) \in \mathcal{X}$$

$$\text{and } \psi \in \Gamma(E_i \rightarrow M) \text{ for some } i \in \{1, \dots, k\}, \text{supp } \psi \subset U, \quad (3.7)$$

$$\text{such that } F(\varphi_1, \dots, \varphi_i + \psi, \dots, \varphi_k) \neq F(\varphi)\} .$$

We have already seen that functional derivatives of Bastiani smooth functionals on \mathcal{E} are compactly supported distributions. Further restrictions on regularity and support of distributions appearing as functional derivatives are obtained for local functionals.

Definition 3.13. *A functional $F \in \mathcal{C}^\infty(\mathcal{E}, \mathbb{C})$ is called local (an element of \mathcal{F}_{loc}) if for each $\varphi \in \mathcal{E}$ there exists $k \in \mathbb{N}$ such that*

$$F(\varphi) = \int_M \alpha(j_x^k(\varphi)) , \quad (3.8)$$

where $j_x^k(\varphi)$ is the k -th jet prolongation of φ and α is a density-valued function on the jet bundle.

Remark 3.2. If F is local then $F^{(n)}(\varphi)$ is a distribution supported on the thin diagonal

$$D_n \doteq \{(x_1, \dots, x_n) \in M^n, x_1 = \dots = x_n\} .$$

We equip the space \mathcal{F}_{loc} of local functionals on the configuration space with the pointwise product using the prescription

$$(F \cdot G)(\varphi) \doteq F(\varphi)G(\varphi) , \quad (3.9)$$

where $\varphi \in \mathcal{E}$. \mathcal{F}_{loc} is not closed under this product, but we can consider instead the space \mathcal{F} of *multilocal functionals*, which is defined as the algebraic closure of \mathcal{F}_{loc} under the product (3.9). We can also introduce the involution operator $*$ on \mathcal{F} using the complex conjugation, i.e.

$$F^*(\varphi) \doteq \overline{F(\varphi)} .$$

This way we obtain a commutative $*$ -algebra. The self-adjoint elements of this algebra are functionals that satisfy $F^* = F$, i.e. these are the real-valued functionals. We will see later on that we can identify such self-adjoint elements with classical observables.

Local and multilocal functionals satisfy some important regularity properties. Firstly, for local functionals the wavefront set of $F^{(n)}(\varphi)$ is orthogonal to TD_n , the tangent bundle of the thin diagonal. In particular, $F^{(1)}(\varphi)$ has empty wavefront set, so is smooth for each fixed $\varphi \in \mathcal{E}$. The latter is true also for multilocal functionals, i.e. $F \in \mathcal{F}$. Note that using the invariant volume form μ_g we can therefore identify $F^{(1)}(\varphi)$ with an element of $\mathcal{E}^{*\text{c}} \doteq \Gamma_c^{\text{c}}(E^* \rightarrow M)$, where E^* is the dual bundle of E .

Actually, one can characterize locality in an abstract way, using Bastiani smoothness and WF set properties as well as support properties of the first and the second derivative. This result has been proven in [BDLGR15], based on the ideas of [BFR12].

Theorem 3.4. *Let U be an open subset of $\mathcal{E}(M) \doteq C^\infty(M, \mathbb{R})$ and $F : U \rightarrow \mathbb{R}$ be smooth in the sense of Bastiani. Assume that*

1. *F is additive and compactly supported in $K \subset M$.*
2. *For every $\varphi \in U$, the differential $F^{(1)}(\varphi)$ of F at φ has empty WF set and the induced map $F^{(1)} : U \rightarrow \mathcal{D}(M)$ is Bastiani smooth.*
3. *For every $\varphi \in U$, the distribution $F^{(2)}(\varphi) \in \mathcal{E}'(M^2)$ induced by the second derivative of F is supported on the diagonal D_2 and $\text{WF}(F^{(2)}(\varphi)) \perp TD_2$.*

Then, for every $\varphi \in U$, there is a neighborhood V of the origin, an integer k and a smooth real-valued function f on the k -jet bundle $J^k(M)$ such that $F(\varphi + \psi) = \int_M f(j_x^k \psi) dx$ for every $\psi \in V$, where $j_x^k \psi$ is the k -jet of ψ at x .

3.3 Fermionic field configurations

Up to now we have considered the configuration spaces that were ordinary infinite dimensional manifolds. Now we will consider a situation, where the configuration space is graded. Physically this becomes relevant when we want to describe Fermionic field configurations, like for example matter fields in QED (Dirac fields). Here we will use the term ‘‘Fermionic’’ in the sense of ‘‘anticommuting’’, rather than ‘‘non-integer spin’’. The later context is related to the spin-statistic theorem [SW00], which, however, doesn’t apply to auxiliary, non-physical field variables like ghosts and antighosts that will be introduced in chapter 7.

There are several ways to define graded manifolds geometrically and in the infinite dimensional context, the approach proposed by [Sac08] seems to be the most appropriate. Here, however, we are not interested in the structure of graded manifolds themselves, but we take the algebraic point of view and focus on their rings of polynomial functions. As in earlier chapters we denote by \mathcal{E} the space of sections of some vector bundle $E \xrightarrow{\pi} M$ (this could be for example the Dirac bundle describing the electrons in QED). Let \mathcal{E}' be its strong dual (i.e. topological dual space equipped with the topology of uniform convergence on bounded sets of \mathcal{E}). We want to give meaning to

the notion of odd \mathcal{E} , i.e. $\mathcal{E}[1]$, where the number in square brackets denotes the degree shift. Like elsewhere in this book, we take the algebraic viewpoint and characterize this space in terms of its ring of functions $\mathcal{O}(\mathcal{E}[1])$. We call the latter *the space of antisymmetric functionals on \mathcal{E}* . Before we can make this notion precise we quote a known result from [Buc72].

Theorem 3.5. *Let \mathcal{X}, \mathcal{Y} be two Fréchet spaces, one of which has the approximation property. Then*

$$i) \mathcal{X}' \hat{\otimes}_{\pi} \mathcal{Y}' \cong (\mathcal{X} \hat{\otimes}_{\epsilon} \mathcal{Y})',$$

$$ii) \mathcal{X}' \hat{\otimes}_{\epsilon} \mathcal{Y}' \cong (\mathcal{X} \hat{\otimes}_{\pi} \mathcal{Y})',$$

where all the duals are meant as the strong duals.

Proof. See [Buc72]. □

We can now prove the following useful result concerning tensor products of \mathcal{E} and its strong dual.

Proposition 3.2. $\Gamma'(E^{\boxtimes n} \rightarrow M^n)$, the strong dual of $\Gamma(E^{\boxtimes n} \rightarrow M^n)$ is isomorphic to $\Gamma'(E)^{\hat{\otimes}_{\pi} n}$

Proof. Since $\Gamma(E)$ is Fréchet and has the approximation property (see [Jar12]), we have $(\Gamma(E)^{\hat{\otimes}_{\pi} n})' \cong \Gamma'(E)^{\hat{\otimes}_{\epsilon} n}$. From the nuclearity of $\Gamma'(E)$ follows that the later can be identified with $\Gamma'(E)^{\hat{\otimes}_{\pi} n}$. As $\Gamma(E)^{\hat{\otimes}_{\pi} n} \cong \Gamma(E^{\boxtimes n} \rightarrow M^n)$, we obtain

$$\Gamma'(E^{\boxtimes n} \rightarrow M^n) \cong \Gamma'(E)^{\hat{\otimes}_{\pi} n}.$$

□

The following definitions make precise the notion of antisymmetric functionals.

Definition 3.14. *Let $E \xrightarrow{\pi} M$ be a vector bundle with fiber V .*

i) We define $\Gamma'_a(E^{\boxtimes n} \rightarrow M^n)$ as the completion of $\underbrace{\Gamma'(E) \wedge \cdots \wedge \Gamma'(E)}_n$ with respect to the topology of $\Gamma'(E)^{\hat{\otimes}_{\pi} n}$. The subscript “a” stands for antisymmetry.

ii) Analogously, $\Gamma'_s(E^{\boxtimes n} \rightarrow M^n)$ is the completion of the symmetric tensor product $\underbrace{\Gamma'(E) \otimes_s \cdots \otimes_s \Gamma'(E)}_n$.

iii) This generalizes also to more complicated statistics. The completion of $\underbrace{\Gamma'(E) \wedge \cdots \wedge \Gamma'(E)}_p \otimes \underbrace{\Gamma'(E) \otimes_s \cdots \otimes_s \Gamma'(E)}_q$ will be denoted by $\Gamma'_{\mathbf{p}|q}(E^{\boxtimes n} \rightarrow M^n)$, where $n = p + q$. In general we can consider labels which are arbitrary finite sequences of the type $p_1|\mathbf{p}_2|\mathbf{p}_3|\dots|p_k$, where integers in boldface indicate the antisymmetric factors and the remaining indices correspond to totally symmetric factors.

Definition 3.15. We define $\mathcal{O}(\mathcal{E}[1])$, the space of odd (antisymmetric) functionals on \mathcal{E} as

$$\mathcal{A} \doteq \prod_{k=0}^{\infty} \mathcal{A}^k \doteq \prod_{k=0}^{\infty} \Gamma'_a(E^{\boxtimes k} \rightarrow M^k) \otimes \mathbb{C},$$

where the notation Γ'_a is clarified in definition 3.14.

The elements of \mathcal{A} are called here the *antisymmetric functionals*, written as (possibly infinite) sequences: $T = (T_k)_{k \in \mathbb{N}}$, where the components $T_k \in \mathcal{A}^k$ are referred to as homogeneous functionals. It is convenient to introduce a notation for such functionals, which is commonly used in physics

Definition 3.16. Let $\mathcal{E} = \Gamma(E \rightarrow M)$ with fiber V . Choose a basis on the fiber labelled by the index set I . For $\alpha \in I$, $x \in M$, define $\Phi_x^\alpha \in \mathcal{E}'$ as the evaluation functional

$$\Phi_x^\alpha(\varphi) \doteq \varphi^\alpha(x).$$

Formally, we can write elements of $T \in \mathcal{A}^k$ in terms of integral kernels

$$\begin{aligned} T(u_1 \otimes \cdots \otimes u_k) &= \\ &= \sum_{\alpha_1, \dots, \alpha_k} \int T(x_1, \dots, x_k)_{\alpha_1, \dots, \alpha_k} u_1(x_1)^{\alpha_1} \cdots u_k(x_k)^{\alpha_k} d\mu(x_1) \cdots d\mu(x_k). \end{aligned}$$

We can write this also in terms of evaluation functionals defined in 3.16 as

$$T = \sum_{\alpha_1, \dots, \alpha_k} \int T(x_1, \dots, x_k)_{\alpha_1, \dots, \alpha_k} \Phi_{x_1}^{\alpha_1} \cdots \Phi_{x_k}^{\alpha_k} d\mu(x_1) \cdots d\mu(x_k).$$

Using this notation we are able to translate some formal expression used in the physics literature into our framework. The key point is that in physics one usually identifies Φ_x^α 's with some abstract Grassmann-valued functions. We refrain from that interpretation, since we want to avoid dealing with infinitely many Grassmann parameters. Instead we treat Φ_x^α 's as honest,

real-valued *functionals*. This viewpoint on functionals of Fermionic fields has been proposed in [Rej11b].

The generalization of the definition 3.15 to the graded case is straightforward.

Definition 3.17. Define $\mathcal{O}(\mathcal{E}_0 \oplus \mathcal{E}_1[1] \oplus \mathcal{E}_2[2])$ as $\mathcal{C}^\infty(\mathcal{E}_0, \mathcal{A})$, where

$$\mathcal{A} \doteq \prod_{\substack{k=0 \\ p+q=k}}^{\infty} \Gamma'_{\mathbf{p}|q}(E_1^{\boxtimes p} \boxtimes E_2^{\boxtimes q} \rightarrow M^k) \otimes \mathbb{C}.$$

Adding further terms to $\mathcal{E}_0 \oplus \mathcal{E}_1[1] \oplus \mathcal{E}_2[2]$ is reflected by adding further factors in $\mathbf{p}|q$, where odd degrees contribute antisymmetric tensor powers and even degrees contribute symmetric tensor powers.

We also introduce the notation

$$\mathcal{O}^k(\mathcal{E}_0 \oplus \mathcal{E}_1[1] \oplus \mathcal{E}_2[2]) \doteq \mathcal{C}^\infty(\mathcal{E}_0, \mathcal{A}^k),$$

where $\mathcal{A}^k \doteq \bigoplus_{p+q=k} \Gamma'_{\mathbf{p}|q}(E_1^{\boxtimes p} \boxtimes E_2^{\boxtimes q} \rightarrow M^k) \otimes \mathbb{C}$.

Remark 3.3. For the future reference, we make a distinction between $\mathcal{O}(\mathcal{E})$ and $\mathcal{O}(\mathcal{E}[0])$. The former is always understood as the space of smooth functionals, while the latter is the space of (potentially infinite) series in symmetric tensor products, without any notion of convergence. We will need this distinction in section 7, where we will need formal objects of the type $\mathcal{O}(\mathcal{E}[0])$.

Next we introduce the notion of a derivative of a graded functional.

Definition 3.18. Let $F \in \mathcal{A}^k$, $u \in \mathcal{E}^{\otimes k-1}$, $h \in \mathcal{E}$.

i) The left derivative of F at u in the direction of h is defined by

$$\left\langle \frac{\delta^l F}{\delta \varphi}(u), h \right\rangle \doteq F(h \wedge u), k > 0$$

$$\frac{\delta^l F}{\delta \varphi} = 0 \quad F \in \mathcal{A}^0.$$

We extend this definition to \mathcal{A} by linearity.

ii) Analogously, the right derivative of F at u in the direction of h is defined by

$$\left\langle \frac{\delta^r F}{\delta \varphi}(u), h \right\rangle \doteq F(u \wedge h), k > 0$$

$$\frac{\delta^r F}{\delta \varphi} = 0 \quad F \in \mathcal{A}^0.$$

Clearly, for $F \in \mathcal{A}$, $\left\langle \frac{\delta^l F}{\delta \varphi}(\cdot), h \right\rangle$ is an element of \mathcal{A} and we can think of $\frac{\delta^l F}{\delta \varphi}$ as a distribution (i.e. continuous linear map) on \mathcal{E} with values in the graded algebra \mathcal{A} . This point of view has been adapted in [Rej11b]. We equip the space of such distributions with the strong topology (the topology of uniform convergence on bounded sets) and use the notation $L_b(\mathcal{E}, \mathcal{A})$. The theory of distribution taking values in general locally convex vector space has been developed in [Sch57, Sch58]. One can define the notions of convolution, Fourier transform and WF set for such objects. We also have the analogue of the theorem 3.3. The following definitions allow to distinguish important classes of graded functionals. For the simplicity of notation we spell out explicitly only the case where the configuration space is $\mathcal{E}_0 \oplus \mathcal{E}_1[1]$.

Definition 3.19. *The support of graded functional $F \in \mathcal{C}^\infty(\mathcal{E}_0, \mathcal{A}^k)$ is defined by*

$$\overline{\sum_{\varphi \in \mathcal{E}_0, u \in \mathcal{E}_1^{\hat{\otimes} \pi^k}} \text{supp} \left(\frac{\delta F}{\delta \varphi_0}(\varphi; u) \right)} \cup \overline{\sum_{\varphi \in \mathcal{E}_0, u \in \mathcal{E}_1^{\hat{\otimes} \pi^{k-1}}} \text{supp} \left(\frac{\delta^l F(\varphi)}{\delta \varphi_1}(u) \right)},$$

where $\frac{\delta}{\delta \varphi_0}$ denotes the usual Bastiani derivative with respect to the variable in \mathcal{E}_0 , while $\frac{\delta}{\delta \varphi_1}$ is the graded derivative on \mathcal{A} , defined in 3.18. The definition of support generalizes to the case of $\mathcal{E}_0 \oplus \mathcal{E}_1[i_1] \oplus \dots \oplus \mathcal{E}_N[i_N]$ by adding further terms in the sum.

Definition 3.20. *A graded functional $F \in \mathcal{C}^\infty(\mathcal{E}_0, \mathcal{A}^k)$ is called local (an element of $\mathcal{C}_{\text{loc}}^\infty(\mathcal{E}_0, \mathcal{A}^k)$) if it is compactly supported and for each $\varphi \in \mathcal{E}$ there exists $k \in \mathbb{N}$ such that*

$$F(\varphi; h_1, \dots, h_k) = \int_M \alpha(j_x^{i_0}(\varphi), j_x^{i_1}(h_1), \dots, j_x^{i_k}(h_k)), \quad (3.10)$$

where $\varphi \in \mathcal{E}_0$, $h_1, \dots, h_k \in \mathcal{E}_1$ and α is a density-valued function on the jet bundle. We denote

$$\mathcal{O}_{\text{loc}}(\mathcal{E}_0 \oplus \mathcal{E}_1[1]) \doteq \prod_{k=0}^{\infty} \mathcal{C}_{\text{loc}}^\infty(\mathcal{E}_0, \mathcal{A}^k).$$

Definition 3.21. We equip $\mathcal{O}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$ with the antisymmetric product:

$$\begin{aligned} & \langle (F \wedge G)(\varphi); h_1, \dots, h_{p+q} \rangle \doteq \\ & \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \text{sgn}(\sigma) \langle F(\varphi); h_{\sigma(1)}, \dots, h_{\sigma(p)} \rangle \langle G(\varphi); h_{\sigma(p+1)}, \dots, h_{\sigma(p+q)} \rangle, \end{aligned} \quad (3.11)$$

where F and G are of degree p and q respectively, $\varphi \in \mathcal{E}_0$ and $h_i \in \mathcal{E}_1$.

Definition 3.22. We define $\mathcal{O}_{\text{ml}}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$ as the algebraic completion of $\mathcal{O}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$ with respect to the graded product \wedge .

By definition, we can differentiate elements of $\mathcal{O}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$ as functionals on \mathcal{E}_0 . The n -th derivative of $F \in \mathcal{O}(\mathcal{E}_0, \mathcal{A})$ at point $\varphi \in \mathcal{E}_0$ will be denoted by $F^{(n)}(\varphi)$ or $\frac{\delta F}{\delta \varphi_0}(\varphi)$, to distinguish it from the graded derivative $\frac{\delta F(\varphi)}{\delta \varphi_1}$ on \mathcal{A} .

The following result allows to characterize derivatives of compactly supported functionals.

Proposition 3.3. Let $F \in \mathcal{O}^k(\mathcal{E}_0 \oplus \mathcal{E}_1[1]) \doteq \mathcal{C}^\infty(\mathcal{E}_0, \mathcal{A}^k)$ be compactly supported, then the n -th derivative of F satisfies

$$F^{(n)}(\varphi) \in \Gamma'^{\mathcal{C}}(E_0^{\boxtimes n} \boxtimes E_1^{\boxtimes k} \rightarrow M^{k+n}) \cong (\mathcal{E}'_0)^{\hat{\otimes}_\pi n} \hat{\otimes}_\pi (\mathcal{E}'_1)^{\hat{\otimes}_\pi k} \otimes \mathbb{C},$$

for all $\varphi \in \mathcal{E}_0$.

Proof. We start with $F^{(1)}$. By definition $F^{(1)}(\varphi)$ is a continuous linear map from \mathcal{E}_0 to $(\mathcal{E}'_1)^{\hat{\otimes}_\pi k} \otimes \mathbb{C}$, so is a vector-valued distribution in $L_b(\mathcal{E}_0, (\mathcal{E}'_1)^{\hat{\otimes}_\pi k}) \otimes \mathbb{C}$. Since \mathcal{E}_0 is a Montel space, the bounded sets are the same as equicontinuous sets, so $L_b(\mathcal{E}_0, (\mathcal{E}'_1)^{\hat{\otimes}_\pi k})$ is identified with $L_\varepsilon(\mathcal{E}_0, (\mathcal{E}'_1)^{\hat{\otimes}_\pi k})$, the space with the topology of uniform convergence on equicontinuous sets. The latter is then isomorphic to $\mathcal{E}'_0 \hat{\otimes}_\varepsilon (\mathcal{E}'_1)^{\hat{\otimes}_\pi k}$, since both arguments are complete and have the approximation property. Next we use the fact that \mathcal{E}'_0 and $(\mathcal{E}'_1)^{\hat{\otimes}_\pi k}$ are nuclear, to conclude that the injective product can be replaced with the projective product. Finally, we use the Buchwalter's theorem 3.5 to conclude that

$$F^{(1)}(\varphi) \in (\mathcal{E}'_0) \hat{\otimes}_\pi (\mathcal{E}'_1)^{\hat{\otimes}_\pi k} \otimes \mathbb{C} \cong \Gamma'^{\mathcal{C}}(E_0 \boxtimes E_1^{\boxtimes k} \rightarrow M^{k+1}),$$

for all $\varphi \in \mathcal{E}_0$. By iterating this procedure we obtain the result for all n . \square

In analogy to the bosonic case, we can conclude that for a local functional $F \in \mathcal{O}_{\text{loc}}^k(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$, the WF set of $F^{(n)}(\varphi) \in \Gamma'(E_0^{\boxtimes n} \boxtimes E_1^{\boxtimes k} \rightarrow M^{k+n})$ is orthogonal to TD_{k+n} , the tangent bundle of the thin diagonal.

3.4 Vector fields

In this section we define some important geometrical structures on \mathcal{E} . Firstly, note that we can view \mathcal{E} (in a trivial way) as an infinite dimensional manifold modeled on a locally convex topological vector space (a Fréchet space in this case). For a precise definition of infinite dimensional manifolds see [KM97, Nee06]. The tangent space to \mathcal{E} is then given by

$$T\mathcal{E} = \mathcal{E} \times \mathcal{E},$$

and smooth vector fields are smooth sections

$$\Gamma(T\mathcal{E}) \cong \mathcal{C}^\infty(\mathcal{E}, \mathcal{E}).$$

As in the finite dimensional situation, vector fields on \mathcal{E} form a Lie algebra, where the Lie bracket is given by the commutator $[\cdot, \cdot]$.

We can also define differential forms on \mathcal{E} , but here the definition is a bit more tricky than with the vector fields. In [Nee06] one uses the following definition

Definition 3.23. *Let X be a differentiable manifold and V a locally convex topological vector space, then a V -valued p -form on X is a function which associates to each $x \in X$ a p -linear alternating map $\omega_x = \omega(x) : T_x(M)^p \rightarrow V$ such that in local coordinates the map $(x, v_1, \dots, v_p) \mapsto \omega_x(v_1, \dots, v_p)$ is smooth.*

It follows from proposition 3.1 that derivatives of Bastiani smooth functionals are n -forms in the above sense. However, later on we will need objects more general than these, so it is better to modify the concepts of vector fields and forms already at this point.

We are interested in the complexification of $\Gamma(T\mathcal{E})$, i.e. in $\Gamma^c(T\mathcal{E})$. This space can be identified with $\mathcal{C}^\infty(\mathcal{E}, \mathcal{E}^c)$. As we did with the functionals, we want to require vector fields to be compactly supported in some sense. There are two possibilities here; we can view X as an element of $\mathcal{C}^\infty(\mathcal{E}, \mathcal{E}^c)$ or as a derivation on $\mathcal{C}^\infty(\mathcal{E}, \mathbb{R})$. The notion of support which we invoke here takes both these aspects into account. In the first, we restrict ourselves to vector fields that are elements of $\mathcal{C}^\infty(\mathcal{E}, \mathcal{E}_c^c)$. We define the support as follows.

Definition 3.24. *Let $X \in \mathcal{C}^\infty(\mathcal{E}, \mathcal{E}_c^c)$ be a vector field. We define*

$$\text{supp } X = \overline{\sum_{\varphi \in \mathcal{E}} \text{supp}(X^{(1)}(\varphi))} \cup \overline{\sum_{\varphi \in \mathcal{E}} \text{supp}(X(\varphi))} \quad (3.12)$$

We restrict ourselves to vector fields that are compactly supported. Note that this implies that in particular they need to induce elements of $\mathcal{C}^\infty(\mathcal{E}, \mathcal{E}_c^c)$. The first part of the formula (3.12) refers to the support of X , as a function on \mathcal{E} . The second term is the support of X seen as a derivation. Next we define locality.

Definition 3.25. *Let $\mathcal{V}_{\text{loc}} \subset \Gamma^c(T\mathcal{E})$ denote the space of compactly supported complexified vector fields X that can be written in the form*

$$X(\varphi)(x) = \tilde{X}(j_x^k(\varphi)) \equiv X_x(\varphi),$$

where $k \in \mathbb{N}$ and \tilde{X} is some \mathcal{E}_c -valued function on the jet bundle. Such vector fields are called **local**.

Note that elements of \mathcal{V}_{loc} are derivations of \mathcal{F}_{loc} and \mathcal{V}_{loc} is a Lie subalgebra (over \mathbb{C}) of $\Gamma^c(T\mathcal{E})$, where the Lie bracket is given by the commutator of vector fields. Note, however, that it is not an \mathcal{F} -submodule of $\Gamma^c(T\mathcal{E})$. In the next step we define multivector fields. We will use here a definition which differs from the standard one used in the literature, but is more natural in our context. Let us start with some motivation. Firstly, we want to be able to insert a differential $F^{(1)}$ of a local functional into a local bi-vector field and the result should be an element of \mathcal{V}_{loc} . Secondly, we want antisymmetry, some smoothness conditions and the compact support requirement. The locality implies that we have to consider objects more general than just elements of $\mathcal{C}^\infty(\mathcal{E}, \mathcal{E}_c^c \wedge \mathcal{E}_c^c)$, which would be the standard notion of the space of bi-vector fields on an infinite dimensional manifold. Instead, we use the framework introduced in section 3.3 and view the complexified multivector fields as $\mathcal{O}(T^*[1]\mathcal{E})$, where $T^*[1]\mathcal{E}$ is the odd cotangent bundle of \mathcal{E} , i.e.

$$T^*[1]\mathcal{E} \doteq \mathcal{E} \oplus \mathcal{E}^*[1],$$

where $\mathcal{E}^* \doteq \Gamma(E^* \rightarrow M)$. The notion of functions on a graded space has been clarified in definition 3.17. Among all elements of $\mathcal{O}(T^*[1]\mathcal{E})$ we distinguish the local ones.

Definition 3.26. *The space of local multivector fields is defined as $\mathcal{O}_{\text{loc}}(\mathcal{E} \oplus \mathcal{E}^*[1])$ in the sense of definition 3.20.*

In particular, we identify \mathcal{V}_{loc} with $\mathcal{O}_{\text{loc}}^1(\mathcal{E} \oplus \mathcal{E}^*[1])$. Note that the notion of the support of a vector field which we have introduced in definition 3.12 is now just a special case of 3.19. The space of local multivector fields defined this way is closed under insertion of differentials $F^{(1)}$ of local functionals, as required.

In the next step we introduce multilocal multivector fields.

Definition 3.27. *The space of multilocal multivector fields $\Lambda\mathcal{V}$ is defined as $\mathcal{O}_{\text{ml}}(\mathcal{E} \oplus \mathcal{E}^*[1])$, in the sense of 3.22. We also denote $\mathcal{V} \doteq \mathcal{O}_{\text{ml}}^1(\mathcal{E} \oplus \mathcal{E}^*[1])$.*

Remark 3.4. Note that \mathcal{V} is just the algebraic completion of \mathcal{V}_{loc} as a \mathcal{F} -module and a Lie subalgebra (over \mathbb{C}) of $\Gamma^c(T\mathcal{E})$. Elements of \mathcal{V} are derivations of \mathcal{F} .

At this point it is convenient to introduce some notation. The action of a vector field $X \in \mathcal{V}$ on a functional $F \in \mathcal{F}$ can be written as

$$(\partial_X F)(\varphi) = \left\langle F^{(1)}(\varphi), X(\varphi) \right\rangle.$$

As $F^{(1)}(\varphi)$ is represented by a certain integration measure formally written as $\frac{\delta F}{\delta\varphi(x)}$, we can use the notation

$$(\partial_X F)(\varphi) = \int_M X_x(\varphi) \frac{\delta F}{\delta\varphi(x)},$$

which motivates the following:

$$X = \int_M X_x \frac{\delta}{\delta\varphi(x)}.$$

This notation is analogous to the one used commonly in the finite dimensional case, i.e. $v = \sum_{i=1}^N v^i \partial_i$, where $v \in \Gamma(T\mathbb{R}^N)$. In the physics literature this formal notation is commonly used, but one replaces $\frac{\delta}{\delta\varphi(x)}$ with a formal generator $\varphi^\ddagger(x)$ called the *antifield*, i.e.

$$X(\varphi, \varphi^\ddagger) = \int_M X_x(\varphi) \varphi^\ddagger(x)$$

This way, vector fields of our approach can be identified with functions of φ and φ^\ddagger present in other approaches. Similarly, we write k -vector fields $Y \in \wedge^k \mathcal{V}$ in the form

$$Y(\varphi) = \int_M Y(\varphi)(x_1, \dots, x_k) \frac{\delta}{\delta\varphi(x_1)} \cdots \frac{\delta}{\delta\varphi(x_k)},$$

where all the indices of the fiber V^* have been suppressed and $Y(\varphi)(x_1, \dots, x_k)$ is a distributional kernel with antisymmetry properties reflecting definition 3.14. If Y is local, then this distribution is supported on the thin diagonal. In the antifield notation we express Y as

$$Y(\varphi, \varphi^\ddagger) = \int_M Y(\varphi)(x_1, \dots, x_k) \varphi^\ddagger(x_1) \cdots \varphi^\ddagger(x_k).$$

This notation is actually not as formal as it seems, if we interpret $\varphi^\ddagger(x)$ as an evaluation functional, i.e. $\varphi^\ddagger_\alpha(x)(v) \doteq v_\alpha(x)$, where $v \in \mathcal{E}^*$. We can then understand the above formula as a definition of an antisymmetric k -form on \mathcal{E}^*

$$\left\langle Y(\varphi, \varphi^\ddagger); v_1, \dots, v_k \right\rangle = \int_M Y(\varphi)(x_1, \dots, x_k) v(x_1) \dots v(x_k),$$

where we have suppressed all the indices of the fiber V^* and we do not need to antisymmetrize, as $Y(\varphi)(x_1, \dots, x_k)$ is already antisymmetric. This agrees with the formal notation introduced in section 3.3.

3.5 Functorial interpretation

All the constructions we have performed can be done covariantly across all the spacetimes, so we can reformulate them in the category theory language.

Definition 3.28. *The configuration space functor is a contravariant functor \mathfrak{E} from \mathbf{Loc} to \mathbf{Vec} , such that for all $\mathcal{M} \in \text{Obj}(\mathbf{Loc})$, $\mathfrak{E}(\mathcal{M})$ is a configuration space according to the definition 3.1.*

Let us consider some examples

Example 3.2. *For the theory of scalar fields, the configuration space functor \mathfrak{E} is a contravariant functor from \mathbf{Loc} to \mathbf{Vec} , defined by*

$$\begin{aligned} \mathfrak{E}(\mathcal{M}) &= \mathcal{C}^\infty(M, \mathbb{R}), \\ \mathfrak{E}\chi &= \chi^*, \end{aligned}$$

where $\mathcal{M} = (M, g) \in \text{Obj}(\mathbf{Loc})$ and for a morphism $\chi \in \text{Hom}(\mathcal{M}, \mathcal{N})$ we have the pullback $\chi^* : \mathfrak{E}(\mathcal{N}) \rightarrow \mathfrak{E}(\mathcal{M})$ defined by

$$\chi^*\varphi \doteq \varphi \circ \chi,$$

where $\varphi \in \mathfrak{E}(\mathcal{N})$

This definition generalizes in a straightforward way to the case where $\mathcal{E}(M)$ is defined as the space of k -forms on M , since the pullback is still well defined. For k -vectors we use the metric g to obtain k -forms, so we can define pullbacks of arbitrary tensor fields.

Example 3.3. *For effective quantum gravity we set*

$$\begin{aligned} \mathfrak{E}(\mathcal{M}) &= \Gamma((T^*M)^{\otimes 2}), \\ \mathfrak{E}\chi &= \chi^*, \end{aligned}$$

where

$$(\chi^*h)(u, v) \doteq h \circ \chi(T\chi(u), T\chi(v)),$$

where $\chi : \mathcal{M} \rightarrow \mathcal{N}$, $u, v \in \Gamma(TM)$, $h \in \mathfrak{E}(\mathcal{N})$ and $\cdot_{\tau}\chi$ is the tangent map.

The generalization to forms taking values in a fixed vector space is also straightforward.

Example 3.4. For Yang-Mills theories with a trivial bundle we set

$$\begin{aligned}\mathfrak{E}(\mathcal{M}) &= \Omega_1(M, g), \\ \mathfrak{E}\chi &= \chi^*,\end{aligned}$$

where

$$(\chi^*A)(u) \doteq h \circ \chi(T\chi(u)),$$

where $\chi : \mathcal{M} \rightarrow \mathcal{N}$, $u \in \Gamma(TM)$, $A \in \mathfrak{E}(\mathcal{N})$.

Given \mathfrak{E} , the spaces of local and multilocal functionals are assigned to spacetimes in a functorial way.

Proposition 3.4. The space of local functionals is a functor $\mathfrak{F}_{\text{loc}} : \mathbf{Loc} \rightarrow \mathbf{Vec}$, so is the space of multilocal functionals $\mathfrak{F} : \mathbf{Loc} \rightarrow \mathbf{Vec}$.

Proof. We set $\mathfrak{F}_{\text{loc}}(\mathcal{M}) \doteq \mathcal{F}_{\text{loc}}(M)$ and $\mathfrak{F}(\mathcal{M}) \doteq \mathcal{F}(M)$ for the objects and $\mathfrak{F}_{\text{loc}}\chi(F)(\varphi) \doteq F(\mathfrak{E}\chi\varphi)$, $\mathfrak{F}\chi(G)(\varphi) \doteq F(\mathfrak{E}\chi\varphi)$ for the morphisms, where $\chi \in \text{Hom}(\mathcal{M}, \mathcal{N})$, $F \in \mathfrak{F}_{\text{loc}}(\mathcal{M})$, $G \in \mathfrak{F}(\mathcal{M})$ and $\chi \in \mathfrak{E}(\mathcal{N})$. \square

The space of configurations is in a natural way a contravariant functor, but the space of compactly supported configurations can be assigned to a spacetime in a covariant way. As an example, consider the space of test functions in the definition 2.50. This motivates the following definition

Definition 3.29. The space of compactly supported configurations \mathfrak{E}_c is a covariant functor from \mathbf{Loc} to \mathbf{Vec} , which acts on the objects as $\mathfrak{E}_c(\mathcal{M}) = \mathcal{E}_c(\mathcal{M})$.

The vector field on \mathcal{E} which we consider are maps from \mathcal{E} to \mathcal{E}_c , so they transform covariantly.

Proposition 3.5. Given \mathfrak{E} and \mathfrak{E}_c , the space of multi-local vector fields is a functor $\mathfrak{V} : \mathbf{Loc} \rightarrow \mathbf{Vec}$, the same holds for local vector fields.

Proof. We set $\mathfrak{V}(\mathcal{M}) \doteq \mathcal{V}(\mathcal{M})$ and $\mathfrak{V}\chi(X) \doteq \mathfrak{E}_c\chi \circ X \circ \mathfrak{E}\chi$, where $\chi \in \text{Hom}(\mathcal{M}, \mathcal{N})$, $X \in \mathfrak{V}(\mathcal{M})$. The same for local vector fields. \square

Chapter 4

Classical theory

Having defined the essential kinematical structure we are now ready to introduce the dynamics. To this end we will use a generalization of the Lagrange formalism.

4.1 Dynamics

We start with introducing some definitions.

Definition 4.1. *A generalized Lagrangian on a fixed spacetime $\mathcal{M} = (M, g)$ is a map $L : \mathcal{D}(M) \rightarrow \mathcal{F}_{\text{loc}}$ such that*

*i) $L(f + g + h) = L(f + g) - L(g) + L(g + h)$ for $f, g, h \in \mathcal{D}$ with $\text{supp } f \cap \text{supp } h = \emptyset$ (**Additivity**).*

*ii) $\text{supp}(L(f)) \subseteq \text{supp}(f)$ (**Support**).*

iii) Let \mathcal{G} be the isometry group of the spacetime \mathcal{M} (for the Minkowski spacetime we set \mathcal{G} to be the proper orthochronous Poincaré group \mathcal{P}_+^\uparrow).

We require that $L(f)(g^\varphi) = L(g_*f)(\varphi)$ for every $g \in \mathcal{G}$ (**Covariance**).*

Proposition 4.1. *Let L be a generalized Lagrangian. The additivity of L in the test function implies that:*

i) $L(f)$ is an additive functional on \mathcal{E} .

ii) For any fixed test function $f \in \mathcal{D}$, $L(f)$ can be written as a finite sum of additive functionals of arbitrarily small space-time support.

Proof. See [BFR12]. □

Definition 4.1 formalizes the idea that the generalized Lagrangian associates to a test function f the local functional $L(f)$ obtained by integrating f with the Lagrangian density $\mathcal{L}(x)[\varphi]$ that depends locally on the field configuration φ . Introducing the cutoff function f is necessary because the manifold M , being globally hyperbolic, is non-compact. Moreover, it is not possible to restrict ourselves to compactly supported configurations φ , since later on we will have to impose equations of motion that are normally hyperbolic and non-trivial solutions to such equations cannot be compactly supported. Let us consider some examples.

Example 4.1. *Examples of generalized Lagrangians:*

i) *Free scalar field:*

$$L_0(f)[\varphi] = \frac{1}{2} \int_M (\nabla_\nu \varphi \nabla^\nu \varphi - m^2 \varphi^2) f d\mu_g, \quad (4.1)$$

where $d\mu_g$ is the invariant measure on \mathcal{M} induced by the metric g , ∇ is the covariant derivative, and we use the Einstein summation convention for the indices, so $\nabla_\nu \varphi \nabla^\nu \varphi \equiv \sum_{\nu=0}^3 \nabla_\nu \varphi \nabla^\nu \varphi$.

ii) *Interaction term in the φ^4 theory:*

$$L_I(f)[\varphi] = \int_M \frac{1}{4!} \varphi^4 f d\mu_g.$$

iii) *Yang-Mills Lagrangian:*

$$L^{YM}(f)(A) = -\frac{1}{2} \int_M f \operatorname{tr}(F \wedge *F),$$

where $A \in \Omega_1(M, g)$, $F = dA + \frac{1}{2}[A, A]$ and $*$ is the Hodge operator.

iv) *Einstein-Hilbert Lagrangian:*

$$L^{EH}(f)[h] \doteq \int R[g+h] f d\mu_{g+h},$$

where $h \in \Gamma((T^*M)^{\otimes 2})$.

The cutoff function f is only an auxiliary tool, which allows to formulate the problem in the mathematically rigorous way, but it has no physical meaning. Therefore, the crucial structures in our classical model cannot depend on the choice of f . This is achieved by means of the following definition:

Definition 4.2. *The Euler-Lagrange derivative of L is a map $S' : \mathcal{E} \rightarrow \mathcal{E}'_c$ defined by*

$$\langle S'(\varphi), h \rangle \doteq \langle L(f)^{(1)}[\varphi], h \rangle,$$

where $h \in \mathcal{E}_c$ and $f \in \mathcal{D}$ is chosen in such a way that $f \equiv 1$ on $\text{supp } h$.

Since $L(f)$ is a local functional, S' doesn't depend on the choice of f . Moreover, S' would not change if we add to L a generalized Lagrangian that is supported in the region where f is not constant. Therefore, the dynamical structure is not encoded in L 's, but rather in equivalence classes of generalized Lagrangians. This leads to the following definition:

Definition 4.3. *An action $S[L]$ is an equivalence class of Lagrangians under the following equivalence relation [BDF09]:*

$$L_1 \sim L_2 \quad \text{iff} \quad \text{supp}((L_1 - L_2)(f)) \subset \text{supp } df. \quad (4.2)$$

Remark 4.1. For convenience we write S instead of $S[L]$, when it's clear from which Lagrangian the given action comes from. For example, the action corresponding to a Lagrangian denoted by L_0 will be written as S_0 rather than $S[L_0]$.

Physical meaning of (4.2) is to identify Lagrangians that “differ by a total divergence”. Note that two Lagrangians equivalent under the relation (4.2) induce the same Euler-Lagrange derivative, so dynamics is a structure coming from actions rather than Lagrangians. We are now ready to introduce the equations of motion EOM's.

Definition 4.4. *The equation of motion (EOM) corresponding to the action S is*

$$S'(\varphi) \equiv 0, \quad (4.3)$$

understood as a condition on $\varphi \in \mathcal{E}$.

The space of solutions to (4.3) will be denoted by \mathcal{E}_S and it is a submanifold of \mathcal{E} . Physically, classical observables should be modeled as multilocal functionals on \mathcal{E}_S . Let us denote the space of such functionals by \mathcal{F}_S . Note that it can be characterized as the quotient

$$\mathcal{F}_S = \mathcal{F}/\mathcal{I}_S,$$

where \mathcal{I}_S is the ideal of \mathcal{F} consisting of functionals that vanish on \mathcal{E}_S .

Definition 4.5. *The second variational derivative S'' of the action S is defined by*

$$\langle S''(\varphi), h_1 \otimes h_2 \rangle \doteq \langle L^{(2)}(f)(\varphi), h_1 \otimes h_2 \rangle,$$

where $f \equiv 1$ on $\text{supp } h_1$ and $\text{supp } h_2$.

By definition S'' is a linear map

$$S'' : \mathcal{E} \rightarrow L(\mathcal{E}_c, \mathcal{E}_c; \mathbb{R}).$$

From the locality of the Lagrangian follows that in fact $S''(\varphi)$ can be extended to a linear map on $\mathcal{E}_c \times \mathcal{E}$ and the Schwarz kernel theorem (see chapter 5 of [?]) implies that this induces a continuous linear operator $P_S(\varphi) : \mathcal{E} \rightarrow \mathcal{E}^*$. For details concerning the proofs of these statements, see [BDLGR15]. One also shows that P_S is map $\mathcal{V} \rightarrow \mathcal{V}^*$, so geometrically it can be interpreted as a symmetric covariant 2-tensor. Note that if S is quadratic then $P_S(\varphi) \equiv P$ is the same for all φ and $S'(\varphi) = P\varphi$. This is the case for the free scalar field described by the Lagrangian L_0 from example 4.1, where $P = \square + m^2$.

The crucial assumption in the pAQFT approach is that $P_S(\varphi)$ is a normally hyperbolic operator. We recall that an operator on \mathcal{E} is normally hyperbolic if its principal symbol is of the metric type, i.e. it is given by

$$\sigma_{P_S(\varphi)}(\xi, \xi) = g(\xi, \xi)\text{id}_{E_x},$$

for all $\xi \in T_x^*M$ and all $x \in M$. Here id_{E_x} denotes the identity on the fiber. For more details on normally hyperbolic operators see [BGP07]. In the same reference it is also shown that for such operators there exist unique retarded and advanced Green's functions (fundamental solutions) $\Delta_S^R(\varphi), \Delta_S^A(\varphi) : \mathcal{E}_c^* \rightarrow \mathcal{E}$ defined by the requirements

$$\begin{aligned} P_S(\varphi) \circ \Delta_S^{R/A} &= \text{id}_{\mathcal{E}_c^*}, \\ \Delta_S^{R/A} \circ P_S(\varphi)|_{\mathcal{E}_c} &= \text{id}_{\mathcal{E}_c}, \end{aligned}$$

and the support properties

$$\begin{aligned} \text{supp } \Delta_A(f) &\subset J^+(\text{supp } f), \\ \text{supp } \Delta_R(f) &\subset J^-(\text{supp } f), \end{aligned}$$

where $f \in \mathcal{E}_c^*$. Note that, by the Schwarz kernel theorem, these operators can be written in terms of their integral kernels, which then satisfy appropriate support properties and

$$\Delta_S^R(y, x) = \Delta_S^A(x, y). \quad (4.4)$$

Let us define the *causal propagator* as

$$\Delta_S(\varphi) \doteq \Delta_S^A(\varphi) - \Delta_S^R(\varphi). \quad (4.5)$$

Due to (4.4) the causal propagator is antisymmetric, i.e. its integral kernel satisfies

$$\Delta_S(\varphi)(y, x) = -\Delta_S(\varphi)(x, y).$$

4.2 Natural Lagrangians

There is an elegant way to describe generalized Lagrangians using the language of category theory.

Definition 4.6. *A natural Lagrangian L is a natural transformations between the functor \mathfrak{D} and $\mathfrak{F}_{\text{loc}}$, such that for all $\mathcal{M} \in \text{Obj}(\mathbf{Loc})$ we have:*

- i) $L_{\mathcal{M}}(f + g + h) = L_{\mathcal{M}}(f + g) - L_{\mathcal{M}}(g) + L_{\mathcal{M}}(g + h)$ for $f, g, h \in \mathfrak{D}$ with $\text{supp } f \cap \text{supp } h = \emptyset$ (**Additivity**).*
- ii) $\text{supp}(L_{\mathcal{M}}(f)) \subseteq \text{supp}(f)$ (**Support**).*

Note that L is fixed by a family of maps $L_{\mathcal{M}} : \mathfrak{D}(\mathcal{M}) \rightarrow \mathfrak{F}_{\text{loc}}(\mathcal{M})$ satisfying the covariance condition

$$L_{\mathcal{M}}(f)(\mathfrak{E}_{\chi}(\varphi)) = L_{\mathcal{N}}(\mathfrak{D}_{\chi}f)(\varphi), \quad (4.6)$$

where $\chi \in \text{Hom}(\mathcal{M}, \mathcal{N})$, $f \in \mathfrak{D}(\mathcal{M})$, $\varphi \in \mathfrak{E}(\mathcal{N})$. The following result shows the relation between natural Lagrangians and generalized Lagrangians introduced earlier in this chapter.

Proposition 4.2. *Let L be a natural Lagrangian from definition 4.6, then for each $\mathcal{M} \in \text{Obj}(\mathbf{Loc})$, $L_{\mathcal{M}}$ is a generalized Lagrangian in the sense of definition 4.1.*

Proof. Since the additivity and support conditions are included in the definition, it suffices to show covariance under isometries of the spacetime. This, however, follows from the general local covariance of L , expressed by the condition (4.6). \square

Example 4.2. *All the Lagrangians from example 4.1 give rise to natural Lagrangians.*

In the following we will always assume that our generalized Lagrangians arise in this way.

4.3 Homological characterization of the solution space

We have already introduced the space of multilocal functionals on the space \mathcal{E}_S of solutions to EOM's as the quotient $\mathcal{F}_S = \mathcal{F}/\mathcal{I}_S$. Now we look for a nice homological interpretation of this quotient. Since \mathcal{E}_S is the zero locus of S' , it is natural to use at this point the derived critical locus construction.

Note that if $X \in \mathcal{V}$ is a multilocal vector field, then the multilocal functional $\langle S', X \rangle$, obtained by contracting this vector field with the one-form S' , obviously vanishes on \mathcal{E}_S . Let us define a map $\delta_S : \mathcal{V} \rightarrow \mathcal{F}$ by

$$\delta_S \doteq \langle S', \cdot \rangle .$$

Clearly, $\delta_S(\mathcal{V}) \subset \mathcal{I}_S$. In general the opposite inclusion can hold only locally, since the structure of the global solution space of nonlinear PDE's can be very complicated. Since our ultimate goal is the quantum theory, we will avoid these complications by *defining*, from now on, \mathcal{I}_S as $\delta_S(\mathcal{V})$.

Definition 4.7. *The ideal $\mathcal{I}_S \subset \mathcal{F}$ is defined as $\delta_S(\mathcal{V})$ and we call it “the ideal generated by the equations of the motion”. The space of on-shell functionals is defined as*

$$\mathcal{F}_S \doteq \mathcal{F}/\mathcal{I}_S .$$

Since QFT models are often constructed by means of some quantization procedure from classical field theory models, the space of solutions to classical EOM's is bound to appear. However, in the approach to pAQFT which we advocate in this book, we use \mathcal{F}_S rather than \mathcal{E}_S , so it is natural to give up the traditional point of view on the space of solutions. This allows now for a natural algebraic interpretation of \mathcal{F}_S as the 0-th homology of the following complex:

$$\dots \rightarrow \bigwedge^2 \mathcal{V} \xrightarrow{\delta_S} \mathcal{V} \xrightarrow{\delta_S} \mathcal{F} \rightarrow 0 , \quad (4.7)$$

$$\qquad \qquad \qquad 2 \qquad \qquad 1 \qquad \qquad 0$$

where δ_S is extended to the exterior algebra $\bigwedge \mathcal{V}$ by requiring the graded Leibniz rule with respect to the exterior product \wedge and by continuity.

Let us now discuss $H_1(\bigwedge \mathcal{V}, \delta_S)$. The kernel of $\delta_S : \mathcal{V} \rightarrow \mathcal{F}$ (here denoted by $\ker \delta_S|_{1 \rightarrow 0}$) consists of those vector fields X which satisfy

$$\partial_X S(\varphi) \doteq \langle S'(\varphi), X(\varphi) \rangle = 0, \quad \forall \varphi \in \mathcal{E} .$$

Geometrically, these vector fields correspond to directions in the configuration space in which the action S is constant. In this sense, we interpret them

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as *symmetries of the action*. Among all the symmetries we can distinguish those which are of the form $\delta_S Z$ for some $Z \in \bigwedge^2 \mathcal{V}$. In order to understand the meaning of such symmetries, let us consider a bivector field of the form $Z = X \wedge Y$ for some $X, Y \in \mathcal{V}$. We have

$$\delta_S (X \wedge Y) = (\delta_S X)Y - (\delta_S Y)X.$$

Note that the vector field obtained this way vanishes identically on \mathcal{E}_S . For this reason, such symmetries are called in physics literature *trivial symmetries*. As a result, one interprets

$$H_1 \left(\bigwedge \mathcal{V}, \delta_S \right) \doteq \frac{\ker \delta_S|_{1 \rightarrow 0}}{\text{Im } \delta_S|_{2 \rightarrow 1}} \quad (4.8)$$

as the space of *non-trivial local symmetries*. Theories that don't possess non-trivial local symmetries include the scalar field theory with a polynomial interaction (e.g. φ^4). One of the simplest examples of a theory with non-trivial $H_1(\bigwedge \mathcal{V}, \delta_S)$ is QED. Also Yang-Mills theories and gravity fall into this category. We will discuss these in more detail in sections 7 and 8. A simple criterion to decide that the theory has no non-trivial local symmetries has been provided in [FR12b].

Proposition 4.3. *The action S possesses no non-trivial symmetries if the linearised equation of motion*

$$P_S(\varphi)\psi = 0$$

doesn't have any non-trivial compactly supported solutions ψ for all $\varphi \in \mathcal{E}$.

Proof. See the discussion at the end of section 2 in [FR12b]. □

To end this section, we will introduce one more algebraic structure on $\bigwedge \mathcal{V}$. As noted before, \mathcal{V} is a Lie subalgebra of $\Gamma(T\mathcal{E})$, where the bracket is just the commutator of vector fields. Now, using the graded Leibniz rule and continuity, one can extend this structure to the Schouten bracket $\{.,.\}$ on $\bigwedge \mathcal{V}$ fixed uniquely by the following properties:

1. $\{X, F\} \doteq \partial_X F$, for $F \in \mathcal{F}$ and $X \in \mathcal{V}$,
2. $\{X, Y\} \doteq [X, Y]$, for $X, Y \in \mathcal{V}$,
3. $\{.,.\}$ fulfills the graded Leibniz rule.

In physics literature this structure is called the *antibracket*, where it is usually expressed with the use of the antifields notation introduced in section 3.4. Recall that $X \in \bigwedge \mathcal{V}$ is a smooth functional on $\mathcal{E} \times (\mathcal{E}^{*\mathbb{C}})^k$, so can be differentiated both with respect to $\varphi \in \mathcal{E}$ and $v_i \in \mathcal{E}^{*\mathbb{C}}$, $i = 1, \dots, k$. We denote the n -th derivative with respect to φ by $\frac{\delta^n X}{\delta \varphi^n}$ and the left derivative of X in the direction of $v \in \mathcal{E}^{*\mathbb{C}}$ is defined as

$$\left\langle \frac{\delta_l X}{\delta \varphi^\ddagger}(\varphi), v \right\rangle \doteq \langle X(\varphi); v, \cdot \rangle,$$

where ι_v is the insertion of v from the left into the antisymmetric k -linear map $X(\varphi)$. Similarly, we define the right derivative as

$$\left\langle \frac{\delta_r X}{\delta \varphi^\ddagger}(\varphi), v \right\rangle \doteq \langle X(\varphi); \cdot, v \rangle,$$

Using this notation we write

$$\{X, Y\}(\varphi) \doteq \left\langle \frac{\delta}{\delta \varphi} X(\varphi), \frac{\delta_l}{\delta \varphi^\ddagger} Y(\varphi) \right\rangle - \left\langle \frac{\delta_r}{\delta \varphi^\ddagger} X(\varphi), \frac{\delta_l}{\delta \varphi} Y(\varphi) \right\rangle.$$

This expression is well defined, since the first derivatives of a multilocal vector field with respect to both φ and φ^\ddagger are, by definition, smooth compactly supported sections.

Note that the BV differential δ_S is locally generated by the bracket in the sense that

$$\delta_S X = \{S, X\} \doteq \{L(f), X\},$$

where $f \equiv 1$ on the support of X and L is the Lagrangian defining the theory.

4.4 The net of topological Poisson algebras

In this section we show how to construct a causal net of topological Poisson algebras in the sense of 2.38. First we introduce a Poisson bracket on an appropriate space of functionals and next we equip this space of functionals with a topology that makes this bracket sequentially continuous.

4.4.1 The Peierls bracket and microcausal functionals

We equip \mathcal{F} with a Poisson bracket called the *Peierls bracket* [Pei52]. It was shown in [FR14] that this bracket is equivalent to the canonical bracket

commonly used in classical mechanics, if the latter exists. The advantage of using the Peierls bracket is that it can be defined completely within the Lagrangian formalism, without the need to pass to the Hamiltonian. This is important for example in cases when the Hamiltonian vanishes. Another advantage of the Peierls bracket is that it is defined in a completely covariant way, so it doesn't require to introduce a foliation of M with Cauchy surfaces.

Definition 4.8. *Let $F, G \in \mathcal{F}$. Then the Peierls bracket is defined by*

$$[F, G](\varphi) \doteq \left\langle F^{(1)}(\varphi), \Delta_S^c(\varphi)G^{(1)}(\varphi) \right\rangle, \quad (4.9)$$

where $\Delta_S^c(\varphi)$ denotes the natural extension of $\Delta_S(\varphi)$ to a map on \mathcal{E}_c^c with values in \mathcal{E}^c .

Remark 4.2. For the simplicity of notation we will write $\Delta_S(\varphi)$ instead of $\Delta_S^c(\varphi)$, when it is clear from the context that we mean the extension to the complex-valued sections. We will do the same also for other bilinear forms like for example $\Delta_A(\varphi)$ and $\Delta_R(\varphi)$.

The bracket defined in 4.8 is antisymmetric (by the antisymmetry of $\Delta_S(\varphi)$), bilinear and satisfies the Jacobi identity [Jak09]. Since $F^{(1)}(\varphi)$ and $G^{(1)}(\varphi)$ are smooth, it is clear that $[\cdot, \cdot]$ is well defined on multilocal functionals. However, \mathcal{F} is closed under this bracket. The natural question to ask is how to extend the domain of definition of $[\cdot, \cdot]$, so that the resulting space is closed under this bracket. To answer this question, it is useful to look at the WF set of $\Delta_S(\varphi)$.

$$\text{WF}(\Delta_S(\varphi)) = \{(x, k; x', -k') \in \dot{T}^*M^2 | (x, k) \sim (x', k')\},$$

where the equivalence relation \sim means that there exists a null geodesic strip such that both (x, k) and (x', k') belong to it. Recall that a null geodesic strip is a curve in T^*M of the form $(\gamma(\lambda), k(\lambda))$, $\lambda \in I \subset \mathbb{R}$, where $\gamma(\lambda)$ is a null geodesic parametrized by λ and $k(\lambda)$ is given by $k(\lambda) = g(\dot{\gamma}(\lambda), \cdot)$. The form of the WF set of $\Delta_S(\varphi)$ follows from the theorem on the propagation of singularities together with the initial conditions and the antisymmetry of $\Delta_S(\varphi)$. (See [Rad96] for a details).

We can now use Hörmander's criterion 3.3 to determine a class of distribution that can be point-wise multiplied with $\Delta_S(\varphi)$ and hence the condition that has to be satisfied by the differentials $F^{(1)}(\varphi)$ and $G^{(1)}(\varphi)$ in (4.9). This leads to the following definition

Definition 4.9. A functional $F \in \mathcal{C}^\infty(\mathcal{E}, \mathbb{R})$ is called *microcausal* if it is compactly supported and satisfies

$$\text{WF}(F^{(n)}(\varphi)) \subset \Xi_n, \quad \forall n \in \mathbb{N}, \quad \forall \varphi \in \mathcal{E}, \quad (4.10)$$

where Ξ_n is an open cone defined as

$$\Xi_n \doteq T^*M^n \setminus \{(x_1, \dots, x_n; k_1, \dots, k_n) \mid (k_1, \dots, k_n) \in (\overline{V}_+^n \cup \overline{V}_-^n)_{(x_1, \dots, x_n)}\}, \quad (4.11)$$

where $(\overline{V}_\pm)_x$ is the closed future/past lightcone understood as a conic subset of T_x^*M .

In [BFR13] it is additionally required that the first derivative $F^{(1)}(\varphi)$ is smooth for all $\varphi \in \mathcal{E}$ and $\varphi \mapsto F^{(1)}(\varphi)$ is smooth as a map $\mathcal{E} \rightarrow \mathcal{E}^c$. We will call functionals satisfying this additional property **strongly microcausal**. We denote the space of microcausal functionals by $\mathcal{F}_{\mu c}$ and the space of the strongly microcausal ones by $\mathcal{F}_{s\mu c}$.

At this point it is convenient to introduce a notation for spaces of distributions with WF sets contained in open and closed cones.

Definition 4.10. Let $\mathcal{D}'_\Gamma(M^n)$ denote the space of distributions whose WF sets are contained in a closed cone $\Gamma \subset \dot{T}^*M^n$. Similarly, $\mathcal{E}'_\Lambda(M^n)$ denotes the space of distributions whose WF sets are contained in an open cone $\Lambda \subset \dot{T}^*M^n$.

We can now rephrase definition 4.9 by saying that n -th derivatives of elements of $\mathcal{F}_{\mu c}$ are distributions belonging to the corresponding spaces $\mathcal{E}'_{\Xi_n}(M^n)$. The generalization of the definition 4.9 to the graded case is straightforward. We spell out the definition for $\mathcal{O}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$

Definition 4.11. Let $F \in \mathcal{O}^k(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$. We say that F is *microcausal*, i.e. an element of $\mathcal{O}_{\mu c}(\mathcal{E}_0 \oplus \mathcal{E}_1[1])$ if it is compactly supported and

$$F^{(n)}(\varphi) \in \mathcal{E}'_{\Xi_{n+k}}(M^{n+k}),$$

for all $\varphi \in \mathcal{E}_0$, $n \in \mathbb{N}$.

F is said to be *strongly microcausal* if in addition

- i) $F(\varphi, u)$ has an empty WF set for all $\varphi \in \mathcal{E}_0$, $u \in \mathcal{E}_1^{\hat{\otimes} \pi^{k-1}}$ and the map $(\varphi, u) \mapsto F(\varphi, u)$ is smooth as a map from $\mathcal{E}_0 \times \mathcal{E}_1^{\hat{\otimes} \pi^{k-1}}$ to \mathcal{E}_1^c ,
- ii) $F^{(1)}(\varphi, u)$ has an empty WF set for all $\varphi \in \mathcal{E}_0$, $u \in \mathcal{E}_1^{\hat{\otimes} \pi^k}$ and the map $(\varphi, u) \mapsto F^{(1)}(\varphi, u)$ is smooth as a map from $\mathcal{E}_0 \times \mathcal{E}_1^{\hat{\otimes} \pi^k}$ to \mathcal{E}_0^c .

The following proposition is crucial for the construction of the local net of Poisson algebras.

Proposition 4.4. $(\mathcal{F}_{s\mu c}, [\cdot, \cdot])$ is closed under the bracket and is a Poisson algebra. If Δ_S doesn't depend on φ , then also $(\mathcal{F}_{\mu c}, [\cdot, \cdot])$ is a Poisson algebra.

Proof. See [BFR12, BFR13] □

Remark 4.3. The stronger version of microcausality is needed if Δ_S depends on φ , because otherwise the proof of the Jacobi identity given in [Jak09] would fail. Alternatively, following Dabrowski [Dab14b], one can use a more refined definition microcausality – called *operadic microcausality* – that involves the notion of dual WF sets. We will give more details on that in section 4.4.2

If the discussion applies to all the notion of microcausality introduced above, we use the notation $\mathcal{F}_{*\mu c}$.

4.4.2 Topologies on the space of microcausal functionals

We come now to the important problem of introducing on $\mathcal{F}_{*\mu c}$ a topology that will be appropriate for construction models of classical and quantum theories in the sense of definitions 2.38 and 2.39. We have already seen that the regularity of smooth functionals is governed by the regularity of their derivatives. The latter is measured using the notion of a WF set. Clearly we need a topology that controls all these regularity properties. In the light of the discussion from section 2.5 it would be desirable to use a topology that is nuclear. Other useful properties are completeness and being bornological. At the moment there is no definite consensus in the literature as to which choice is the most natural, so we will review the proposals which are most relevant for the scope of this book.

The basic idea is to introduce a topology on the space $\mathcal{E}'_{\Xi_n}(M^n)$ of distributions with WF sets contained within the open cone Ξ_n defined by (4.11) and use this topology to control the regularity of derivatives of functionals. At this point there are several possibilities. The simplest one is to invoke the topology of pointwise convergence of all the functional derivatives [BDF09], but the resulting space is not complete, so it might be better to use some weak notion of uniform convergence instead (see [Dab14b]).

We start with reviewing the proposal made by [BDF09], where the authors equip $\mathcal{D}'_{\Gamma}(M^n)$ the locally convex topology proposed by Duistermaat [Dui96]. For simplicity we state the definition for \mathbb{R}^n , but it generalizes to manifolds in a straightforward way with the use of local charts.

Definition 4.12. We define τ_H on $\mathcal{D}'_{\Gamma}(\mathbb{R}^n)$ as the locally convex topology given by the following systems of seminorms:

- i) All the seminorms on $\mathcal{D}'(\mathbb{R}^n)$ for the weak topology: $\|u\|_f = |\langle u, f \rangle|$ for all $f \in \mathcal{D}(\mathbb{R}^n)$.
- ii) The seminorms of the form

$$\|u\|_{m,V,\chi} = \sup_{k \in V} (1 + |k|)^m |\widehat{u\chi}(k)|,$$

where $m \geq 0$, $\chi \in \mathcal{D}(\mathbb{R}^n)$, and $V \in \mathbb{R}^n$ is a closed cone with $\text{supp } \chi \times V \cap \Gamma = \emptyset$.

This is also called the Hörmander topology, since the seminorms ii) of the definition above were already present in [Hör71], where they were used to define a pseudo-topology rather than topology. In [BDH14] it had be shown that it defines in fact a *bornology*. In short, bornology is a family of bounded sets, just as a topology is a family of open sets. More precisely:

Definition 4.13. A *bornology* on a set X is a family \mathcal{B} of subsets of X (called the **bounded (sub)sets** of X) such that:

- i) every one-element subset of X belongs to \mathcal{B} ,
- ii) if $A \in \mathcal{B}$ and $B \subset A$ then $B \in \mathcal{B}$,
- iii) if A and B are in \mathcal{B} then $A \cup B \in \mathcal{B}$.

In [BDH14] it is proven that the convergence in the bornological sense (Mackey convergence) in $\mathcal{D}'_{\Gamma}(\mathbb{R}^n)$ is the same as the convergence in the sense of the Hörmander pseudotopology.

Proposition 4.5 (after [BDH14]). A sequence u_j in \mathcal{D}'_{Γ} converges to u in the sense of Hörmander iff it Mackey-converges to u for the bornology of \mathcal{D}'_{Γ} .

Proof. See proposition 3.1 of [BDH14]. □

There are several indications that the notion of bornology is more natural for the applications in physics than the notion of topology. Another argument is that the family of smooth curves in a topological vector space is determined by the by the bornology rather than topology. In the convenient setting of global analysis [KM97] a map is smooth if it maps smooth curves into smooth curves. This is a very elegant and robust notion of smoothness and allows to do geometry on infinite dimensional manifolds of the type that

appear commonly in physics. In [Mey04] it is pointed out that bornological vector spaces are also very useful in non-commutative geometry and representation theory. At the moment the formalism of bornological vector spaces hasn't been fully embraced by mathematical physicists studying QFT, so we refrain from formulating everything in these terms here as well. Nevertheless, it seems tempting to explore the consequences of taking a purely bornological viewpoint on the foundations of QFT in the future.

Let us come back to the main problem of the present section. For the definition of a topology on microcausal functionals we still need to make one more step. The topology defined above was meant for distributions with WF sets contained in a *closed* cone. However, in the definition 4.9 one uses *open cones* instead. It was proposed in [BDF09] to introduce a topology on $\mathcal{E}'_{\Xi_n}(M^n)$ as the inductive limit topology over a countable family of spaces $(\mathcal{D}'_C(M^n), \tau_H)$ with closed cones contained in Ξ_n . The proof has been spelled out in detail in [BFR12, Lemma 4.0.18]. By a slight abuse of notation, we denote the resulting topological spaces by $(\mathcal{E}'_{\Xi_n}(M^n), \tau_H)$.

Remark 4.4. It is crucial, that the inductive limit in the definition of $(\mathcal{E}'_{\Xi_n}(M^n), \tau_H)$ is countable, since this allows to conclude that $(\mathcal{E}'_{\Xi_n}(M^n), \tau_H)$ is nuclear.

In [BDF09] $(\mathcal{E}'_{\Xi_n}(M^n), \tau_H)$ is used to define a topology on $\mathcal{F}_{\mu c}$.

Definition 4.14. Equip $\mathcal{F}_{\mu c}$ with the topology τ_{BDF} defined as the initial topology with respect to all the maps

$$\begin{aligned} \mathcal{F}_{\mu c} &\rightarrow (\mathcal{E}'_{\Xi_n}(M^n), \tau_H), \\ F &\mapsto F^{(n)}(\varphi), \end{aligned}$$

where $n \in \mathbb{N}$, $\varphi \in \mathcal{E}$.

Definition 4.15. To obtain a topology on $\mathcal{F}_{s\mu c}$, we replace in 4.14 $\mathcal{E}'_{\Xi_1}(M)$ with $\mathcal{D}(M)$, equipped with its standard topology.

The topology τ_{BDF} is nuclear, but is not complete and it has been argued in [DB14] that from the functional analytic viewpoint this is not the most optimal choice. In [Sch57] it is argued that the spaces of distributions most optimal for applications are the *normal spaces of distributions*

Definition 4.16. A Hausdorff locally convex space \mathcal{X} is said to be a *normal space of distributions* if there are continuous injective linear maps $i : \mathcal{D}(\Omega) \hookrightarrow \mathcal{X}$ and $j : \mathcal{X} \hookrightarrow \mathcal{D}'(\Omega)$, where $\mathcal{D}'(\Omega)$ is equipped with its strong topology, such that:

- i) The image of i is dense in \mathcal{X} ,
- ii) for any f and g in $\mathcal{D}(\Omega)$, $\langle j \circ i(f), g \rangle = \int_{\Omega} f(x)g(x)dx$.

The motivation for using normal spaces of distributions is that they have a better behaviour under duality, in particular, the dual space can also be equipped with a normal topology. In order to make $\mathcal{D}'_{\Gamma}(\mathbb{R}^n)$ into a normal space of distributions one needs to refine its topology. It was shown in [DB14] that this can be done simply by replacing in 4.12 the seminorms of the weak topology with the seminorms of the strong topology.

Definition 4.17 (following [DB14]). *We define τ_N on $\mathcal{D}'_{\Gamma}(\mathbb{R}^n)$ as the locally convex topology given by the following systems of seminorms:*

- i) All the seminorms on $\mathcal{D}'(\mathbb{R}^n)$ for the strong topology: $p_B(u) = \sup_{f \in B} |\langle u, f \rangle|$, where B runs over the bounded sets of $\mathcal{D}(\Omega)$.
- ii) The seminorms $\|u\|_{m, V, \chi}$, where $m \geq 0$, $\chi \in \mathcal{D}(\mathbb{R}^n)$, and $V \in \mathbb{R}^n$ is a closed cone with $\text{supp } \chi \times V \cap \Gamma = \emptyset$.

With this choice of topologies one obtains a duality between $\mathcal{D}'_{\Gamma}(M^n)$ and $\mathcal{E}'_{\Gamma}(M^n)$.

Proposition 4.6 (following [DB14]). *The dual of $\mathcal{D}'_{\Gamma}(M^n)$ for its normal topology τ_N is $\mathcal{E}'_{\Gamma}(M^n)$.*

Proof. See Proposition 7 in [DB14]. □

The space $\mathcal{E}'_{\Gamma}(M^n)$ can be equipped with an inductive limit topology similar to the one proposed in [BDF09] and spelled out in [BFR12]. It was shown in [DB14] that this topology is equivalent to the strong topology coming from the duality with $\mathcal{D}'_{\Gamma}(M^n)$. In the same reference it was also proven that both $\mathcal{E}'_{\Gamma}(M^n)$ and $\mathcal{D}'_{\Gamma}(M^n)$ are nuclear and that $\mathcal{D}'_{\Gamma}(M^n)$ is complete. The latter property unfortunately doesn't hold for $\mathcal{E}'_{\Gamma}(M^n)$, which was the motivation for further study, done in [Dab14a].

Definition 4.18. *Let τ_{BDH} denote the topology on the space of (strongly) microcausal functionals by replacing in definition 4.14 the Hörmander topology τ_H with the normal topology τ_N .*

This topology has much better functional properties than τ_{BDF} , but is not complete.

In order to obtain a complete topological space one needs to modify the definition of the space of microcausal functionals as well. The idea in [Dab14a] is to control not only the WF set of distributions but also the dual WF set defined as follows.

Definition 4.19 (after [Dab14a]). *Let $u \in \mathcal{D}'(\Omega, \Omega \subset \mathbb{R}^n)$. The dual WF set is defined as $DWF(u) = \cup_{s>0}, WF_s(u)$, i.e the union of Sobolev H^s -wave front sets.*

A point $(x, k_0) \notin WF_s(u)$ means that there is a neighborhood U of x and a conic neighborhood C of k_0 such that for any $f \in \mathcal{D}(U)$

$$(1 + |k|^2)^{s/2} \widehat{fu} \in L^2(C).$$

Note that if the H^s -wave front set of a distribution u is empty, it means that u is of Sobolev type s , just like the distribution with the empty WF set is smooth.

The notion of the DWF set introduced above allows for an explicit characterisation of the completion of $\mathcal{E}'_\Lambda(M^n)$. It was shown in [Dab14a] that the completion of $(\mathcal{E}'_\Lambda(M^n), \tau_n)$ is the space of distributions in \mathcal{E}' with DWF sets contained in Λ .

In order to obtain a space of functionals with nice functional analytic properties, in [Dab14b] the following modifications are made, with respect to the ansatz of [BDF09]:

1. In the definition 4.9, view derivatives $F^{(n)}(\varphi)$ not as distributions in $\mathcal{D}'(M^n)$, but rather as linear maps between spaces of distributions with control on both the WF set and DWF set,
2. Use a topology on the space of distributions with control on both the WF set and the DWF set, which is complete, nuclear and bornological (the latter means essentially that it is compatible with the bornology),
3. Replace the pointwise convergence of all the derivatives with the uniform convergence on images of compact subsets of \mathbb{R} under smooth curves (this is a natural choice from the point of view of the convenient setting [KM97]).

The resulting space of functionals is called the space of *operadically micro-causal functionals*, denoted by \mathcal{F}_{omc} and we will denote the topology on this space by τ_D . It has been shown in [Dab14b] that $(\mathcal{F}_{\text{omc}}, \tau_D)$ is complete, nuclear and bornological.

4.4.3 The classical causal net

We are now ready to construct a classical field theory model in the sense of definition 2.38. The only missing ingredient is localization, but this can be

easily introduced in our case, using the notion of the spacetime support of a functional. We obtain the following result

Proposition 4.7. *We define the classical field theory model on a spacetime $\mathcal{M} = (M, g)$ with the configuration space $\mathcal{E} = \Gamma(E \xrightarrow{\pi} M)$ and the generalized Lagrangian L as the net*

$$\mathcal{O} \mapsto (\mathcal{F}_{s\mu c}, \tau_{BDH}, [\cdot, \cdot]_S).$$

The involution is the pointwise complex conjugation of complex-valued functionals and self-adjoint elements are those functionals, which take values in \mathbb{R} .

Proof. The proof has been outlined in [BDF09] and [BFR13]. The main idea is to reduce the problem to the problem involving basic operation on distribution. The fact that $[\cdot, \cdot]$ is well defined on the space of strongly microcausal functionals and that $\mathcal{F}_{s\mu c}$ is stable under this bracket follow from the WF set properties of derivatives of microcausal functionals and of $\Delta_S^+(\varphi)$. The only missing step, not commented on in [BDF09] and [BFR13], is the (sequential) continuity. We will fill this gap here. For simplicity of notation, we spell out the proof for the example of the scalar field, i.e. $\mathcal{E} = \mathcal{E}$. First we note that with the initial topology we are using it is sufficient to show the continuity pointwise in φ , so it reduces to the continuity of operations on distributions of the form

$$(f, g) \mapsto \langle f, \Delta_S g \rangle,$$

where $f, g \in \mathcal{E}'_{\Xi_1}(M)$. This in turn is reduced to proving the sequential continuity of the tensor product and the sequential continuity of the distributional pullback. In the closed cone case these two results are well known (see [CP81, p. 511] and [Hör03, 8.2.4] respectively). For the open cone, we will use the strategy proposed by [Dab15], based on the results of [BDH14, DB14, Dab14a, Dab14b].

We start with the sequential continuity of the tensor product. It was proven in [DB14, Prop. 28] that $\mathcal{E}'_{\Xi_n}(M^n)$ is a barrelled space, so, following [Trè06, Th. 41.2] we conclude that the separate continuity implies hypocontinuity¹ thus sequential continuity. By the definition of the inductive limit, the separate continuity follows from the separate continuity in the closed cone case, which has been proven in [BDH14]. Another way to see it is to

¹Hypocontinuity of a bilinear map is a notion stronger than sequential continuity on the product space, but is weaker than the joint continuity

use the fact that for linear maps Hörmander's sequential continuity implies boundedness, hence continuity between the bornologifications. It was shown in [Dab14a, Prop. 33] that the inductive limit topology on $\mathcal{E}'_{\Xi_n}(M^n)$ is also an inductive limit of bornologifications, hence it is itself bornological.

As for the continuity of the pull-back, see [Dab14a, Prop. 36]. \square

Remark 4.5. In fact the argument presented above proves something stronger than sequential continuity, namely the *hypocontinuity* of the product. This result would not be possible for τ_H replacing τ_N , as shown by counterexamples provided in [BDH14]. Therefore, to show the joint sequential continuity of the bracket with the topology τ_{BDF} one needs to use a different, indirect argument (work in progress).

A major improvement of the above result has been obtained recently in [Dab14b].

Proposition 4.8 (after [Dab14b]). *($\mathcal{F}_{\text{omc}}, \tau_D$) equipped with the Peierls bracket $[\cdot, \cdot]$ is a complete, nuclear and bornological topological Poisson algebra with hypocontinuous operations.*

Proof. See [Dab14b]. \square

Chapter 5

Deformation quantization

The reformulation of classical theory done in section 3 served as a preparation for constructing QFT models. The framework which we are going to use is deformation quantization combined with causal perturbation theory. To quantize a given theory described by the action S we first need to split S into a free part S_0 (at most quadratic in field configurations) and the interaction term S_I . Then, we quantize the theory defined by S_0 , using deformation quantization based on a Moyal-type formula and in the final step we will re-introduce the interaction using causal perturbation theory. This last step will be discussed in chapter 6, while the present chapter deals with deformation quantization.

The idea of deformation quantization goes back to Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [BFF⁺78a, BFF⁺78b] and the first attempt to use these structures in quantum field theory is due to Dito [Dit90]. Based on these ideas Brunetti, Dütsch, and Fredenhagen developed a formalism, which we present here [DF01a, DF01b, BDF09]. At the moment this quantization method is known to work only perturbatively, but the ultimate aim is to obtain some convergence results as well. As for the deformation quantization part, this is already possible in some examples like the free scalar fields. The interacting theory is a bigger challenge, but at least some weak convergence in the coupling constant seems to be within reach in the nearest future.

5.1 Star products

In this chapter we will focus on quantizing theories where no local symmetries are present. We will also restrict ourselves to even (bosonic) field configura-

rations, to avoid extra complication with the signs. Let $\mathcal{E} = \Gamma(E \rightarrow M)$ be the configuration space and S an arbitrary action that doesn't possess non-trivial local symmetries and for which $S''(\varphi)$ induces a normally hyperbolic operator for every $\varphi \in \mathcal{E}$. In the first step we first split $S = S_0 + S_I$, where S_0 is at most quadratic. This can be done by means of Taylor expansion around any field configuration φ_0 , i.e.

$$L(f)(\varphi) = L(f)(\varphi_0) + \underbrace{\left\langle L(f)^{(1)}(\varphi_0), \varphi \right\rangle + \frac{1}{2} \left\langle L(f)^{(2)}(\varphi_0), \varphi \otimes \varphi \right\rangle}_{L_0(f)} + \text{Rest term},$$

The constant term can be neglected, as it doesn't affect the dynamics. Note that if we choose φ_0 to be a solution to the equations of motion (i.e. $S'(\varphi_0) = 0$), then

$$\left\langle L(f)^{(1)}(\varphi_0), \varphi + \psi \right\rangle = \left\langle L(f)^{(1)}(\varphi_0), \varphi \right\rangle,$$

if $\psi \in \mathcal{E}_c$ is supported in the region, where $f = \text{const}$. Hence $\text{supp} \langle L(f)^{(1)}(\varphi_0), \cdot \rangle \subset \text{supp}(df)$ and by means of the equivalence relation (4.2) we conclude that in such a situation S_0 contains only the quadratic term. Otherwise, S_0 has both a quadratic and a linear term. For simplicity, we will consider here only the situation where φ_0 is a solution, so S_0 is quadratic and we denote the linear operator induced by S_0'' by P .

Starting from the Poisson algebra $(\mathcal{F}_{\mu c}, [\cdot, \cdot]_{S_0})$, a formal deformation quantization means constructing an associative algebra $(\mathcal{F}_{\mu c}[[\hbar]], \star)$, where the product \star is expressed as

$$F \star G = \sum_{n=0}^{\infty} \hbar^n B_n(F, G), \quad (5.1)$$

in terms of some differential (in the sense of calculus on \mathcal{E}) operators B_n such that

$$\begin{aligned} B_0(F, G) &= F \cdot G, \\ B_1(F, G) - B_1(G, F) &= i\hbar [F, G]_{S_0}. \end{aligned}$$

Note that the second condition corresponds to Dirac's idea that in order to quantize a classical theory one should "replace canonical brackets with commutators". Including terms of higher order in \hbar is necessary to avoid the Groenewald-van Hove no-go theorem, which states that (also in the finite dimensional case) a Dirac type quantization prescription is not possible in the strict sense [Gro46, VH51].

More precisely (see [?]), let \mathfrak{h} be the Lie algebra spanned by the canonical coordinate and momenta functions $q^1, \dots, q^N, p_1, \dots, p_N$ and 1, with the canonical Poisson bracket $\{.,.\}_{\text{can}}$. This algebra is a Lie subalgebra of $\mathfrak{g} \doteq (\text{Pol}(T^*\mathbb{R}^N), \{.,.\}_{\text{can}})$ (polynomials on the phase space). According to the Groenewald-van Hove Theorem, there exists no faithful irreducible representation of \mathfrak{h} by operators on a dense domain of some Hilbert space which can be extended to a representation of \mathfrak{g} . As a result, one cannot have a Dirac quantization map \mathfrak{Q} from \mathfrak{g} to the space of operators on some Hilbert space \mathcal{H} , such that

$$[\mathfrak{Q}(f), \mathfrak{Q}(g)] = i\hbar\mathfrak{Q}(\{f, g\}). \quad (5.2)$$

There is, however a way out. Deformation quantization [BFF⁺78a, BFF⁺78b] allows to avoid this no-go result, by weakening the condition (5.2) to

$$[\mathfrak{Q}(f), \mathfrak{Q}(g)] = \mathfrak{Q}([f, g]_{\star}) = i\hbar\mathfrak{Q}(\{f, g\}) + \mathcal{O}(\hbar^2).$$

A stronger notion than formal deformation quantization is *strict deformation quantization*. In this case, instead of constructing a space of formal power series, one aims at constructing a C^* -algebra. This fits well with the algebraic framework for quantum theory described in section 2.3. The notion of strict deformation quantization has been introduced in [Rie94]. For a review on the current status of the subject refer to [Rie98], see also [Haw08, BMS94].

As for the formal deformation quantization, the most significant recent result is the one of Kontsevich, who has proven in [Kon03] the existence of formal deformation quantization for arbitrary Poisson manifolds. Unfortunately this cannot be applied directly in field theory, as the configuration space \mathcal{E} is infinite dimensional. However, if S_0 is at most quadratic, there exists an explicit formula for the star product and we will focus on this construction for the rest of the present chapter.

To understand the algebraic structure, it is helpful to put the functional analytic subtleties aside for the time being. To this end, we will start with a class of functionals which is much smaller than $\mathcal{F}_{\mu\mathbb{C}}$ and exhibits some nice regularity properties.

Definition 5.1. *A functional $F \in \mathcal{C}^\infty(\mathcal{E}, \mathbb{C})$ is called regular if for all $\varphi \in \mathcal{E}$ and $n \in \mathbb{N}$ the WF set of $F^{(n)}(\varphi)$ (seen as a distribution in $\Gamma_c^{\prime\mathbb{C}}(E^{\boxtimes n} \rightarrow M^n)$ by means of the Proposition 3.1) is empty. Equivalently this means that we can identify*

$$F^{(n)}(\varphi) \in \Gamma_c^{\mathbb{C}}((E^*)^{\boxtimes n} \rightarrow M^n).$$

The space of regular functionals is denoted by \mathcal{F}_{reg} .

We are now ready to introduce the star product. We define

$$(F \star G)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), \left(\frac{i}{2}\Delta_{S_0}\right)^{\otimes n} G^{(n)}(\varphi) \right\rangle, \quad (5.3)$$

for $F, G \in \mathcal{F}_{\text{reg}}$.

Example 5.1 (Weyl algebra). *For the free scalar field we have $\mathcal{E} = \mathcal{C}^\infty(M, \mathbb{R})$ and the Lagrangian is given by 4.1i). Consider regular functionals of the form*

$$F_f(\varphi) = \int_M f(x)\varphi(x)d^4x \equiv \int f\varphi d\mu, \quad \text{where } f \in \mathcal{D}.$$

We define $\mathcal{W}(f) \doteq \exp(iF_f)$ and verify that

$$\begin{aligned} \left\langle (\mathcal{W}(f))^{(1)}(\varphi), h \right\rangle &= \frac{d}{d\lambda} e^{i \int f(\varphi+\lambda h) d\mu} \Big|_{\lambda=0} = \\ &= \left(i \int f h d\mu \right) \mathcal{W}(f)(\varphi). \end{aligned}$$

Hence

$$\left\langle (\mathcal{W}(f))^{(n)}(\varphi), h^{\otimes n} \right\rangle = \left(i \int f h d\mu_g \right)^n \mathcal{W}(f)(\varphi)$$

and we obtain the following formula for the star product:

$$\begin{aligned} \mathcal{W}(f) \star \mathcal{W}(\tilde{f}) &= \sum_{n=0}^{\infty} \left(\frac{i\hbar}{2}\right)^n \frac{(-1)^n}{n!} \left(\int \Delta_S(x, y) \tilde{f}(y) f(x) d\mu_g(x) d\mu_g(y) \right)^n \mathcal{W}(f + \tilde{f}) \\ &= e^{-\frac{i\hbar}{2}\Delta_S(f, \tilde{f})} \mathcal{W}(f + \tilde{f}), \end{aligned} \quad (5.4)$$

which reproduces the Weyl relations from example 2.1, with the difference that now we are dealing with a bilinear form which is Poisson, but not symplectic (has a non-trivial kernel).

In the next step we extend the star product to the space of microcausal functionals. To prepare for this task we take another look at the singularity structure of the distribution Δ_{S_0} . Note that the WF set of this distribution is composed of two parts: one with $k \in (\overline{V}_+)_x$ and another with $k \in (\overline{V}_-)_x$, where \overline{V}_\pm is (the dual of) the closed future/past lightcone. As shown in

[Rad96], one can decompose Δ_{S_0} into two distributions with WF sets corresponding to these two components

$$\frac{i}{2}\Delta_{S_0} = \Delta_{S_0}^+ - H, \quad (5.5)$$

where the WF set of $\Delta_{S_0}^+$ is

$$\text{WF}(\Delta_{S_0}^+) = \{(x, k; x, -k') \in \dot{T}^*M^2 | (x, k) \sim (x', k'), k \in (\bar{V}_+)_x\}, \quad (\mathbf{H} 0)$$

and in addition the following properties hold:

$$(\mathbf{H} 1) \quad \Delta_{S_0} = 2\text{Im}(\Delta_{S_0}^+)$$

$$(\mathbf{H} 2) \quad \Delta_{S_0}^+ \text{ is a distributional bisolution to the field equation, i.e. } \langle \Delta_{S_0}^+; Pf, g \rangle = 0 \text{ and } \langle \Delta_{S_0}^+; f, Pg \rangle = 0 \text{ for all } f, g \in \mathcal{E}_c^c.$$

$$(\mathbf{H} 3) \quad \Delta_{S_0}^+ \text{ is of positive type, meaning that } \langle \Delta_{S_0}^+, \bar{f} \otimes f \rangle \geq 0, \text{ where } \bar{f} \text{ is the complex conjugation of } f \in \mathcal{E}_c^c.$$

Example 5.2. *On Minkowski spacetime it is natural to choose $\Delta_{S_0}^+$ as the Wightman 2-point function, given by*

$$\Delta_{S_0}^+(x, y) = \int d^3\mathbf{p} \frac{e^{-i\omega(\mathbf{p})(x^0-y^0)+i\mathbf{p}(\mathbf{x}-\mathbf{y})}}{2\omega(\mathbf{p})},$$

with $\omega(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + m^2}$. For this choice we will use the notation $\Delta_{S_0}^+ = \frac{i}{2}\Delta_{S_0} + \Delta_1$.

On general globally hyperbolic spacetimes a decomposition (5.5) with properties (H 0)–(H 3) always exists but is not unique. If H and H' corresponds to two different choices of the split (5.5), then their difference $H - H'$ is a smooth symmetric bisolution to the field equations (a smooth symmetric function with $P_x(H - H')(x, y) = P_y(H - H')(x, y) = 0$). Physically, the split of the causal propagator into $\Delta_{S_0}^+$ and H is interpreted as “taking the positive frequency part” of the propagator. In flat spacetime this corresponds to the spectrum condition, which is one of the Haag-Kastler axioms listed in section 2.3.

The WF set of Δ_S^+ has better properties than the WF set on Δ_S . If we now replace $\frac{i}{2}\Delta_S$ with Δ_S^+ in (5.3), then the new product, denoted by \star_H can be extended from \mathcal{F}_{reg} to $\mathcal{F}_{\mu c}$ [BDF09]. On $\mathcal{F}_{\text{reg}}[[\hbar]]$ the two star products \star

and \star_H are isomorphic. To see this, consider a map $\alpha_H : \mathcal{F}_{\text{reg}}[[\hbar]] \rightarrow \mathcal{F}_{\text{reg}}[[\hbar]]$ given by

$$\alpha_H \doteq e^{\frac{\hbar}{2} \langle H, \frac{\delta^2}{\delta \varphi^2} \rangle}. \quad (5.6)$$

It is easy to check that

$$F \star_H G = \alpha_H \left((\alpha_H^{-1} F) \star (\alpha_H^{-1} G) \right), \quad F, G \in \mathcal{F}_{\text{reg}}. \quad (5.7)$$

We say that α_H provides a gauge transformation between \star and \star_H .

Definition 5.2. *A gauge transformation is a map $F \mapsto F + \sum_{\hbar \geq 1} \hbar^n D_n(f)$, where each D_n is a differential operator. If there exists a gauge transformation relating two star products \star and \star' , then they define the same deformation quantization.*

In the case of α_H , $D_n = \frac{1}{n!} \left\langle \frac{1}{2}(H - H'), \frac{\delta^2}{\delta \varphi^2} \right\rangle^n$. Physically, we can identify the transition between \star and \star_H with normal ordering, so passing to the \star_H -product is the algebraic version of Wick's theorem. Note that the codomain of $\alpha_H : \mathcal{F}_{\text{reg}} \rightarrow \mathcal{F}_{\text{reg}}$ is sequentially dense in a larger space $\mathcal{F}_{\mu c}$ (with respect to the topology τ_{BDF} described in section 4.4.2, definition 4.14) and we can also build a corresponding (sequential) completion $\alpha_H^{-1}(\mathcal{F}_{\mu c})$ of the domain. This amounts to extending \mathcal{F}_{reg} with all elements of the form $\lim_{n \rightarrow \infty} \alpha_H^{-1}(F_n)$, where (F_n) is a convergent sequence in $\mathcal{F}_{\mu c}$. This motivates the following definition

Definition 5.3. *The quantum algebra of the free theory \mathfrak{A} is defined as the extension of $\mathcal{F}_{\text{reg}}[[\hbar]]$ by limits $\lim_{n \rightarrow \infty} \alpha_H^{-1}(F_n)$, where (F_n) is a convergent sequence in $\mathcal{F}_{\mu c}$. \mathfrak{A} is equipped with the star product defined by*

$$A \star B = \alpha_H^{-1} \left(\alpha_H(A) \star_H \alpha_H(B) \right).$$

We introduce an involution on \mathfrak{A} by setting

$$A^* \doteq \alpha_H^{-1} \left(\overline{\alpha_H(A)} \right),$$

where bar denotes the complex conjugation of a functional. The support of $A \in \mathfrak{A}$ is defined by

$$\text{supp } A \doteq \text{supp } \alpha_H(A).$$

Remark 5.1. Note that (\mathfrak{A}, \star) as an abstract unital involutive topological algebra is unique up to an isomorphism, since different choices of H are related by

$$F \star_{H'} G = \alpha_{H-H'}^{-1} \left(\alpha_{H-H'}(F) \star_H \alpha_{H-H'}(G) \right),$$

and $H - H'$ is a smooth function, so $\alpha_{H-H'}$ is an isomorphism. This isomorphism doesn't change the support of a functional, so the notion of support introduced in definition 5.3 is also independent of the choice of H .

Remark 5.2. We can also characterize \mathfrak{A} as the space of families (F_H) , labeled by possible choices of H , where $F_H \in \mathfrak{A}^H \doteq (\mathcal{F}_{\mu c}[[\hbar]], \star_H)$ fulfill the relations

$$F_{H'} = \alpha_{H'-H} F_H,$$

and the product is defined by

$$(F \star G)_H = F_H \star_H G_H.$$

We think of \mathfrak{A} as the abstract algebra of observables, while the choice of α_H corresponds to a choice of realization of \mathfrak{A} as an algebra of formal power series with coefficients in some space of functionals. Although we are primarily interested in the abstract structure, the concrete realization is important if we want to make some computations. The diagram below summarizes the algebraic structures introduced so far.

$$\begin{array}{ccc} (\mathcal{F}_{\text{reg}}, \star) & \xrightarrow{\alpha_H} & (\mathcal{F}_{\text{reg}}, \star_H) \\ \text{dense} \downarrow \cap & & \text{dense} \downarrow \cap \\ \mathfrak{A} & \xleftarrow{\alpha_H^{-1}} & (\mathcal{F}_{\mu c}, \star_H) \end{array}$$

We use the notion of support defined for the elements of \mathfrak{A} to build a net of involutive topological algebras.

Proposition 5.1. *We obtain a free quantum theory field theory model in the sense of definition 2.39 on a spacetime $\mathcal{M} = (M, g)$ with the configuration space $\mathcal{E} = \Gamma(E \xrightarrow{\pi} M)$ and the generalized Lagrangian L as the net*

$$\mathcal{O} \mapsto (\mathfrak{A}(\mathcal{O}), \star, *),$$

where $\mathfrak{A}(\mathcal{O})$ is generated by elements with support contained in \mathcal{O} . This net satisfies also the axiom of covariance and the time-slice axiom.

Proof. The properties of covariance and causality are clear from the construction. The time-slice axiom has been proven in [CF08]. \square

The construction is fully covariant, so the assignment of unital $*$ -algebras to spacetime defined above induces a functor from **Loc** to **Obs_p**.

Proposition 5.2. *A locally covariant QFT model in the sense of definition 2.51 is obtained by assigning $\mathfrak{A}(\mathcal{M})$ to objects of **Loc** and for morphisms $\chi \in \text{Hom}(\mathcal{M}, \mathcal{N})$, \mathfrak{A}_χ is defined with the use of the pullback of functionals, as in proposition 3.4.*

Let us now discuss the existence of states. One obtains a family of states on \mathfrak{A} by setting

$$\omega_{H,\varphi}(F) \doteq \alpha_H(F)(\varphi) = F_H(\varphi),$$

where $\varphi \in \mathcal{E}_S$. This is well defined, since F_H is a functional in $\mathcal{F}_{\mu c}$, hence the evaluation at a field configuration φ makes sense.

Example 5.3. *We continue with the example of the free scalar field. We define $\tilde{\mathfrak{A}}$ as the subalgebra of \mathfrak{A} generated by the Weyl generators $\mathcal{W}(f) \doteq \exp(iF_f)$. Note that*

$$\left\langle H, \frac{\delta^2}{\delta\varphi^2} \right\rangle \left(i \int f\varphi d\mu \right)^n = -\frac{n!}{(n-2)!} H(f, f) \left(i \int f\varphi d\mu \right)^{n-2}.$$

Hence

$$\alpha_H(\mathcal{W}(f)) = e^{-\frac{\hbar}{2}H(f,f)}\mathcal{W}(f).$$

It follows now that

$$\omega_{H,0}(\mathcal{W}(f)) = e^{-\frac{\hbar}{2}H(f,f)},$$

so $\omega_{H,0}$ is a quasi-free state with covariance H (compare with the definition 2.19).

5.2 The star product on the space of multivector fields

In the classical theory we have defined “going on-shell” as taking the quotient of the algebra of classical observables by the ideal \mathfrak{I}_S generated by the equations of motion. Now we want to do something similar in the quantum theory.

Definition 5.4. *The equations of motion ideal $\mathfrak{I}_{S_0}^H$ of \mathfrak{A}^H is defined as the \star_H -ideal of \mathfrak{A}^H generated by elements of the form $\langle S'_0, X \rangle$, where $X \in \mathcal{V}_{\text{loc}}$. The ideal \mathfrak{I}_{S_0} is then defined by the requirement that $F \in \mathfrak{I}_{S_0}$ if and only if $F_H \in \mathfrak{I}_{S_0}^H$ for all H .*

The on-shell quantum algebra of the free theory is then understood as the quotient

$$\mathfrak{A}_{S_0} \doteq \mathfrak{A}/\mathfrak{I}_{S_0}.$$

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Remark 5.3. Note that \mathfrak{A}_{S_0} can be characterized as the space of families $F = (F_H)$, where

$$F_H \in \mathfrak{A}_{S_0}^H \doteq \mathfrak{A}^H / \mathfrak{I}_{S_0}^H.$$

The definition used here suggests that we can express it also in terms of the differential δ_{S_0} , as defined in section 4.3. The star product \star can be easily extended to the space $\bigwedge \mathcal{V}$ of vector fields, if one replaces the pointwise product in (5.3) with the graded product \wedge . Before we give the explicit formula for \star_H of two multivector fields, it is convenient to introduce some notation. From the definition of smoothness for multilocal vector fields and by proposition 3.1, we know that $\frac{\delta^n X}{\delta \varphi^n}(\varphi)$ induces an element of $\Gamma^c(E^{\boxtimes n} \boxtimes (E^*)^{\boxtimes k} \rightarrow M^{n+k})$. We can now generalize the class of multivector fields of interest by allowing for maps X such that $\frac{\delta^n X}{\delta \varphi^n}(\varphi)$ (understood as a distributional section) satisfies a condition analogous to (6.25).

Definition 5.5. *The space $\bigwedge^k \mathcal{V}_{\mu c}$ of microcausal k -vector fields is identified with $\mathcal{O}_{\mu c}(T^*[1]\mathcal{E})$.*

Using this definition, we can now introduce the star product of multivector fields:

$$\begin{aligned} & \langle (X \star_H Y)(\varphi); v_1, \dots, v_{p+q} \rangle \doteq \\ & \sum_{n=0}^{\infty} \frac{\hbar^n}{n!p!q!} \sum_{\sigma \in S_{p+q}} \left\langle \left\langle \frac{\delta^n X}{\delta \varphi^n}(\varphi); v_{\sigma(1)}, \dots, v_{\sigma(p)} \right\rangle, (\Delta_{S_0}^+)^{\otimes n} \left\langle \frac{\delta^n Y}{\delta \varphi^n}(\varphi); v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)} \right\rangle \right\rangle, \end{aligned}$$

where $X, Y \in \bigwedge \mathcal{V}_{\mu c}$ are of degree p and q respectively. In the 0-th order, the star product gives just the wedge product of two multivector fields. In the first order, one obtains the extension of $[\cdot, \cdot]$ to $\bigwedge \mathcal{V}_{\mu c}$.

We can now characterize the ideal \mathfrak{A}_0 by means of the differential δ_{S_0} , as we did in classical theory. First we note that since $\Delta_{S_0}^+$ is a distributional bisolution for the operator P , we have

$$\delta_{S_0}(X \star_H Y) = (\delta_{S_0} X) \star_H Y + (-1)^{|X|} X \star_H (\delta_{S_0} Y),$$

for $X, Y \in \bigwedge \mathcal{V}_{\mu c}$, i.e. δ_{S_0} is a derivation with respect to the product \star_H . In particular we can write $\mathfrak{A}_{S_0}^H$ as

$$\mathfrak{A}_{S_0}^H = \frac{\text{Ker } \delta_{S_0} \upharpoonright_{\mathcal{F}_{\mu c}}}{\text{Im } \delta_{S_0} \upharpoonright_{\mathcal{V}_{\mu c}}}$$

Note that in our case $\text{Ker } \delta_{S_0} \upharpoonright_{\mathcal{F}_{\mu c}} = (\mathcal{F}_{\mu c}, \star_H)$, but we will see later on that this is no longer the case in gauge theories (chapter 7).

Example 5.4. Consider the algebra $\tilde{\mathfrak{A}}$ from example 5.3. Let $\tilde{\mathfrak{I}}_{S_0} \doteq \mathfrak{I}_{S_0} \cap \tilde{\mathfrak{A}}$. Note that $S'_0(\varphi) = P\varphi = (\square + m^2)\varphi$, and using integration by parts, we conclude that $F_{(\square+m^2)f}(\varphi) = \langle S'_0, f \rangle$. It is then easy to verify that $\tilde{\mathfrak{I}}_{S_0}$ is generated by the elements

$$\mathcal{W}((\square + m^2)f) - 1, \quad f \in \mathcal{D} \quad (5.8)$$

We denote $\tilde{\mathfrak{A}}/\tilde{\mathfrak{I}}_{S_0}$ by $\tilde{\mathfrak{A}}_{S_0}$. This algebra is then (algebraically) isomorphic to the Weyl algebra $\mathcal{W}(L, \sigma)$, with $L = \mathcal{D}/P\mathcal{D}$ and $\sigma = \Delta_S$. States of the form $\omega_{H,0}$ are well defined on $\tilde{\mathfrak{A}}_{S_0}$, as H is a bisolution for the operator P .

5.3 Kähler structure

Objects introduced in the previous section have interpretation in terms of a Kähler structure, as in definition 2.20. H is a symmetric, non-degenerate bilinear form on $L = \mathcal{D}/P\mathcal{D}$, Δ_S is the symplectic structure and $\Delta_S^\dagger = \frac{i}{2}\Delta_S + H$ is a Hermitian 2-form on L^c , as in formula (2.2).

We have seen in example 5.3 that H is the covariance of a quasi-free state $\omega_{H,0}$ on $\tilde{\mathfrak{A}}_{S_0}$. If this state is pure, then the pair $(\Delta_S, 2H)$ is Kähler. In general one can always use Δ_S and H to define an anti-involution J , but $\Delta_S \circ J = 2H$ holds only in the case of pure $\omega_{H,0}$.

Let us now recall briefly the construction of J . We will essentially follow [DG13b]. First note that since H is a covariance of a quasi-free state, we know from theorem (2.3) that

$$|\Delta_S(f_1, f_2)| \leq 2\sqrt{H(f_1, f_2)}\sqrt{H(f_1, f_2)}, \quad \forall f_1, f_2 \in L. \quad (5.9)$$

We complete L with the product $(\cdot, \cdot)_H \doteq \langle \cdot, H \cdot \rangle$ to a real Hilbert space \mathcal{H} and the inequality (5.9) implies that Δ_S is a bilinear form on \mathcal{H} with norm less or equal 2. Therefore, there exists an operator $A \in \mathcal{B}(\mathcal{H})$ with $\|A\| \leq 1$ such that

$$\langle f_1, \Delta_S f_2 \rangle = 2\langle f_1, A f_2 \rangle_H$$

If the kernel of A is trivial, then we take the polar decomposition $A = -J|A|$ and J satisfies $J^2 = -1$. More generally, as in the proof of theorem 17.12 of [DG13b], we define $L_{\text{sg}} \doteq \text{Ker } A$ and $L_{\text{reg}} \doteq L_{\text{sg}}^\perp$. Then we set $A_{\text{reg}} \doteq A|_{L_{\text{reg}}}$ and construct the polar decomposition $A_{\text{reg}} = -J_{\text{reg}}|A_{\text{reg}}|$. If the dimension of L_{sg} is even or infinite (which is the case in the situation we are interested in), then there exist an orthogonal anti-involution J_{sg} on L_{sg} and we set $J = J_{\text{reg}} \oplus J_{\text{sg}}$. J constructed this way defines an almost-complex structure on L .

We define the holomorphic and anti-holomorphic subspaces of $L^{\mathbb{C}}$ as

$$\begin{aligned}\mathcal{Z} &\doteq \{(f - iJf) | f \in L\}, \\ \overline{\mathcal{Z}} &\doteq \{(f + iJf) | f \in L\},\end{aligned}$$

respectively. Projections onto these subspaces are defined as $\mathbf{1}_{\mathcal{Z}} = \frac{1}{2}(\mathbf{1} - iJ^{\mathbb{C}})$ and $\mathbf{1}_{\overline{\mathcal{Z}}} = \frac{1}{2}(\mathbf{1} + iJ^{\mathbb{C}})$

If $\omega_{H,0}$ is pure then the quadruple $(L, 2H, \Delta_S, J)$ is a Kähler structure. We decompose Δ_S^+ in the holomorphic basis to obtain

$$\langle \mathbf{1}_{\overline{\mathcal{Z}}} f_1, \Delta_S^+(\mathbf{1}_{\mathcal{Z}} f_2) \rangle = \langle f_1, \Delta_S^+ f_2 \rangle,$$

where $f_1, f_2 \in L^{\mathbb{C}}$ and remaining components vanish. As a result, Δ_S^+ is represented in the holomorphic basis as

$$\begin{pmatrix} 0 & 0 \\ \Delta_S^+ & 0 \end{pmatrix},$$

so it acts only on the holomorphic part of the first argument and the anti-holomorphic part of the second argument.

Chapter 6

Interaction and renormalization of the scalar field theory

6.1 Outline of the approach

In the previous chapter we have covered the quantization of free theories (quadratic actions), now is the time to introduce the interactions. This is where we have to start working perturbatively. The ultimate goal of AQFT is to be able to construct interacting models in 4 spacetime dimensions non-perturbatively, but at the moment no such models are known. The perturbative approach, on the other hand, has proven to be successful in describing many phenomena in particle physics, so it is worthwhile to try to understand its mathematical foundations. It turns out that a careful analysis of the problem and employing some tools from functional analysis allows to avoid dealing with not well defined “divergent” expressions, as is often done in physics textbooks.

We will follow the ideas on renormalization developed by [BP57, BS59, Hep66, EG73, ?, Ste71]. The approach is motivated by the interaction picture of quantum mechanics, as outlined in section 2.1.3. We begin with a heuristic argument and then we will show how to make it rigorous. Let H_0 be the free Hamiltonian and let $H_{t,I} = - \int_K : \mathcal{L}_I(0, \mathbf{x}) : d\sigma_t$ be the interaction Hamiltonian, where $: \mathcal{L}_I :$ is the normal-ordered Lagrangian density (the precise definition of normal-ordering will be introduced later in this chapter), constructed from the classical quantity $\mathcal{L}_I(x)$ and K is some compact subset of a Cauchy surface Σ .

We would like to use the Dyson formula (2.10) for the interacting time evolution operator $U_I(t, s)$, so formally we write

$$U_I(t, s) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int_{([s,t] \times \mathbb{R}^3)^n} T(:\mathcal{L}_I(x_1): \dots : \mathcal{L}_I(x_n):) d^{4n}x,$$

where

$$x \mapsto \mathcal{L}_I(x) = e^{iH_0x^0} : \mathcal{L}_I(0, \mathbf{x}) : e^{-iH_0x^0}$$

is an operator-valued function and T denotes time-ordering. Heuristically, one would use the unitary map defined above to obtain interacting fields as

$$\varphi_I(x) = U(x^0, s)^{-1} \varphi(x) U(x^0, s) = U(t, s)^{-1} U(t, x^0) \varphi(x) U(x^0, s), \quad (6.1)$$

where $s < x^0 < t$.

There are, however, serious problems with this idea. The first obvious difficulty is the fact that typical Lagrangian densities like $:\mathcal{L}_I(x): = :\varphi(x)^4:$ can not be restricted to a Cauchy surface Σ_0 as operator-valued distributions (they are too singular). This is the source of the so called UV problem. Moreover, having the sharp cutoff function in the Lagrangian and Hamiltonian (integrating with the characteristic function of $K \times [s, t]$ leads to additional divergences, called the Stückelberg divergences. Finally there is the problem with taking the adiabatic limit, as the integral of the Lagrangian density over \mathbf{x} does not exist, if Σ is non-compact. Last but not least, the overall sum might not converge.

Fortunately all these problems, apart from the last one, can be easily dealt with by a slight modification of the above ansatz. First, to avoid the Stückelberg divergences, we replace the sharp cutoffs with smooth test functions. Next, we solve the UV problem by using causal perturbation theory in the sense of Epstein and Glaser [EG73], where the interaction is switched on only in a compact region of spacetime. Finally, we take the adiabatic limit algebraically, as a certain inductive limit on the level of interacting observable algebras.

6.2 Scattering matrix and time ordered products

The modifications of the Dyson formula ansatz described at the end of the previous section lead to the definition of the *formal S-matrix* as

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{i^n}{n!} \int g(x_1) \dots g(x_n) T(:\mathcal{L}_I(x_1): \dots : \mathcal{L}_I(x_n):),$$

where g is a test density. Compare this formula with (2.11). In order to make this formula mathematically rigorous, first we need to make sense of the normal ordering operation $:\mathcal{L}_I(x):$, and then we have to define the time-ordered products of $:\mathcal{L}_I(x_i):$. Finally, in order to make sense of the formula (6.1) for the interacting field we interpret it as a definition of a *distribution*, rather than a function. Hence, for a test density f we obtain

$$\begin{aligned} \int f(x)\varphi_I(x) &= S(g)^{-1} \sum_{n=0}^{\infty} \frac{i^n}{n!} \int f(x)g(x_1)\dots g(x_n)T\varphi(x)\mathcal{L}_I(x_1)\dots\mathcal{L}_I(x_n) = \\ &= \frac{d}{d\lambda} S(g)^{-1}S(g, \lambda f)|_{\lambda=0}, \end{aligned} \quad (6.2)$$

where $S(g, f)$ is the formal S-matrix with the Lagrangian density $g\mathcal{L}_I + f\varphi$. This is the so called *Bogoliubov's formula* [BS59].

6.2.1 Wick products

Let us start our construction with defining the Wick-ordered quantities $:\mathcal{L}_I(x):$. In our framework normal (Wick) ordering is a prescription how to identify classical quantities with their quantum counterparts. More precisely, it is a map $\mathcal{T}_1 : \mathcal{F}_{\text{loc}} \rightarrow \mathfrak{A}$, $\mathcal{T}_1 \doteq \alpha_H^{-1} \circ \mathcal{T}_1^H$, where $\mathcal{T}_1^H : \mathcal{F}_{\text{loc}} \rightarrow \mathfrak{A}_H$. Note that both classical and quantum observables are understood as (formal power series in) microcausal functionals, so the easiest way is define the normal ordering is to set $\mathcal{T}_1^H = \text{id}$ and hence

$$:F: \doteq \mathcal{T}_1 F = \alpha_H^{-1}(F), \quad F \in \mathcal{F}_{\text{loc}}. \quad (6.3)$$

Example 6.1 (Algebraic Wick's theorem). *Wick's theorem plays important role in physics, so we will show here how it follows from our algebraic definition of normal ordering. Consider the model of free scalar fields. Let*

$$F_n(\varphi) = \int \varphi(x)\varphi(y)g_n(y-x)f(x)d\mu(x)d\mu(y),$$

where $f \in \mathcal{D}$ and g_n is a sequence of smooth compactly supported functions that converges to which converges to the Dirac delta distribution $\delta(x-y)$ in the topology τ_H defined in 4.12. By applying α_H^{-1} to the sequence (F_n) , we obtain a sequence

$$\alpha_H^{-1}F_n = \int (\varphi(x)\varphi(y)g_n(y-x)f(x) - \hbar H(x,y)g_n(y-x)f(x))d\mu(x)d\mu(y),$$

The limit of this sequence is identified with the normally ordered expression $\int :\varphi(x)^2: f(x) d\mu(x)$, i.e.:

$$\mathcal{T}_1 F = \int :\varphi^2: f d\mu = \lim_{n \rightarrow \infty} \int (\varphi(x)\varphi(y) - \hbar H(x, y)) g_n(y-x) f(x) d\mu(x) d\mu(y).$$

Not that the right-hand side cannot be interpreted as a functional on the configuration space, but is just a formal expression representing an abstract element of the extension of \mathcal{F}_{reg} , as defined in 5.3. We write this expression in a short-hand notation as a coinciding point limit:

$$:\varphi(x)^2: = \lim_{y \rightarrow x} (\varphi(x)\varphi(y) - \hbar H(x, y)).$$

This shows that transforming with α_H^{-1} corresponds formally to the subtraction of $\hbar H(x, y)$.

Now, to recover Wick's theorem, consider a product of two Wick squares $:\varphi(x)^2: :\varphi(y)^2:$. Note that in our setting this has to be understood as the \star -product of two elements of \mathfrak{A} . Using the α_H^{-1} prescription we identify this product as

$$\begin{aligned} \left(\alpha_H^{-1} \int \varphi^2 f_1 d\mu \right) \star \left(\alpha_H^{-1} \int \varphi^2 f_2 d\mu \right) &= \alpha_H^{-1} \left(\left(\int \varphi^2 f_1 d\mu \right) \star_H \left(\int \varphi^2 f_2 d\mu \right) \right) = \\ &= \alpha_H^{-1} \left(\int \varphi^2 f_1 d\mu \int \varphi^2 f_2 d\mu + 4\hbar \langle f_1 \varphi, \Delta_{S_0}^+(f_2 \varphi) \rangle + \right. \\ &\quad \left. + \frac{\hbar^2}{2} \int (\Delta_{S_0}^+(x, y))^2 f_1(x) f_2(y) d\mu(x) d\mu(y) \right). \end{aligned}$$

Omitting the test functions we obtain

$$:\varphi(x)^2: :\varphi(y)^2: = :\varphi(x)^2 \varphi(y)^2: + 4:\varphi(x)\varphi(y): \hbar \Delta_{S_0}^+(x, y) + 2 \left(\hbar \Delta_{S_0}^+(x, y) \right)^2,$$

which is a familiar form of Wick's theorem applied to $:\varphi(x)^2: :\varphi(y)^2:$.

6.2.2 Locally covariant Wick products

The normal ordering prescription (6.3) is not the most optimal one, if we work on curved spacetime. This is because we would like to define the normal ordering on all the globally hyperbolic spacetimes in a coherent way and the choice of H across different spacetimes cannot be made covariantly. More precisely, we want to lift \mathcal{T}_1 to the level of locally covariant fields in the sense of definition 2.51.

Definition 6.1. Let $\mathcal{T}_{1\mathcal{M}}$ be the Wick-ordering prescription on spacetime $\mathcal{M} \in \text{Obj}(\mathbf{Loc})$. Given a natural transformation $\Phi : \mathfrak{D} \rightarrow \mathfrak{F}_{\text{loc}}$ we say that $(\mathcal{T}_{1\mathcal{M}})_{\mathcal{M} \in \text{Obj}(\mathbf{Loc})}$ is covariant if the family $\mathcal{T}_{1\mathcal{M}} \circ \Phi_{\mathcal{M}}$ defines a natural transformation from \mathfrak{D} to \mathfrak{A} .

We will now discuss the existence of such a family of Wick-ordering prescriptions and the remaining renormalization freedom.

Even though Hadamard states on different spacetimes cannot be chosen in a coherent way, it turns out that this is possible for a family of Hadamard parametrices. The difference between $\Delta_{S_0}^+$, a 2-point function of a Hadamard state, and a paramterix is that the latter is a bi-solution of the linearized equations of motion only up to smooth terms. Due to the covariance condition, it is more appropriate to define the normal ordering by a prescription where only the singular part of H enters into α_H^{-1} . We realize this idea by setting

$$\mathcal{T}_{1\mathcal{M}}F \doteq \alpha_H^{-1}(\alpha_w F) = \alpha_{H-w}^{-1}F,$$

where $F \in \mathfrak{F}_{\text{loc}}$ and where w is the smooth part of the Hadamard 2-point function. In other words, we set $\mathcal{T}_1^H \equiv \alpha_w$. To understand what is w , we need to recall some facts about the singularity structure of 2-point functions Hadamard states. We follow here closely [KW91] and a recent review [FR14].

Be begin with introducing some notation. Let $t : \mathcal{M} \rightarrow \mathbb{R}$ be a time function (smooth function with a timelike and future directed gradient field) and let

$$\sigma_\varepsilon(x, y) \doteq \sigma(x, y) + 2i\varepsilon(t(x) - t(y)) + \varepsilon^2,$$

where $\sigma(x, y)$ is half of the square geodesic distance between x and y , i.e.

$$\sigma(x, y) \doteq \frac{1}{2}g(\exp_x^{-1}(y), \exp_x^{-1}(y)).$$

Definition 6.2. We say that a bi-distribution W on \mathcal{M} is of local Hadamard form if, for every $x_0 \in M$, there exists a geodesically convex neighbourhood V of x_0 such that, for every integer N , $W(x, y)$ on $V \times V$ can be written in the form

$$W = \lim_{\varepsilon \downarrow 0} \left(\frac{u}{\sigma_\varepsilon} + \sum_{n=0}^N \sigma^n v_n \log \left(\frac{\sigma_\varepsilon}{\lambda^2} \right) + w_N \right) = W_N^{\text{sing}} + w_N, \quad (6.4)$$

where $u, v_n \in \mathcal{C}^\infty(M^2, \mathbb{R})$, $n = 0, \dots, N$ are solutions of the so-called transport equations and are uniquely determined by the local geometry, λ is a free parameter with the dimension of inverse length and w_N is an $2N + 1$ times continuously differentiable real-valued function.

We define now

$$\alpha_w F \doteq \lim_{N \rightarrow \infty} \alpha_{w_N} F, \quad (6.5)$$

for $F \in \mathcal{F}_{\text{loc}}$. This limit makes sense, because the series converges after finitely many steps. It is crucial that in this formula F is local, since the limit of w_N is well defined only in a geodesically convex neighborhood of the diagonal in $M \times M$. Since functional derivatives of a local functional are supported on the diagonal, $\alpha_{w_N} F$ has a well-defined limit. For details see [HW02a]. Note that with in our definition $\mathcal{T}_{1\mathcal{M}}$ is fixed up to the choice of the energy scale λ in definition 6.4. One can, however allow additional freedom, staying consistent with the requirements of locality, covariance and some regularity properties. Such possibility has been investigated in [HW02a], where it was shown that for the scalar field $\mathcal{T}_{1\mathcal{M}}$ is fixed up to two paramters. The choice of these paramters is then treated as additional renormalization freedom.

6.2.3 Time-ordered products

After giving sense to normally ordered expressions we want now to define their time-ordered products. In the first step we consider only regular functionals, to understand the algebraic structure. We want to define the time ordered product $\cdot_{\mathcal{T}}$ as a binary operation on $\mathcal{F}_{\text{reg}}[[\hbar]]$, which satisfies the condition (2.9), i.e.

$$F \cdot_{\mathcal{T}} G = \begin{cases} F \star G & \text{if } \text{supp } G \prec \text{supp } F, \\ G \star F & \text{if } \text{supp } F \prec \text{supp } G, \end{cases} \quad (6.6)$$

where the relation “ \prec ” means “not later than” i.e. there exists a Cauchy surface which separates $\text{supp } G$ and $\text{supp } F$ and in the first case $\text{supp } F$ is in the future of this surface and in the second case it’s in the past. We postulate

$$F \cdot_{\mathcal{T}} G = \sum_{n=0}^{\infty} \hbar^n B_n(F, G), \quad (6.7)$$

for some functional differential operators B_n . Now let $F = \langle \cdot, f \rangle$ and $G = \langle \cdot, g \rangle$ be two linear functionals with $f, g \in \Gamma_c(E^* \rightarrow M)$. Then condition (6.6) implies

$$B_0(FG) + \hbar B_1(F, G) = \begin{cases} FG + \frac{i}{2} \hbar \langle f, \Delta_{S_0} g \rangle & \text{if } \text{supp } g \prec \text{supp } f, \\ FG + \frac{i}{2} \hbar \langle g, \Delta_{S_0} f \rangle & \text{if } \text{supp } f \prec \text{supp } g, \end{cases}$$

This suggests to set $B_0(F, G) = FG$ for all $F, G \in \mathcal{F}_{\text{reg}}$. To analyze the first order condition in more detail, we note that $\Delta_{S_0} = \Delta_{S_0}^A - \Delta_{S_0}^R$ and from

the support properties of the advanced and retarded Green functions follows that in the example above

$$B_1(F, G) = \begin{cases} \frac{i}{2} \langle f, \Delta_{S_0}^A g \rangle & \text{if } \text{supp } g \prec \text{supp } f, \\ \frac{i}{2} \langle f, \Delta_{S_0}^R g \rangle & \text{if } \text{supp } f \prec \text{supp } g, \end{cases}$$

Now, since B_1 has to be a differential operator, the condition above fixes its coefficients by the first order derivatives up to diagonal, i.e.

$$B_1 = m_2 \circ \left\langle t, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle + \text{higher derivatives},$$

where m_n is the pointwise multiplication operator, i.e. $m_n(F_1, \dots, F_n)(\varphi) \doteq F_1(\varphi) \cdots F_n(\varphi)$ and t is a distribution in $\Gamma'_c((E^*)^{\boxtimes 2} \rightarrow M^2)$ with a kernel satisfying

$$t(x, y) = \begin{cases} \frac{i}{2} \Delta_{S_0}(x, y) = \frac{i}{2} \Delta_{S_0}^A(x, y) & \text{if } y \prec x, \\ -\frac{i}{2} \Delta_{S_0}(x, y) = \frac{i}{2} \Delta_{S_0}^R(x, y) & \text{if } x \prec y. \end{cases}$$

It turns out that a consistent choice is provided by $t = i\Delta_{S_0}^D$, where $\Delta_{S_0}^D \doteq \frac{1}{2}(\Delta_{S_0}^R + \Delta_{S_0}^A)$ is the Dirac propagator. Using the higher order conditions and requiring associativity leads to the following formula for the time-ordering product:

$$\begin{aligned} F \cdot_{\mathcal{T}} G &\doteq m_2 \circ e^{ih \left\langle \Delta_{S_0}^D, \frac{\delta}{\delta\varphi} \otimes \frac{\delta}{\delta\varphi} \right\rangle} (F \otimes G) = \\ &= \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}, (i\Delta_{S_0}^D)^{\otimes n} G^{(n)} \right\rangle, \end{aligned} \quad (6.8)$$

where $F, G \in \mathcal{F}_{\text{reg}}$. The product defined this way in contrast to \star is commutative, since $\Delta_{S_0}^D$ is symmetric. It is also isomorphic to the point-wise product by means of

$$F \cdot_{\mathcal{T}} G = \mathcal{T} (\mathcal{T}^{-1} F \cdot \mathcal{T}^{-1} G), \quad (6.9)$$

where

$$\mathcal{T} = e^{ih \left\langle \Delta_{S_0}^D, \frac{\delta^2}{\delta\varphi^2} \right\rangle}, \quad (6.10)$$

or more precisely

$$(\mathcal{T}F)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle (i\Delta_{S_0}^D)^{\otimes n}, F^{(2n)}(\varphi) \right\rangle.$$

The linear operator \mathcal{T} defined above is interpreted as a map which goes from the classical world to the quantum world, i.e.

$$\begin{array}{ccc} (\mathcal{F}_{\text{reg}}, \cdot) & \xrightarrow{\mathcal{T}} & (\mathfrak{A}_{\text{reg}}, \star, \cdot_{\mathcal{T}}) \\ \text{classical} & & \text{quantum} \end{array} ,$$

where $\mathfrak{A}_{\text{reg}} \subset \mathfrak{A}$ is the range of \mathcal{T} . Note that on the quantum side there are *two* products, the non-commutative \star and commutative $\cdot_{\mathcal{T}}$.

Using the time-ordered product we express the formal S -matrix $\mathcal{S} : \mathcal{F}_{\text{reg}}[[\hbar]] \rightarrow \mathcal{F}_{\text{reg}}[[\hbar]]$ as the time-ordered exponential

$$\mathcal{S}(V) \doteq e_{\mathcal{T}}^{iV/\hbar} = \mathcal{T}(e^{\mathcal{T}^{-1}iV/\hbar}) , \quad (6.11)$$

so \mathcal{S} is a map from the quantum algebra \mathfrak{A} to itself. Interacting fields are obtained by means of the Bogoliubov formula, which reads

$$\begin{aligned} R_V(F) &= -i\hbar \frac{d}{d\lambda} (\mathcal{S}(V)^{\star^{-1}} \star \mathcal{S}(V + \lambda F))|_{\lambda=0} = \\ &= \left(e_{\mathcal{T}}^{iV/\hbar} \right)^{\star^{-1}} \star \left(e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} F \right) . \end{aligned} \quad (6.12)$$

We understand $R_V(F)$ as the interacting observable corresponding to F . We define the interacting star product by

$$F \star_V G \doteq R_V^{-1}(R_V F \star R_V G) .$$

The interacting theory for regular functionals is the algebra $(\mathcal{F}_{\text{reg}}, \star_V)$ and R_V acts as the intertwining map between the free quantum theory and the interacting quantum theory, i.e.

$$\begin{array}{ccc} (\mathcal{F}_{\text{reg}}, \cdot) & \xrightarrow{\mathcal{T}} & (\mathfrak{A}_{\text{reg}}, \star, \cdot_{\mathcal{T}}) & \xrightarrow{R_V^{-1}} & (\mathfrak{A}_{\text{reg}}, \star_V) \\ \text{classical} & & \text{free} & & \text{interacting} \\ & & \text{quantum} & & \text{quantum} \end{array} . \quad (6.13)$$

The formula (6.9) for the time-ordered product make sense if we restrict ourselves to regular functionals. This is, however, not enough, since typical interactions appearing in particle physics are local and non-linear (hence not regular). In the first attempt to fix the problem we pass to a different star product, which amounts to replacing Δ_{S_0} by $\Delta_{S_0}^+$ and $\Delta_{S_0}^D$ by the Feynman propagator $\Delta_{S_0}^F = i\Delta_{S_0}^D + H$, so we obtain now

$$\begin{array}{ccc} (\mathcal{F}_{\text{reg}}, \cdot) & \xrightarrow{\mathcal{T}^H} & (\mathfrak{A}_{\text{reg}}, \star_H, \cdot_{\mathcal{T}^H}) & \xrightarrow{\alpha_H^{-1}} & (\mathfrak{A}_{\text{reg}}, \star, \cdot_{\mathcal{T}}) & \xrightarrow{R_V^{-1}} & (\mathfrak{A}_{\text{reg}}, \star_V) \\ \text{classical} & & & & \text{free} & & \text{interacting} \\ & & & & \text{quantum} & & \text{quantum} \end{array} ,$$

where $\mathcal{T}^H \doteq e^{i\hbar\langle\Delta_{S_0}^F, \frac{\delta^2}{\delta\psi^2}\rangle}$, so $\mathcal{T} = \alpha_H^{-1} \circ \mathcal{T}^H$. Unfortunately this modification is not sufficient to extend the time-ordered products to arbitrary local functionals. The difficulty which we have to face is the fact that the WF set of $\Delta_{S_0}^F$ at 0 behaves like the WF set of the Dirac delta (i.e. contains all non-zero covectors), hence the tensor powers of $\Delta_{S_0}^F$ cannot be contracted with derivatives of local functionals.

The non-trivial extension problem which we want to solve is what physics literature refers to as the *renormalization problem*. A mathematically rigorous solution to this problem has been proposed by Epstein and Glaser in the seminal paper [EG73] from 1973 and it makes use of the causal structure of the spacetime. Here we will use a similar approach.

First we extent the linear operator \mathcal{T}^H to \mathcal{F}_{loc} by setting $\mathcal{T}^H = \alpha_w$, where α_w is defined by (6.5). On Minkowski spacetime we can also set $\mathcal{T} = \text{id}$. The subspace $\mathcal{T}^H(\mathcal{F}_{\text{loc}}) \subset \mathfrak{A}^H$ is denoted by $\mathfrak{A}_{\text{loc}}^H$. Recall that $\mathfrak{A}^H \equiv (\mathcal{F}_{\mu c}[[\hbar]], \star_H)$. We define a map from local functionals into the quantum algebra $\mathcal{T} : \mathcal{F}_{\text{loc}} \rightarrow \mathfrak{A}$ by setting $\mathcal{T} \doteq \alpha_H^{-1} \circ \mathcal{T}^H$ and denote its image by $\mathfrak{A}_{\text{loc}}$.

Definition 6.3. Denote by $(\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes n}$ the subspace of $(\mathcal{F}_{\text{loc}})^{\otimes n}$ spanned by $F_1 \otimes \cdots \otimes F_n$, where $F_1, \dots, F_n \in \mathcal{F}_{\text{loc}}$ have pairwise disjoint supports.

Definition 6.4. On $(\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes n}$ we define the n -fold time-ordered product as a map $\mathcal{T}_n^H : (\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes n} \rightarrow \mathfrak{A}^H$ given by

$$\mathcal{T}_n^H(F_1, \dots, F_n) \doteq F_1 \cdot_{\mathcal{T}_H} \cdots \cdot_{\mathcal{T}_H} F_n.$$

Let $F_1 \otimes \cdots \otimes F_n \in (\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes n}$ such that the supports $\text{supp } F_i$, $i = 1, \dots, k$ of the first k entries do not intersect the past of the supports $\text{supp } F_j$, $j = k+1, \dots, n$ of the last $n-k$ entries. From the definition of the time-ordered product follows that in this case

$$\mathcal{T}_n^H(F_1 \otimes \cdots \otimes F_n) = \mathcal{T}_k^H(F_1 \otimes \cdots \otimes F_k) \star_H \mathcal{T}_{n-k}^H(F_{k+1} \otimes \cdots \otimes F_n), \quad (6.14)$$

This property is called the *causal factorisation property*. It is a crucial feature which we want to require also from the extended n -fold time-ordered products. This motivates the following, axiomatic definition.

Definition 6.5. Renormalized time-ordered products are multilinear maps $\mathcal{T}_n^H : \mathcal{F}_{\text{loc}}^{\otimes n} \rightarrow \mathfrak{A}^H$, $n \in \mathbb{N}$, satisfying:

(T 1) Causal factorisation property

$$\mathcal{T}_n^H(F_1 \otimes \cdots \otimes F_n) = \mathcal{T}_k^H(F_1 \otimes \cdots \otimes F_k) \star_H \mathcal{T}_{n-k}^H(F_{k+1} \otimes \cdots \otimes F_n)$$

if the supports $\text{supp } F_i$, $i = 1, \dots, k$ of the first k entries do not intersect the past of the supports $\text{supp } F_j$, $j = k + 1, \dots, n$ of the last $n - k$ entries.

(T 2) Starting element: $\mathcal{T}_0^H = 1$, $\mathcal{T}_1^H = \text{id}^1$.

(T 3) Symmetry: For a purely bosonic theory \mathcal{T}_n^H s are symmetric in their arguments. If the fermions are present, \mathcal{T}_n^H s are graded-symmetric.

(T 4) Field independence: $\mathcal{T}_n^H(F_1, \dots, F_n)$, as a functional on \mathcal{E} , depends on φ only via the functional derivatives of F_1, \dots, F_n , i.e.

$$\frac{\delta}{\delta\varphi} \mathcal{T}_n^H(F_1, \dots, F_n) = \sum_{i=1}^n \mathcal{T}_n^H \left(F_1, \dots, \frac{\delta F_i}{\delta\varphi}, \dots, F_n \right)$$

(T 5) φ -Locality: $\mathcal{T}_n^H(F_1, \dots, F_n) = \mathcal{T}_n^H(F_1^{[N]}, \dots, F_n^{[N]}) + \mathcal{O}(\hbar^N)$, where $F_i^{[N]}$ is the Taylor series expansion of the functional F_i up to the N -th order.

Remark 6.1. Note that the property **(T 5)** allows one to reduce the problem of constructing the time-ordered products of local functionals to the construction of the time-ordered products of polynomials. Property **(T 4)** is crucial to show that the construction reduces to extensions of numerical distributions.

Given the family of maps $\{\mathcal{T}_n^H\}_{n \in \mathbb{N}}$, we can define time ordered products of Wick-ordered quantities and the finally the S-matrix.

Definition 6.6. We define $\mathcal{T}_n : \mathfrak{A}_{\text{loc}}^{\otimes n} \rightarrow \mathfrak{A}$ as

$$\mathcal{T}_n \doteq \alpha_H^{-1} \circ \mathcal{T}_n^H \circ \alpha_H^{\otimes n},$$

and the renormalized S-matrix, which is a map $\mathcal{S} : \mathfrak{A}_{\text{loc}} \rightarrow \mathfrak{A}$, is given by

$$\mathcal{S}(V) \doteq \sum_{n=0}^{\infty} \frac{i^n}{\hbar^n n!} \mathcal{T}_n(V^{\otimes n}),$$

where $V \in \mathfrak{A}_{\text{loc}}$.

¹One can leave some freedom in the definition of \mathcal{T}_1^H which can be then used to absorb the renormalization ambiguity in defining \mathcal{T} . See [HW05] for an exemple of a situation where this extra freedom turns out to be useful.

We want the S-matrix defined this way to be unitary, i.e. we require that

$$\mathcal{S}(-V)^* \star \mathcal{S}(V) = \mathbb{1},$$

for a real-valued V . This condition can be translated into an additional axiom for the time-ordered product:

(T 6) Unitarity: Let $F_1, \dots, \mathfrak{F}_n$ be real-valued, then $(\mathcal{T}_n^H)^* = \bar{\mathcal{T}}_n^H$, where $\bar{\mathcal{T}}_n^H$ is the anti-time-ordered product defined by

$$\bar{\mathcal{T}}_n^H(F_1 \otimes \dots \otimes F_n) = \sum_{P \in \mathcal{P}_n} (-1)^{n+|P|} \prod_{I \in P}^{\star_H} \mathcal{T}_{|I|}^H \left(\bigotimes_{i \in I} F_i \right),$$

where \mathcal{P}_n denoted the set of partitions of $\{1, \dots, n\}$ and $\prod_{I \in P}^{\star_H}$ means taking the \star_H product of factors labeled by I .

The final consideration is to ensure that the time-ordered products are covariant, i.e. they are defined on all the spacetimes in a coherent way. We adapt a definition analogous to 6.1.

Definition 6.7. Let $\mathcal{T}_{n\mathcal{M}}$ be the n -fold time-ordered product on spacetime $\mathcal{M} \in \text{Obj}(\mathbf{Loc})$. Given locally covariant quantum fields

$$:\Phi_i: \equiv \mathcal{T}_{\mathcal{M}} \Phi_i \in \text{Nat}(\mathcal{D}, \mathfrak{A}_{\text{loc}}), \quad i = 1, \dots, n,$$

we say that $(\mathcal{T}_{n\mathcal{M}})_{\mathcal{M} \in \text{Obj}(\mathbf{Loc})}$ is covariant if the family

$$\mathcal{T}_{n\mathcal{M}} \circ (:\Phi_1:_{\mathcal{M}}, \dots, :\Phi_n:_{\mathcal{M}})$$

defines a natural transformation in $\text{Nat}(\mathcal{D}^n, \mathfrak{A})$.

Finding the right notion of covariance for time-ordered products was a very important step in indertanding quantum field theory on curved spacetime. The idea goes back to Fredenhagen [Fre00] and was developed in a collaboration with R. Brunetti, S. Hollands, R. Verch and R. Wald [HW01, Ver01, BFV03]. The requirement of covariance can be translated into another axiom for \mathcal{T}_n^H 's.

(T 7) Covariance: $\mathcal{T}_n \doteq \alpha_H^{-1} \circ \mathcal{T}_n^H \circ \alpha_H^{\otimes n}$ is covariant in the sense of definition 6.7, i.e. $\alpha_H^{-1} \circ \mathcal{T}_n^H \circ \alpha_w^{\otimes n}$ maps an n -tuple of classical fields in $\text{Nat}(\mathcal{D}, \mathfrak{F}_{\text{loc}})$ to a quantum field in $\text{Nat}(\mathcal{D}^n, \mathfrak{A})$.

Using the inductive method of [EG73], generalized to curved spacetimes by [BF00, HW01, HW02a], one shows that a family of maps satisfying axioms (T 1)-(T 7) exists and the non-uniqueness of the construction is fully absorbed into adding multilinear maps $\mathcal{Z}_n^H : \mathcal{F}_{\text{loc}}^{\otimes n} \rightarrow \mathcal{F}_{\text{loc}}[[\hbar]]$, i.e.

$$\widetilde{\mathcal{T}}_n^H(F_1, \dots, F_n) = \mathcal{T}_n^H(F_1, \dots, F_n) + \mathcal{Z}_n^H(F_1, \dots, F_n), \quad (6.15)$$

where $\{\mathcal{T}_n^H\}_{n \in \mathbb{N}}$ and $\{\widetilde{\mathcal{T}}_n^H\}_{n \in \mathbb{N}}$ are two choices of families of n -fold time-ordered products that coincide up to order $n - 1$. A detailed argument for the existence of time-ordered products is given in sections 6.5.

The causal factorisation property for time-ordered products implies that the S-matrix satisfies Bogoliubov's factorization relation

$$\mathfrak{S}(V_1 + V_2 + V_3) = \mathfrak{S}(V_1 + V_2)\mathfrak{S}(V_2)^{-1}\mathfrak{S}(V_2 + V_3) \quad (6.16)$$

if the support of V_1 does not intersect the past of the support of V_3 .

Definition 6.8. We define the renormalized map $\mathcal{T} : \mathcal{F} \rightarrow \mathfrak{A}$ by $\mathcal{T} \doteq \bigoplus_n \alpha_H^{-1} \circ \mathcal{T}_n^H \circ \alpha_w \circ m^{-1}$, where $m^{-1} : \mathcal{F} \rightarrow S^\bullet \mathcal{F}_{\text{loc}}^{(0)}$ is the inverse of the multiplication, as defined in [FR12a] and $\mathcal{F}_{\text{loc}}^{(0)}$ is the space of local functionals that vanish at 0.

Using \mathcal{T} we conclude that the renormalized time-ordered product $\cdot_{\mathcal{T}}$ is a binary operation defined on the domain $D_{\mathcal{T}} \doteq \mathcal{T}(\mathcal{F})$. Analogously to the diagram (6.13), we obtain

$$\begin{array}{ccc} (\mathcal{F}, \cdot) & \xrightarrow{\mathcal{T}} & (\mathfrak{A}, \star, \cdot_{\mathcal{T}}) \\ \text{classical} & & \begin{array}{c} \text{free} \\ \text{quantum} \end{array} \end{array} \xrightarrow{R_V^{-1}} \begin{array}{c} (\mathfrak{A}, \star_V) \\ \text{interacting} \\ \text{quantum} \end{array}, \quad (6.17)$$

with the caveat that $\cdot_{\mathcal{T}}$ is well defined on $D_{\mathcal{T}} \subset \mathfrak{A}$.

6.3 Renormalization group

In this section we discuss in detail the ambiguity arising in defining \mathcal{T}_n 's. In physics this is known as the *renormalization ambiguity* and is related to the notion of the *renormalization group*. First note that the ambiguities \mathcal{Z}_n^H appearing in formula (6.15) give rise to maps $\mathcal{Z}_n : \mathfrak{A}_{\text{loc}}^{\otimes n} \rightarrow \mathfrak{A}_{\text{loc}}$ defined by

$$\mathcal{Z}_n \doteq \alpha_H^{-1} \circ \mathcal{Z}_n^H \circ \alpha_H^{\otimes n}.$$

Next we define a map $\mathcal{Z} : \mathfrak{A}_{\text{loc}} \rightarrow \mathfrak{A}_{\text{loc}}$ by summing up all the \mathcal{Z}_n 's relating two chosen prescriptions to define the time-ordered products, i.e.

$$\mathcal{Z}(V) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{Z}_n(V^{\otimes n}), \quad V \in \mathfrak{A}_{\text{loc}}.$$

For any two choices of the families $(\mathcal{T}_n)_{n \in \mathbb{N}}$, the corresponding map \mathcal{Z} , which relates them, has the following properties:

(Z 1) $\mathcal{Z}(0) = 0$,

(Z 2) $\mathcal{Z}^{(1)}(0) = \text{id}$,

(Z 3) $\mathcal{Z} = \text{id} + \mathcal{O}(\hbar)$,

(Z 4) $\mathcal{Z}(F + G + H) = \mathcal{Z}(F + G) + \mathcal{Z}(G + H) - \mathcal{Z}(G)$, if $\text{supp } F \cap \text{supp } G$,

(Z 5) $\frac{\delta \mathcal{Z}}{\delta \varphi} = 0$.

The group of formal diffeomorphisms of $\mathfrak{A}_{\text{loc}}[[\hbar]]$ that fulfill **(Z 1) – (Z 5)** is called the *Stückelberg-Petermann renormalization group* \mathcal{R} . The relation between the formal S-matrices and the elements of \mathcal{R} is provided by the *main theorem of renormalization*. This theorem, originally formulated in an unpublished preprint by Stora and Popineau [PS82], was later generalized and improved, in particular by Pinter [Pin01]. Its final version, which relies heavily on a proof of Stora's ‘‘Action Ward Identity’’ [DF04, DF07], was obtained in [HW02b, DF04] and was then further analyzed in [BDF09].

Theorem 6.1. *Let \mathcal{S} and $\hat{\mathcal{S}}$ be two formal S-matrices, built from time ordered products satisfying the axioms **(T 1)– (T 7)**. Then there exists $\mathcal{Z} \in \mathcal{R}$ such that*

$$\hat{\mathcal{S}} = \mathcal{S} \circ \mathcal{Z}, \tag{6.18}$$

where $\mathcal{Z} \in \mathcal{R}$. Conversely, if \mathcal{S} is an S-matrix satisfying the axioms **(T 1) – (T 7)** and $\mathcal{Z} \in \mathcal{R}$ then also $\hat{\mathcal{S}}$ defined by (6.18) fulfills the axioms.

Proof. For proof see [DF04, BDF09]. □

The abstract notion of the renormalization group can be made more concrete when we consider a specific model. This is done by fixing a natural Lagrangian (see definition 4.6) $L_I \in \text{Nat}(\mathfrak{D}, \mathfrak{F}_{\text{loc}})$, which defines the interaction term of the theory. Since we defined the Wick-ordering in a covariant way (in the sense of definition ??), the normally-ordered quantity $:L_I:$ is a natural transformation $\in \text{Nat}(\mathfrak{D}, \mathfrak{A}_{\text{loc}})$. For a fixed spacetime \mathcal{M} the map $:L_I:_{\mathcal{M}} : \mathfrak{D} \rightarrow \mathfrak{A}_{\text{loc}}$ satisfies **additivity**, the **support property** and **covariance** from definition 4.1. This motivates the following definition.

Definition 6.9. A quantum Lagrangian on a fixed spacetime $\mathcal{M} = (M, g)$ is a map $L : \mathcal{D}(M) \rightarrow \mathcal{F}_{\text{loc}}$ such that

i) $L(f + g + h) = L(f + g) - L(g) + L(g + h)$ for $f, g, h \in \mathcal{D}$ with $\text{supp } f \cap \text{supp } h = \emptyset$ (**Additivity**).

ii) $\text{supp}(L(f)) \subseteq \text{supp}(f)$ (**Support**).

iii) Let \mathcal{G} be the isometry group of the spacetime \mathcal{M} (for the Minkowski spacetime we set \mathcal{G} to be the proper orthochronous Poincaré group \mathcal{P}_+^\uparrow). We require that $L(f)(\alpha^*\varphi) = L(\alpha_*f)(\varphi)$ for every $\alpha \in \mathcal{G}$ (**Covariance**).

In a given pQFT model we are interested in computing expectation values of such quantum Lagrangians L_i , $i = 1, \dots, N$ in a given state ω . The following result is crucial for our framework.

Theorem 6.2 (Proposition 6.2. of [BDF09]). *The space \mathcal{L} of quantum Lagrangians is invariant under the renormalization group \mathcal{R} .*

Proof. For the details of the proof see [BDF09]. Note that the additivity property for the transformed Lagrangians is a direct consequence of the property (**Z 4**) of the renormalization group. \square

It is convenient to choose the state ω as a quasi-free state of the form $\omega_{H,0}$. On the Minkowski spacetime there is a distinguished choice of H as Δ_1 , so that $\Delta_{S_0}^+ = \frac{i}{2}\Delta_{S_0} + \Delta_1$ is the Wightman 2-point function, as in example 5.2. In this case $\omega_{\Delta_1,0}$ is just the usual *vacuum expectation value* that can be compared with the standard physics literature.

We are now ready to give an abstract definition of a *theory*.

Definition 6.10. A theory in pQFT on a spacetime \mathcal{M} is defined by fixing the space \mathcal{E} of field configurations and a complex vector space Φ of quantum Lagrangians, which is closed under the action of the renormalization group \mathcal{R} . A theory defined by (\mathcal{E}, Φ) is called **power-counting renormalizable** if Φ is finite-dimensional (as a vector space).

6.4 Interacting local nets

In this section we show how to construct a perturbative model of the interacting quantum theory, in the sense of definition 2.39. We follow closely the construction proposed in [FR15]. We start with the vector space Φ of quantum Lagrangians of a power-counting renormalizable theory. Let N be

the vector space dimension of Φ . We define \mathcal{D}^N to be the space of compactly supported functions on M with values in \mathbb{R}^N . We choose a basis in Φ consisting of quantum Lagrangians L_1, \dots, L_N in such a way that $L_0 = L_I$ is the interaction term of the theory. The remaining quantum Lagrangians correspond to other quantities we wish to model (e.g. conserved currents) and to terms that appear after applying maps $Z \in \mathcal{R}$ on the interaction term and these quantities.

For a fixed $f = (f^1, \dots, f^N) \in \mathcal{D}^N$ we construct the S-matrices corresponding to $L_1(f^1), \dots, L_N(f^N)$ and their linear combinations, using the Epstein-Glaser renormalization. This allows to define a map from \mathcal{D}^N to \mathfrak{A} by means of

$$f \mapsto \mathcal{S}(f) \doteq \mathcal{S} \left(\sum_{i=1}^N L_i(f^i) \right) \quad (6.19)$$

This way we obtain a family of unitaries $\{\mathcal{S}(f) | f \in \mathcal{D}^N\}$ with $\mathcal{S}(0) = 0$, which generate a *-subalgebra \mathfrak{A}_Φ of \mathfrak{A} and satisfy for $f, g, h \in \mathcal{D}$ Bogoliubov's factorization relation

$$S(f + g + h) = S(f + g)S(g)^{-1}S(g + h) \quad (6.20)$$

if the past J_- of $\text{supp } h$ does not intersect $\text{supp } f$ (or, equivalently, if the future J_+ of $\text{supp } f$ does not intersect $\text{supp } h$).

Proposition 6.1. *Assigning to bounded simply-connected regions $\mathcal{O} \subset \mathcal{M}$ subalgebras $\mathfrak{A}_\Phi(\mathcal{O}) \subset \mathfrak{A}_\Phi$ generated by $\{\mathcal{S}(f) | \text{supp } f \subset \mathcal{O}\}$ defines a **Local, Covariant** net in the sense of definition 2.39.*

Proof. The isotony condition, needed for the assignment $\mathcal{O} \mapsto \mathfrak{A}_\Phi(\mathcal{O})$ to be a net, is straightforward to verify. All the structures used in constructing this net are covariant, so $\mathcal{O} \mapsto \mathfrak{A}_\Phi(\mathcal{O})$ is covariant in the sense of the definition 2.39. The **Locality** axiom follows from the fact that for functions f, g with spacelike separated supports

$$\text{supp } f \cap J_\pm(\text{supp } g) = \emptyset \quad (6.21)$$

and hence

$$S(f)S(g) = S(f + g) = S(g)S(f) . \quad (6.22)$$

□

The net $\{\mathfrak{A}_\Phi(\mathcal{O})\}$ is not yet the interacting net we are looking for. However, it turns out that there is another way to assign subalgebras of \mathfrak{A}_Φ to

bounded regions of spacetime, which takes the interaction into account. Let us come back to the heuristic Bogoliubov formula (6.2) and its mathematically rigorous version (6.12). Note that the intertwining operator R_V , which relates the free and interacting theory is written as the derivative of the *relative S-matrix* $\mathfrak{S}(V)^{\star^{-1}} \star \mathfrak{S}(V + \lambda F)$. Here $V \equiv L_0(g^0) = L_I(g^0)$ for a test function $g^0 \in \mathcal{D}$ and we choose $F = L_i(f^i)$, for some $i \in \{1, \dots, N\}$ and $f^i \in \mathcal{D}$, so we can express the relative S-matrix using the elements of \mathfrak{A}_Φ and we obtain a map

$$f \mapsto \mathfrak{S}_g(f) \doteq \mathfrak{S}(g)^{-1} \mathfrak{S}(g + f) \quad (6.23)$$

where $f, g \in \mathcal{D}^N$ and $g = (g^0, 0, \dots, 0)$. $\mathfrak{S}(g)$ as well as $\mathfrak{S}(f + g)$ are defined by means of (6.19). We can now prove a crucial result concerning the relative S-matrices.

Proposition 6.2 (after [FR15]). *All the maps \mathfrak{S}_g labelled by different choices of g (or more precisely g^0), satisfy Bogoliubov's factorisation relation (6.20).*

Proof. We follow the proof given in [FR15]. Let $f, h \in \mathcal{D}^N$ such that $\text{supp } f$ does not intersect $J_-(\text{supp } h)$. Let $g, g' \in \mathcal{D}^N$. Then

$$\begin{aligned} \mathfrak{S}_g(f + g' + h) &= S(g)^{-1} S(f + (g + g') + h) \\ &= S(g)^{-1} S(f + (g + g')) S(g + g')^{-1} S((g + g') + h) \\ &= \mathfrak{S}_g(f + g') \mathfrak{S}_g(g')^{-1} \underbrace{S(g)^{-1} S(g)}_{=1} \mathfrak{S}_g(g' + h). \quad \square \end{aligned}$$

□

Remark 6.2. Note that we have constructed things in such a way that $S(g + \lambda f) = \mathfrak{S}(L_I(g^0 + \lambda f^0) + \sum_{i=1}^N L_i(\lambda f^i))$, so the derivative of $S_g(\lambda f)$ with respect to λ is just the retarded field

$$R_{L_I(g^0)} \left(\frac{d}{d\lambda} L_I(g^0 + \lambda f^0) \Big|_{\lambda=0} + \sum_{i=1}^N \frac{d}{d\lambda} L_i(\lambda f^i) \Big|_{\lambda=0} \right),$$

and if all the quantum Lagrangians L_i are linear in the test function, this reduces to

$$R_{L_I(g^0)} \left(\sum_{i=0}^N L_i(\lambda f^i) \right).$$

We interpret $\mathfrak{S}_g(f)$ as the generating function for retarded observable described by $\sum_{i=1}^N L_i(f^i)$, under the influence of the interaction $L_I(g^0)$. We are now ready to define the interacting net of observables.

Definition 6.11. *The interacting quantum net $\mathfrak{A}_{\Phi,g}$ corresponding to the theory Φ is defined by assigning to bounded simply-connected regions \mathcal{O} the local algebras $\mathfrak{A}_{\Phi,g}(\mathcal{O})$ which are generated by the relative S -matrices $S_g(f)$ with $\text{supp } f \subset \mathcal{O}$.*

Note that the algebras $\mathfrak{A}_{\Phi,g}$ are subalgebras of \mathfrak{A}_{Φ} but the net defined by 6.11 differs from the net 5.3. This is a very subtle point, since one could suspect that building interacting fields out of free fields should imply that the resulting theories are identical. Note, however, that in the algebraic approach physical information is contained not only in the global algebra, but also in the net structure, i.e. in the way we assign the elements of this algebra to bounded regions of spacetime. The next proposition shows that elements of $\mathfrak{A}_{\Phi,g}(\mathcal{O})$ can be interpreted as retarded observables,

Proposition 6.3 (after [FR15]). *The relative S -matrix has the following properties:*

1. $S_g(f)$ depends only on the behavior of g in the past of $\text{supp } f$.
2. $S_g(f)$ depends on the behavior of g outside of the future of $\text{supp } f$ via a (formal) unitary transformation which does not depend on f .

Proof. i) Let g and g' be such that $\text{supp}(g - g') \cap J_-(\text{supp } f) = \emptyset$, i.e. they differ in the regions outside the past of $\text{supp } f$. This implies

$$\begin{aligned} S_g(f) &= S(g)^{-1}S((g - g') + g' + f) = \\ &= S(g)^{-1}S((g - g') + g')S(g')^{-1}S(g' + f) = S_{g'}(f) . \end{aligned}$$

ii) For $\text{supp}(g - g') \cap J_+(\text{supp } f) = \emptyset$ we have

$$\begin{aligned} S_g(f) &= S(g)^{-1}S(f + g' + (g - g')) = \\ &= S(g)^{-1}S(f + g')S(g')^{-1}S(g' + (g - g')) = \\ &= S(g)^{-1}S(g')S(g')^{-1}S(f + g')S_{g'}(g - g') = \\ &= \text{Ad}S_{g'}(g - g')^{-1}(S_{g'}(f)) , \end{aligned}$$

where we use the notation $\text{Ad}B(A) \doteq B^{-1}AB$ for $A, B \in \mathfrak{A}_{\Phi}$. □

The above proposition shows that the structure of local algebras depends only locally on the interaction. This allows to perform the adiabatic limit directly on the level of local algebras as a certain inductive limit. In the first step we remove the restriction to interactions with compact support. Let

$G \in \mathcal{C}^\infty(M, \mathbb{R}^N)$ and let \mathcal{O} be a bounded simply connected region in M . We define

$$[G]_{\mathcal{O}} = \{g \in \mathcal{D}^N \mid g \equiv G \text{ on a neighborhood of } J_+(\mathcal{O}) \cap J_-(\mathcal{O})\} .$$

Next we consider the \mathfrak{A}_{Φ} -valued maps

$$\mathfrak{S}_{G,\mathcal{O}}(f) : [G]_{\mathcal{O}} \ni g \mapsto \mathfrak{S}_g(f) \in \mathfrak{A}_{\Phi} .$$

We are now ready to extend the definition 6.11 to functions with arbitrary support.

Definition 6.12. *The local net of algebras $\mathcal{O} \mapsto \mathfrak{A}_{\Phi,G}(\mathcal{O})$ is defined by assigning to \mathcal{O} the algebra generated by $\mathfrak{S}_{G,\mathcal{O}}(f)$, where $\text{supp } f \subset \mathcal{O}$. For $\mathcal{O}_1 \subset \mathcal{O}_2$ the embedding $i_{\mathcal{O}_2\mathcal{O}_1}$ of corresponding algebras is defined by*

$$i_{\mathcal{O}_2\mathcal{O}_1} : \mathfrak{S}_{G,\mathcal{O}_1}(f) \mapsto \mathfrak{S}_{G,\mathcal{O}_2}(f)$$

for $f \in \mathcal{D}^N$ with $\text{supp } f \subset \mathcal{O}_1$.

Let \mathfrak{A}_G be the inductive limit of algebras $\mathfrak{A}_{\Phi,G}(\mathcal{O})$ constituting the net defined in 6.12. Let us denote the corresponding canonical embedding into \mathfrak{A}_G by

$$i_{\mathcal{O}} : \mathfrak{A}_G(\mathcal{O}) \rightarrow \mathfrak{A}_G$$

We set

$$\mathfrak{S}_G(f) = i_{\mathcal{O}}(\mathfrak{S}_{G,\mathcal{O}}(f)) .$$

The following result shows that the net defined in 6.12 is covariant in the sense of definition 2.40.

Theorem 6.3. *Let*

$$\alpha_{\beta}^{G,\mathcal{O}}(\mathfrak{S}_G(f)) \doteq \mathfrak{S}_G(\beta^* f) , \tag{6.24}$$

where $\mathfrak{S}_G(f) \in \mathfrak{A}_{\Phi,G}(\mathcal{O})$, $\beta \in \mathcal{G}$ is an element of the isometry group of \mathcal{M} and $\beta^* f \doteq f \circ \beta$. Then $\alpha_{\beta}^{G,\mathcal{O}}$, $\beta \in \mathcal{G}$ extends to a family of isomorphisms $\alpha_{\beta}^{G,\mathcal{O}} : \mathfrak{A}_{\Phi,G}(\mathcal{O}) \rightarrow \mathfrak{A}_{\Phi,G}(\beta\mathcal{O})$ satisfying the conditions of definition 2.40.

Before we prove the theorem we need a lemma that allows to reduce the problem to the problem that involves the net $\{\mathfrak{A}_{\Phi,g}(\mathcal{O})\}$.

Lemma 6.1. *The evaluation maps*

$$\gamma_{g,G} : \mathfrak{S}_{G,\mathcal{O}}(f) \rightarrow \mathfrak{S}_g(f)$$

extend to isomorphisms of $\mathfrak{A}_{\Phi,G}(\mathcal{O})$ and $\mathfrak{A}_{\Phi,g}(\mathcal{O})$ for every $g \in [G]_{\mathcal{O}}$.

Proof. The only thing that we need to check is that the map $\gamma_{g,G}$ is injective. Assume that $\gamma_{g,G}(\mathcal{S}_{G,\mathcal{O}}(f)) = \gamma_{g,G}(\mathcal{S}_{G,\mathcal{O}}(f'))$ for some f, f' with supports contained in \mathcal{O} . Then $\mathcal{S}_{g,\mathcal{O}}(f) = \mathcal{S}_{G,\mathcal{O}}(f')$ and using the proposition 6.3 we conclude that $\mathcal{S}_{g,\mathcal{O}}(f) = \mathcal{S}_{G,\mathcal{O}}(f')$ for every $g' \in [G]_{\mathcal{O}}$, so $\mathcal{S}_{G,\mathcal{O}}(f) = \mathcal{S}_{G,\mathcal{O}}(f')$. \square

Proof of the theorem. We have to prove that the map $\alpha_{\beta}^{G,\mathcal{O}}$ given by (6.24) extends to an isomorphism $\mathfrak{A}_{\Phi,G}(\mathcal{O}) \rightarrow \mathfrak{A}_{\Phi,G}(\beta\mathcal{O})$. Let $\mathcal{O}_1 \supset \mathcal{O} \cup \beta\mathcal{O}$ and $g \in [G]_{\mathcal{O}_1}$. Then $g, \beta^*g \in [G]_{\mathcal{O}}$ and $\beta^*g = g + h_+^{\beta} + h_-^{\beta}$ with $\text{supp } h_{\pm}^{\beta} \cap J_{\mp}(\mathcal{O}) = \emptyset$. By the causal factorization property (6.20) we obtain

$$\alpha_{\beta}^{G\mathcal{O}} = \gamma_{g,G}^{-1} \circ \text{Ad}U_g(\beta) \circ \alpha_{\beta}^{\mathcal{O}} \circ \gamma_{g,G}$$

where $U_g(\beta) = \mathcal{S}_g(h_-^{\beta})$ and

$$\alpha_{\beta}^{\mathcal{O}}(\mathcal{S}(f)) = \mathcal{S}(\beta^*f).$$

It is now apparent that (6.24) extends to an isomorphism. From the definition (6.24) it is also clear that if $\mathcal{O}_1 \subset \mathcal{O}_2$, then the restriction of $\alpha_{\beta}^{G\mathcal{O}_2}$ to $\mathfrak{A}_{\Phi,G}(\mathcal{O}_1)$ coincides with $\alpha_{\beta}^{G\mathcal{O}_1}$ and for any $\beta, \beta' \in \mathfrak{G}$, we have $\alpha_{\beta'}^{G\mathcal{O}} \circ \alpha_{\beta}^{G\mathcal{O}} = \alpha_{\beta' \circ \beta}^{G\mathcal{O}}$. \square

6.5 Construction of time-ordered products

In this section we review the abstract Epstein-Glaser construction of time-ordered products on Minkowski spacetime following [EG73, BF00, HW02a]. We will also show how to obtain more explicit formulas using regularization prescriptions and combinatorics involving Feynman diagrams. First we need to introduce some notation. Time-ordered products \mathcal{T}_n^H are maps from $\mathcal{F}_{\text{loc}}^{\otimes n}$ to $\mathcal{F}_{\mu\nu}[[\hbar]]$ and, as indicated in the previous section, they are obtained by extending non-renormalized expressions that are originally defined only on $(\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes n}$. Let us consider $F \equiv F_1 \otimes \cdots \otimes F_n \in (\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes n}$. Note that F induces a map from \mathcal{E}^n to \mathbb{C} by $F(\varphi_1, \dots, \varphi_n) = F_1(\varphi_1) \cdots F_n(\varphi_n)$. When we talk about functionals on \mathcal{E} we will denote the variable by φ and for functionals on \mathcal{E}^n , the variable is an n -tuple $(\varphi_1, \dots, \varphi_n)$. For a multiindex $\beta \in \mathbb{N}_0^n$ we denote

$$\delta^{\beta} F(\varphi_1, \dots, \varphi_n) \doteq \frac{\delta^{|\beta|} F}{\delta \varphi_1^{\beta_1} \cdots \delta \varphi_n^{\beta_n}}(\varphi_1, \dots, \varphi_n),$$

where F is a smooth functional on \mathcal{E}^n and $(\varphi_1, \dots, \varphi_n) \in \mathcal{E}^n$. Clearly $\delta^{\beta} F(\varphi_1, \dots, \varphi_n) \in L(\mathcal{E}^{|\beta|}; \mathbb{C})$ and by proposition 3.1 it induces an element of $\Gamma'(\mathcal{E}^{\boxtimes |\beta|} \rightarrow M^{|\beta|})$.

Definition 6.13. We say that a smooth functional F on \mathcal{E}^n is microcausal if it is compactly supported and satisfies

$$\text{WF} \left(\delta^\beta F(\varphi_1, \dots, \varphi_n) \right) \subset \Xi_{|\beta|}, \quad \forall \beta \in \mathbb{N}_0^n, \quad \forall (\varphi_1, \dots, \varphi_n) \in \mathcal{E}^n, \quad \forall n \in \mathbb{N}. \quad (6.25)$$

We denote the space of such functionals by $\mathcal{F}_{\mu\text{c}}^n$.

6.5.1 Existence of time-ordered products

We have seen in section 6.2.3 that the renormalization problem amounts to extension of maps \mathcal{T}_n^H to local functionals with arbitrary supports. In fact, we will see that it reduces to extending numerical distributions defined everywhere outside certain subdiagonals in M^n (see [BF00]). The construction proceeds recursively and, having constructed the time-ordered products of order $k < n$, at order n one is left with the problem of extending a distribution defined everywhere outside the thin diagonal of M^n . One way of constructing explicitly such distributional extensions relies on the splitting method (see for example [Sch95]). Here we take a different approach, based on the notion of Steinmann's scaling degree [Ste71].

Definition 6.14. Let $U \subset \mathbb{R}^n$ be a scale invariant open subset (i.e. $\lambda U = U$ for $\lambda > 0$), and let $t \in \mathcal{D}'(U)$ be a distribution on U . Let $t_\lambda(x) = t(\lambda x)$ be the scaled distribution. The scaling degree sd of t is

$$\text{sd } t = \inf \{ \delta \in \mathbb{R} \mid \lim_{\lambda \rightarrow 0} \lambda^\delta t_\lambda = 0 \}. \quad (6.26)$$

The degree of divergence, another important concept used often in the literature, is defined as:

$$\text{div}(t) \doteq \text{sd}(t) - n.$$

The crucial result which allows us to construct time-ordered products is stated in the following theorem:

Theorem 6.4. Let $t \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ with scaling degree $\text{sd } t < \infty$. Then there exists an extension of t to an everywhere defined distribution with the same scaling degree. The extension is unique up to the addition of a derivative $P(\partial)\delta$ of the delta function, where P is a polynomial with degree bounded by $\text{div}(t)$ (hence vanishes for $\text{sd } t < n$).

In order to apply theorem (6.4), we need to be able to reduce the problem of constructing the time-ordered products to the problem of extension of numerical distributions defined everywhere outside the thin diagonal. In

this section we work on the level of quantum Lagrangian in the sense of definition 6.9. A construction performed on the level of functionals will be presented in section 6.5.2.

In the first step we restrict ourselves to Lagrangians, which are linear maps on the space of test functions. The general case is obtained by re-defining the test functions. For example if $L(f) = \int_M f^2(x)\mathcal{L}(x)d\mu(x)$, then we obtain a Lagrangian linear in test function by setting $L'(f') \doteq \int_M f'\mathcal{L}(x)d\mu(x)$ and the original Lagrangian is $L'(f^2)$.

An n -fold time-ordered product of Lagrangians L_1, \dots, L_n induces a linear map

$$f_1 \otimes \dots \otimes f_n \mapsto \mathcal{T}_n^H(L_1(f_1), \dots, L_n(f_n)), \quad (6.27)$$

We denote $\mathbf{L} \doteq (L_1, \dots, L_n)$ and the map (6.27) by $\mathcal{T}_n^H(L_1, \dots, L_n)$ or by $\mathcal{T}_{n,\mathbf{L}}^H$. This maps extends to $\mathcal{D}(M^n, \mathbb{R})$ and we denote this extension by the same symbol $\mathcal{T}_{n,\mathbf{L}}^H$.

Similarly, let $I = \{i_1, \dots, i_k\}$ be a subset of $\{1, \dots, n\}$ we introduce the notation $\mathcal{T}_{I,\mathbf{L}}^H$ for the map

$$f_1 \otimes \dots \otimes f_k \mapsto \mathcal{T}_k^H(L_{i_1}(f_1), \dots, L_{i_k}(f_k)).$$

Ultimately we need maps $\mathcal{T}_{n,\mathbf{L}}^H$ to construct the S-matrix and interacting local nets, as outlined in section 6.4. For this purpose, it is convenient to formulate the problem of constructing the time-ordered products as the problem of constructing maps $\mathcal{T}_{n,\mathbf{L}}^H$. A method formulated on the level of local functionals, rather than such maps will be presented in the next subsection.

Let us start with formulating the axioms that maps $\mathcal{T}_{n,\mathbf{L}}^H$ have to fulfill.

(TL 1) Causal factorisation property

$$\mathcal{T}_{n,\mathbf{L}}^H(f_1 \otimes \dots \otimes f_n) = \mathcal{T}_{I,\mathbf{L}}^H(f_1 \otimes \dots \otimes f_k) \star_H \mathcal{T}_{I^c,\mathbf{L}}^H(f_{k+1} \otimes \dots \otimes f_n)$$

where $I = \{1, \dots, k\}$ and the supports $\text{supp } f_i$, $i = 1, \dots, k$ of the first k arguments do not intersect the past of the supports $\text{supp } f_j$, $j = k + 1, \dots, n$ of the last $n - k$ entries.

(TL 2) Starting element: $\mathcal{T}_0^H = 1$, $\mathcal{T}_{1,L}^H(f) = L(f)$.

(TL 3) Symmetry: For a bosonic theory $\mathcal{T}_{I,\mathbf{L}}^H = \mathcal{T}_{I,\sigma(\mathbf{L})}^H$, where σ is a permutation of n elements and we define $\sigma(L_1, \dots, L_n) \doteq (L_{\sigma(1)}, \dots, L_{\sigma(n)})$. In the presence of fermions, each permutation of two elements introduces an extra minus sign.

(TL 4) Field independence.

(TL 5) φ -Locality.

(TL 6) Unitarity.

(TL 7) Covariance.

The construction of time-ordered products proceeds inductively. We assume that the maps $\mathcal{T}_{k,\mathbf{L}'}^H$, with $k < n$ are constructed for all possible k -tuples \mathbf{L}' of generalized Lagrangians of the theory. For a power-counting renormalizable theory this set is finite dimensional, see definition 6.10. Using the causal factorisation property (T 1) we fix the values of $\mathcal{T}_{n,\mathbf{L}}^H$ on test functions $f \in \mathcal{D}(M^n \setminus D_n)$. Maps constructed this way are called *non-renormalized time-ordered products* and we denote them by $\mathring{\mathcal{T}}_{n,\mathbf{L}}^H$.

Definition 6.15. Let I be a proper subset of $\{1, \dots, n\}$ and we denote by C_I the subset of M^n defined by

$$C_I = \{(x_1, \dots, x_n) | x_i \notin J^+(x_j) \text{ for all } i \in I, j \in I^c\},$$

where I^c is the complement of I .

Sets C_I are open and they constitute an open covering of $M^n \setminus D_n$.

Definition 6.16. Let I be a proper subset of $\{1, \dots, n\}$. We define a functional-valued linear map $\mathcal{T}_{I|I^c,\mathbf{L}}^H$ on $\mathcal{D}(M^n \setminus D_n)$ by fixing its values on test functions that arise as products

$$\mathcal{T}_{I|I^c,\mathbf{L}}^H(f_1 \otimes \dots \otimes f_n) \doteq \mathcal{T}_{I,\mathbf{L}}^H(f_I) \star \mathcal{T}_{I^c,\mathbf{L}}^H(f_{I^c}),$$

where for $I = \{i_1, \dots, i_k\}$ we define $f_I \doteq f_{i_1} \otimes \dots \otimes f_{i_k}$.

Let $\{g_I\}$ be a partition of unity subordinate to the open covering defined in 6.15. We define the non-renormalized time-ordered products by

$$\mathring{\mathcal{T}}_{n,\mathbf{L}}^H(f) \doteq \sum_{\substack{I \subseteq \{1, \dots, n\} \\ I \neq \emptyset}} \mathcal{T}_{I|I^c,\mathbf{L}}^H(g_I f),$$

where $f \in \mathcal{D}(M^n \setminus D_n)$. The following proposition has been proven in [HW01].

Proposition 6.4. Assuming that time ordered products $\mathring{\mathcal{T}}_{k,\mathbf{L}'}^H$, with less or equal than $n - 1$ arguments have been defined in such a way that they satisfy properties (TL 1) – (TL 7) as maps on $\mathcal{D}^{\otimes n}$. Then the maps $\mathring{\mathcal{T}}_{n,\mathbf{L}}^H$ automatically satisfy the restrictions of properties (TL 1) – (TL 7) to $\mathcal{D}(M^n \setminus D_n)$.

Proof. See the proof in [HW01]. \square

In the next step we reduce the problem of defining the time-ordered products $\mathcal{T}_{n,\mathbf{L}}^H$ to the problem involving only \mathbb{C} -valued distributions. For this we use the properties **(TL 4)** and **(TL 5)** and write the n -fold time-ordered product $\mathcal{T}_{n,\mathbf{L}}^H$ in terms of its Taylor expansion around 0.

$$\begin{aligned} \mathcal{T}_{n,\mathbf{L}}^H(f_1 \otimes \cdots \otimes f_n)[\varphi] &= \\ &= \mathcal{T}_{n,\mathbf{L}}^H(f_1 \otimes \cdots \otimes f_n)[0] + \\ &+ \sum_{|\beta| > 0} \left\langle \mathcal{T}_n^H(\delta^{\beta_1} L_1(f_1), \dots, \delta^{\beta_n} L_n(f_n))[0], \varphi^{\otimes \beta_1} \otimes \cdots \otimes \varphi^{\otimes \beta_n} \right\rangle, \end{aligned}$$

where $\beta \in \mathbb{N}_0^n$. In physics literature this is called the *Wick expansion*. Note that this expansion converges as a power series in \hbar , because due to **(TL 5)**, in each order in \hbar the sum contains only finitely many terms. Therefore it is sufficient to know the time-ordered products of polynomials.

Among all the Lagrangians linear in test functions we distinguish the monomial ones,

$$L(f)[\varphi] = \int_M \varphi^k(x) f(x) d\mu(x) \equiv \Phi^k(f)[\varphi], \quad (6.28)$$

We will now construct time ordered products of Lagrangians that are of this form. The advantage of working with such objects is the fact that computing functional derivatives reduces to simple algebraic manipulation.

$$\begin{aligned} \left\langle \frac{\delta^l}{\delta \varphi^l} \Phi^k(f)[\varphi_0], \psi^{\otimes l} \right\rangle &= \frac{k!}{(k-l)!} \left\langle \Phi^{k-l}(\cdot)[\varphi_0], f \psi^l \right\rangle \\ &\equiv \frac{k!}{(k-l)!} \left\langle \Phi^{k-l}[\varphi_0], f \psi^l \right\rangle \end{aligned}$$

Hence the Wick expansion takes the form

$$\begin{aligned} \mathcal{T}_n^H(\Phi^{k_1}(f_1), \dots, \Phi^{k_n}(f_n))(\varphi) &= \\ = \sum_{l_1=0}^{k_1} \cdots \sum_{l_n=0}^{k_n} \binom{k_1}{l_1} \cdots \binom{k_n}{l_n} \left\langle t_n^H(\Phi^{k_1-l_1}, \dots, \Phi^{k_n-l_n}), f_1 \varphi^{l_1} \otimes \cdots \otimes f_n \varphi^{l_n} \right\rangle, \end{aligned} \quad (6.29)$$

We chose the convention where test functions are put together with configurations on the right-hand side of the pairing, and the remaining numerical distributions are denoted by small t 's.

Remark 6.3 (Hopf algebraic interpretation). The combinatorics appearing in our formulas can be significantly simplified in the Hopf algebra language [Bro06, Dan13, BFK06]. Let \mathcal{H} denote the algebra generated by the Lagrangians Φ^k , where the product is defined by

$$\Phi^k \Phi^l \doteq \Phi^{k+l}.$$

Note that this is *not* the pointwise product we have introduced before on the space of functionals. We identify Φ^0 with the unit $\mathbf{1}$, i.e. a Lagrangian of the form $\Phi(f)[\varphi] = \int_M f d\mu$. We define the co-product $\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ by

$$\Delta(\Phi^k) = \sum_{l=0}^k \binom{k}{l} \Phi^{k-l} \otimes \Phi^l,$$

and the counit

$$\varepsilon(\Phi^k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{else.} \end{cases}$$

Using Swindler notation we can now write

$$\left\langle \frac{\delta^l}{\delta \varphi^l} \Phi^k(f)[\varphi_0], \psi^{\otimes l} \right\rangle = \left\langle \Phi_{(1)}^k[\varphi_0], f \Phi_{(2)}^k[\psi] \right\rangle.$$

where $\Delta(\Phi^k) \equiv \sum \Phi_{(1)}^k \otimes \Phi_{(2)}^k$. Hence

$$\begin{aligned} \mathcal{T}_n^H(\Phi^{k_1}(f_1), \dots, \Phi^{k_n}(f_n))(\varphi) &= \\ &= \sum \left\langle t_n^H(\Phi_{(1)}^{k_1}, \dots, \Phi_{(1)}^{k_n}), f_1 \Phi_{(2)}^{k_1}(\varphi) \otimes \dots \otimes f_n \Phi_{(2)}^{k_n}(\varphi) \right\rangle, \end{aligned} \quad (6.30)$$

On the left-hand side of the pairing we have a distribution on M^n and on the left-hand side a compactly supported function on M^n , so the expression above is well defined.

We have reduced the extension problem to a problem that involves only numerical distributions $t_n^H(\Phi^{k_1}, \dots, \Phi^{k_n})$. Combining with the earlier construction of maps $\mathring{\mathcal{J}}_n^H$, we can conclude that the renormalization problem reduces to extending numerical distributions $\mathring{t}_n^H(\Phi^{k_1}, \dots, \Phi^{k_n}) \in \mathcal{D}(M^n \setminus D_n)$. Using the axioms (TL 1)-(TL 7) we can formulate the corresponding axioms for \mathring{t}_n^H 's (see [HW01]) for details. We label these axioms by (t 1)-(t 7)

It is actually helpful to impose some additional, more technical requirements, which control the regularity of distributions t_n^H . These are:

(t 8) Almost homogeneous scaling,

(t 9) **Micolocal spectrum condition,**

(t 10) **Smoothness,**

(t 11) **Analyticity.**

The final two axioms concern the dependence on the background metric. The precise statement of these axioms can be found in section 3.3 of [HW01]. The existence of the distributional extensions with desired properties is guaranteed by the following result.

Theorem 6.5 (following [HW01]). *There exist distribution $t_n^H(\Phi^{k_1}, \dots, \Phi^{k_n})$ satisfying the axioms (t 1)-(t 11).*

Proof. See [BF00, HW01]. A more robust and mathematically cleaner method for constructing such distributional extension has been provided in [Dan13]. \square

Let us now discuss Lagrangians containing derivatives. We generalize the class of objects we consider to Lagrangians of the form

$$\Phi_a^{\mathbf{m}}(f)[\varphi] \doteq \int_M \langle a, (\nabla)^{m_1} \varphi \otimes \dots \otimes (\nabla)^{m_k} \varphi \rangle f d\mu,$$

where $f \in \mathcal{D}$, $\mathbf{m} = (m_1, \dots, m_k)$ is a multiindex, $(\nabla)^{m_1} \varphi$ denotes the rank m_1 covariant tensor obtained by applying m_1 covariant derivatives to φ , a is a rank $|\mathbf{m}|$ contravariant tensor and $\langle \cdot, \cdot \rangle$ denotes the pairing between covariant and contravariant tensors of matching rank. We can also generalize Lagrangians to maps on arbitrary tensor fields, i.e. $L : \text{Tens} \rightarrow \mathcal{F}_{\text{loc}}$, where $\text{Tens} \doteq \bigoplus_{p,q} \Gamma(T_q^p M)$. We introduce the notation

$$\Phi_{a,\mathbf{c}}^{\mathbf{m}}(f)[\varphi] \doteq \int_M \langle a, f \otimes (\nabla)^{m_1} \varphi \otimes \dots \otimes (\nabla)^{m_k} \varphi \rangle_{\mathbf{c}} d\mu, \quad (6.31)$$

where f is an arbitrary rank (q, r) tensor field², the rank of a is $\text{rk}(a) = (r, q + |\mathbf{m}|)$, $\mathbf{c} = (c_1, \dots, c_{|\mathbf{m}|})$ is an ordered $(|\mathbf{m}| + q)$ -tuple of numbers that indicate in which slots contractions of tensor indices have to be performed and $\langle \cdot, \cdot \rangle_{\mathbf{c}}$ is the contraction itself. This way, after applying $\langle \cdot, \cdot \rangle_{\mathbf{c}}$ we are left with a tensor of rank $\text{rk}(a) - (0, |\mathbf{m}|)$, which is subsequently contracted with f .

²We use convention where a tensor field of type $(q, 0)$ is a q -form and $(0, r)$ is an r -vector.

The l -th derivative of $\Phi_{a,c}^m$ can be expressed as

$$\left\langle \frac{\delta^l}{\delta\varphi^l} \Phi_{a,c}^m(f)[\varphi_0], \psi^{\otimes l} \right\rangle = \sum_I \left\langle \Phi_{a,c_I}^{m/I}[\varphi_0], f \otimes (\nabla)^{m_{i_1}} \psi \otimes \cdots \otimes (\nabla)^{m_{i_l}} \psi \right\rangle_{c_I}, \quad (6.32)$$

where $I = \{i_1, \dots, i_l\}$ is an ordered subset of $\{1, \dots, k\}$ and the notation $\Phi_a^{m/I}$ means that we “take out” the corresponding factors from the monomial in (6.32). The notation gets a bit complicated at this point, but we can still write an analog of (6.29) with the appropriate combinatorics of $\Phi_{a,c}^m$ ’s. This can be simplified with the use of the Hopf algebra language, since (6.32) induces again a co-product. This Hopf algebra, however is non-commutative, because a in (6.31) is not necessary symmetric and the operation of putting together such objects using the tensor product is non-commutative. The framework needed to handle non-commutative Hopf algebras that arise this way can be found for example in [BFK06]. The non-commutativity appears also if we want to generalize from scalar fields to field configurations which are sections of an arbitrary vector-bundle.

The general procedure can be summarized as follows. We fix a theory Φ and we choose a suitable family of Lagrangians $\{\Phi^\alpha\}_{\alpha \in A}$, where A is some suitable label set (for example in (6.28) $A = \mathbb{N}_0$). We allow for test functions f which are compactly supported sections of some vector bundle E_0 over M . The family $\{\Phi^\alpha\}_{\alpha \in A}$ has to be chosen in such a way that functional derivatives of Φ^α ’s can be written in terms of other elements of $\{\Phi^\alpha\}_{\alpha \in A}$, as it was done in (6.29). The concrete expression for the l -th functional derivative of Φ^α determines the choice of the product and the co-product in the set $\{\Phi^\alpha\}_{\alpha \in A}$ and we define \mathcal{H} to be the bi-algebra generated with respect to these structures. The crucial requirement is that

$$\left\langle \frac{\delta^l}{\delta\varphi^l} \Phi^\alpha(f)[\varphi_0], \psi^{\otimes l} \right\rangle = \sum \left\langle \Phi_{(1)}^\alpha[\varphi_0], f \otimes \Phi_{(2)}^\alpha[\psi] \right\rangle_{(1)(2)},$$

with a suitable choice of pairings $\langle \cdot, \cdot \rangle_{(1)(2)}$. The choice of the unit and the co-unit is the same as in remark 6.3 and we introduce a grading in such a way that the degree 0 component is one-dimensional, spanned by the unit $\mathbb{1}$. The latter guarantees the existence of the antipode and \mathcal{H} becomes a Hopf algebra. The Wick expansion takes the form

$$\begin{aligned} \mathcal{T}_n^H(\Phi^{\alpha_1}(f_1), \dots, \Phi^{\alpha_n}(f_n))(\varphi) &= \\ &= \sum \left\langle t_n^H \left(\Phi_{(1)}^{\alpha_1}, \dots, \Phi_{(1)}^{\alpha_n} \right), f_1 \Phi_{(2)}^{\alpha_1}(\varphi) \otimes \cdots \otimes f_n \Phi_{(2)}^{\alpha_n}(\varphi) \right\rangle, \quad (6.33) \end{aligned}$$

The problem is again reduced to construction of numerical-valued distributions t_n^H .

The potential non-uniqueness in representing functionals in terms of Φ^α 's implies some extra conditions on t 's. For example, if we consider just the scalar field in the presence of field derivatives in Lagrangians, the non-uniqueness arises due to the possibility of integration by parts. For consistency, distributions $t_n^H(\Phi_{q^1}^{m_1}, \dots, \Phi_{q^n}^{m_n})$ have to satisfy the condition:

(t 12) Action Ward Identity

$$k^i \circ t_n^H(\Phi_{q^1}^{m_1}, \dots, \Phi_{q^n}^{m_n}) = t_n^H(\Phi_{q^1}^{m_1}, \dots, k \circ \Phi_{q^i}^{m_i}, \dots, \Phi_{q^n}^{m_n}), \quad (6.34)$$

where $k^i = \sum_\mu k^\mu \nabla_\mu^i$, the notation ∇_μ^i indicates that the derivative is applied in the i -th slot and

$$k \circ \Phi_{a,c}^m(f)[\varphi] \doteq \sum_{j=1}^k \int_M \langle a, f \otimes (\nabla)^{m_1} \varphi \otimes k \circ (\nabla)^{m_j} \varphi \otimes \dots \otimes (\nabla)^{m_k} \varphi \rangle_c d\mu.$$

It was shown in [DF07] that this condition can be fulfilled also for the extensions of t 's to the diagonal. Before AWI has been proven, the construction of time-ordered products was known only on the level of polynomials $\Phi_{a,c}^m$, treated as formal expressions, not as functionals on the configuration space. It was then suggested by Stora [Sto02] to impose the additional requirement (6.34). The proof of this requirement achieved in [DF07] was a major breakthrough in mathematical pQFT since it opened the way for the functional formalism. In [HW05] this condition is called *the Leibniz rule*, and its proof is generalized to curved spacetimes.

Having constructed the numerical distributions $t_n^H(\Phi_{q^1}^{m_1}, \dots, \Phi_{q^n}^{m_n})$ satisfying conditions (t 1)–(t 12) we use the formula (6.33) to construct maps $T_n^H(\Phi^{\alpha_1}, \dots, \Phi^{\alpha^n})$ satisfying properties (TL 1)–(TL 7). Using linearity and appropriate redefinitions of test functions this allows to construct time-ordered products of arbitrary Lagrangians in Φ . Note that all the way throughout this section we have worked purely on the level of topological algebras of functionals, without the need to refer to a concrete Hilbert space representation. This functional viewpoint is very useful in QFT on curved spacetimes, where there is no unique vacuum state and hence no distinguished vacuum representation. We can push this idea even further and formulate the problem of construction the time ordered product as a problem of extending vector-valued distributions. We expect the following to hold true:

Conjecture 6.1. *Equip $\mathcal{F}_{\mu c}(\mathcal{M})$ with a topology that controls the WF set properties of functional derivatives. Then the time-ordered products $T_n^H(L_1, \dots, L_n)$ constructed in this section, in each order in \hbar , are $\mathcal{F}_{\mu c}(\mathcal{M})$ -valued distributions.*

6.5.2 Explicit construction and Feynman graphs

In the section (6.5.1) we have reviewed the existence proof for time ordered products, reformulating it in the language of the functional formalism. The drawback of the method presented there was that it relied on (potentially non-unique) parametrisation of local functionals in terms of Lagrangians and also, that the existence proof is rather abstract and doesn't provide an explicit expression for time-ordered products. There is a more direct method to construct \mathcal{T}_{nH} , which is formulated on the level of functionals, rather than the natural Lagrangians. For simplicity we will present it only in the context of Minkowski spacetime. The generalization to arbitrary globally hyperbolic spacetimes is not difficult and can be done using the methods of [BF00, Dan13].

In this section we work on the Minkowski spacetime, so we set $H = \Delta_1$ and $\Delta_{S_0}^+ = \frac{i}{2}\Delta_{S_0} + \Delta_1$ is the Wightman 2-point function. In this case $\Delta_{S_0}^F$ is the “standard” Feynman propagator.

Theorem 6.4 allows in principle to extend all the numerical distributions we need for the construction of time-ordered products. Although it is possible to argue on the level of \mathcal{T}_n^H s, here we use a perturbative expansion of time-ordered products in terms of Feynman graphs, to make the relation to other approaches to renormalization more apparent. Note, however, that in the pAQFT framework, Feynman graphs are not fundamental objects, but instead they are derived (together with the corresponding Feynman rules) from the definition of time-ordered products.

Let us denote $D_{ij} \doteq i\hbar \langle \Delta_{S_0}^F, \frac{\delta^2}{\delta\varphi_i \delta\varphi_j} \rangle$ and $D \doteq i\hbar \langle \Delta_{S_0}^F, \frac{\delta^2}{\delta\varphi^2} \rangle$. The Leibniz rule for differentiation can be formulated as

$$\frac{\delta}{\delta\varphi} \circ m_n = m_n \circ \left(\sum_{i=1}^n \frac{\delta}{\delta\varphi_i} \right), \quad (6.35)$$

where m_n is the pointwise multiplication of n arguments, or in other words, a pullback through the diagonal map $\mathcal{E} \rightarrow \mathcal{E}^n$, $\varphi \mapsto (\varphi, \dots, \varphi)$.

Proposition 6.5. *The unrenormalized n -fold time ordered product $\mathcal{T}_n^H : (\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes n} \rightarrow \mathcal{F}_{\mu c}[[\hbar]]$ can be expressed as*

$$\mathcal{T}_n^H(F_1, \dots, F_n) = m_n \circ e^{\sum_{i < j} D_{ij}}(F_1, \dots, F_n).$$

Proof. By definition we have

$$\mathcal{T}_n^H(F_1, \dots, F_n) = F_1 \cdot_{\mathcal{T}^H} \dots \cdot_{\mathcal{T}^H} F_n = e^{\frac{D}{2}} \circ m_n(e^{-\frac{1}{2}D_{11}} F_1, \dots, e^{-\frac{1}{2}D_{nn}} F_n).$$

The Leibniz rule implies that

$$e^{\frac{D}{2}} \circ m_n = m_n \circ e^{\sum_{i<j} D_{ij} + \sum_i \frac{1}{2} D_{ii}},$$

hence

$$\mathcal{T}_n^H(F_1, \dots, F_n) = m_n \circ e^{\sum_{i<j} D_{ij} + \sum_i \frac{1}{2} D_{ii}} \circ (e^{-\frac{1}{2}D_{11}} F_1, \dots, e^{-\frac{1}{2}D_{nn}} F_n).$$

It is now clear that all the factors $e^{-\frac{1}{2}D_{ii}}$ cancel, which proves the result. \square

Finally, let us denote

$$T_n \doteq e^{\sum_{i<j} D_{ij}},$$

so T_n is a map from $(\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes n}$ to $\mathcal{F}_{\mu c}^n$. The combinatorial identity

$$e^{\sum_{i<j} D_{ij}} = \prod_{i<j} \sum_{l_{ij}=0}^{\infty} \frac{D_{ij}^{l_{ij}}}{l_{ij}!} \quad (6.36)$$

allows us to express time ordered products in terms of graphs. Let \mathcal{G}_n be the set of all graphs with vertex set $V(\Gamma) = \{1, \dots, n\}$ and l_{ij} the number of lines $e \in E(\Gamma)$ connecting the vertices i and j . We set $l_{ij} = l_{ji}$ for $i > j$ and $l_{ii} = 0$. If e connects i and j we set $\partial e := \{i, j\}$. Then

$$T_n = \sum_{\Gamma \in \mathcal{G}_n} T_{\Gamma}, \quad (6.37)$$

where

$$T_{\Gamma} = \frac{1}{\text{Sym}(\Gamma)} \langle t_{\Gamma}, \delta_{\Gamma} \rangle, \quad (6.38)$$

with

$$\delta_{\Gamma} = \frac{\delta^{2|E(\Gamma)|}}{\prod_{i \in V(\Gamma)} \prod_{e: i \in \partial e} \delta \varphi_i(x_{e,i})}$$

and

$$t_{\Gamma} = \prod_{e \in E(\Gamma)} \hbar \Delta_F(x_{e,i}, i \in \partial e) \quad (6.39)$$

The symmetry factor Sym is the number of possible permutations of lines joining the same two vertices, $\text{Sym}(\Gamma) = \prod_{i<j} l_{ij}!$. Note that T_{Γ} is a map

from $(\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes V(\Gamma)}$ to $\mathcal{F}^{|V(\Gamma)|}[[\hbar]]$, where $\otimes V$ means that the factors in the tensor product are numbered by vertices and to a vertex $v \in V(\Gamma)$ we assign the variable φ_v . The renormalization problem to extend T_n 's to maps on $(\mathcal{F}_{\text{loc}})^{\otimes n}$ is now reduced to extending all the maps T_Γ . For the latter we can use methods relying on the combinatorics of Feynman graphs, as is done in other approaches to pQFT. In particular, one can establish a relation with the Connes-Kreimer approach [CK00, CK01] (see [Pin00, GBL00, DFKR14]).

First we note that functional derivatives of local functionals are of the form

$$F^{(l)}(\varphi)(x_1, \dots, x_l) = \int \sum_{j=1}^N g_j[\varphi](y) p_j(\partial_{x_1}, \dots, \partial_{x_l}) \prod_{i=1}^l \delta(y - x_i) d\mu(y), \quad (6.40)$$

where $N \in \mathbb{N}$, p_j 's are polynomials in partial derivatives and $g_j[\varphi]$ are φ -dependent test functions. The representation above is not unique, since some of the partial derivatives ∂_{x_i} can be replaced with ∂_y and applied to $g_j[\varphi]$. Another representation of $F^{(l)}(\varphi)$ is obtained by performing the integral above and using the centre of mass and relative coordinates:

$$F^{(l)}(\varphi)(x_1, \dots, x_l) = \sum_{\beta} f_{\beta}[\varphi](z) \partial^{\beta} \delta(x^{\text{rel}}) \quad (6.41)$$

where $\beta \in \mathbb{N}_0^{4(l-1)}$, test functions $f_{\beta}[\varphi](x) \in \mathcal{D}$ are now φ -dependent functions of the center of mass coordinate $z = (x_1 + \dots + x_k)/k$ and $x^{\text{rel}} = (x_1 - z, \dots, x_k - z)$ denotes the relative coordinates.

Using (6.40) we see that the functional differential operator δ_Γ applied to $F \in \mathcal{F}_{\text{loc}}^{\otimes n}$ yields, at any n -tuple of field configurations $(\varphi_1, \dots, \varphi_n)$, a compactly supported distribution in the variables $x_{e,i}, i \in \partial e, e \in E(\Gamma)$ with support on the partial diagonal $\Delta_\Gamma = \{x_{e,i} = x_{f,i}, i \in \partial e \cap \partial f, e, f \in E(\Gamma)\} \subset \mathbb{M}^{2|E(\Gamma)|}$ and with a wavefront set perpendicular to $T\Delta_\Gamma$. Note that the partial diagonal Δ_Γ can be parametrized using the center of mass coordinates

$$z_v \doteq \frac{1}{\text{valence}(v)} \sum_{e|v \in \partial e} x_{e,v},$$

assigned to each vertex. The remaining relative coordinates are $x_{e,v}^{\text{rel}} = x_{e,v} - z_v$, where $v \in V(\Gamma)$, $e \in E(\Gamma)$ and $v \in \partial e$. Obviously, we have $\sum_{e|v \in \partial e} x_{e,v}^{\text{rel}} = 0$ for all $v \in V(\Gamma)$. In this parametrization $\delta_\Gamma F$ can be written as a finite sum

$$\delta_\Gamma F = \sum_{\text{finite}} f^\beta \partial_\beta \delta_{\text{rel}},$$

where $\beta \in \mathbb{N}_0^{4|V(\Gamma)|}$, each $f^\beta(\varphi_1, \dots, \varphi_n)$ is a test function on Δ_Γ and δ_{rel} is the Dirac delta distribution in relative coordinates, i.e. $\delta_{\text{rel}}(g) = g(0, \dots, 0)$, where g is a function of $(x_{e,v}^{\text{rel}}, v \in V(\Gamma), e \in E(\Gamma))$.

We can simplify our notation even further. Let Y_Γ denote the vector space spanned by derivatives of the Dirac delta distributions $\partial_\beta \delta_{\text{rel}}$, where $\beta \in \mathbb{N}_0^{4|V(\Gamma)|}$. Obviously, Y_Γ is graded by $|\beta|$. Let $\mathcal{D}(\Delta_\Gamma, Y_\Gamma)$ denote the graded space of test functions on Δ_Γ with values in Y_Γ . With this notation we have $\delta_\Gamma F \in \mathcal{D}(\Delta_\Gamma, Y_\Gamma)$ and if $F \in (\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes n}$, then $\delta_\Gamma F$ is supported on $\Delta_\Gamma \setminus \text{DIAG}$, where DIAG is the large diagonal:

$$\text{DIAG} = \{z \in \Delta_\Gamma \mid \exists v, w \in V(\Gamma), v \neq w : z_v = z_w\}.$$

We can now write (6.38) in the form

$$\frac{1}{\text{Sym}(\Gamma)} \langle t_\Gamma, \delta_\Gamma \rangle = \sum_{\text{finite}} \langle f^\beta \partial_\beta \delta_{\text{rel}}, t_\Gamma \rangle$$

where t_Γ is now written in terms of centre of mass and relative coordinates. To see that this expression is well defined, note that we can move all the partial derivatives ∂_β to t_Γ by formal partial integration. Then the contraction with δ_{rel} is just the pullback through the diagonal map $\rho_\Gamma : \Delta_\Gamma \rightarrow \mathbb{M}^{2|E(\Gamma)|}$ by

$$(\rho_\Gamma(z))_{e,v} = z_v \quad \text{if } v \in \partial e.$$

From the wavefront set properties of $\Delta_{S_0}^F$, we deduce that the pullback ρ_Γ^* of each $t_\Gamma^\beta \doteq \partial_\beta t_\Gamma$ is a well defined distribution on $\Delta_\Gamma \setminus \text{DIAG}$, so (6.38) makes sense if $F \in (\mathcal{F}_{\text{loc}})_{\text{pds}}^{\otimes n}$, as expected. We conclude that $t_\Gamma \in \mathcal{D}'(\Delta_\Gamma \setminus \text{DIAG}, Y_\Gamma)$, where the duality between t_Γ and a test function $f = \sum_{\text{finite}} f^\beta \partial_\beta \delta$ is given by

$$\langle t_\Gamma, f \rangle \doteq \sum_{\beta} \langle t_\Gamma^\beta, f^\beta \rangle.$$

The renormalization problem now reduces to finding the extensions of t_Γ^β , so that t_Γ^β gets extended to an element of $\mathcal{D}'(\Delta_\Gamma, Y_\Gamma)$. The solution to this problem is obtained by using the inductive procedure of Epstein and Glaser. The induction step works as follows: if $t_{\Gamma'}$ is known for all graphs Γ' with fewer vertices than Γ , then t_Γ can be uniquely defined for all *disconnected*, all *connected one particle reducible* and all *one particle irreducible one vertex reducible graphs*. Graphs which are irreducible and do not contain any non-trivial irreducible subgraphs are called *EG-primitive*. For the remaining

graphs, called *EG-irreducible*, t_Γ is defined uniquely on all $f \in \mathcal{D}(\Delta_\Gamma, Y_\Gamma)$ of the form above where f_β vanishes together with all its derivatives of order $\leq \omega_\Gamma + |\beta|$ on the thin diagonal of Δ_Γ . Here

$$\omega_\Gamma = (d - 2)|E(\Gamma)| - d(|V(\Gamma)| - 1)$$

is the degree of divergence of the graph Γ . We denote this subspace by $\mathcal{D}_{\omega_\Gamma}(\Delta_\Gamma, Y_\Gamma)$. Graphs which are irreducible and do not contain any non-trivial irreducible subgraphs are called *EG-primitive*. Renormalization amounts to project a generic f to this subspace by a translation invariant projection $W_\Gamma : \mathcal{D}(\Delta_\Gamma, Y_\Gamma) \rightarrow \mathcal{D}_{\omega_\Gamma}(\Delta_\Gamma, Y_\Gamma)$. Different renormalization schemes differ by different choices of the projections W_Γ (see [DFKR14] for details).

By exploiting the translation invariance in Minkowski spacetime we find that, at each step of the recursive construction of time-ordered products, the renormalization problem reduces to the problem of extension of some distribution defined everywhere outside the origin, so this is what we will focus on now.

6.5.3 Regularization of distributions

There is an important conceptual difference between the Epstein Glaser framework and other approaches to renormalization; namely from the EG point of view one constructs objects (e.g. time-ordered products, the S-matrix) which are already renormalized and can be physically interpreted. In other approaches, one first introduces some *regularization*, which renders the Feynman graphs well defined and then, in the next steps performs renormalization, which is some procedure that allows to recover physically relevant information after the regularization parameters are removed by some limiting process.

In this section we show how introducing an explicit regularization procedure is related to the problem of extension of distributions.

Definition 6.17. *We define*

$$\mathcal{D}_\lambda(\mathbb{M}^{n-1}) := \{f \in \mathcal{D}(\mathbb{M}^{n-1}) \mid (\partial^\alpha f)(0) = 0 \ \forall |\alpha| \leq \lambda\} \quad (6.42)$$

to be the space of functions with derivatives vanishing up to order λ and $\mathcal{D}'_\lambda(\mathbb{M}^{n-1})$ is the corresponding space of distributions.

Theorem 6.6 ([Ste71, BF00]). *A distribution $t \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ with scaling degree $\text{sd}(t)$ has a unique extension $\bar{t} \in \mathcal{D}'_\lambda(\mathbb{R}^n)$, $\lambda = \text{sd}(t) - n$, which satisfies the condition $\text{sd}(\bar{t}) = \text{sd}(t)$.*

Definition 6.18 ([DFKR14]). Let $\tilde{t} \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ be a distribution with degree of divergence λ , and let $\bar{t} \in \mathcal{D}'_\lambda(\mathbb{R}^n)$ be the unique extension of \tilde{t} with the same degree of divergence. A family of distributions $\{t^\zeta\}_{\zeta \in \Omega \setminus \{0\}}$, $t^\zeta \in \mathcal{D}'(\mathbb{R}^n)$, with $\Omega \subset \mathbb{C}$ a neighborhood of the origin, is called a regularization of \tilde{t} , if

$$\forall g \in \mathcal{D}_\lambda(\mathbb{R}^n) : \lim_{\zeta \rightarrow 0} \langle t^\zeta, g \rangle = \langle \bar{t}, g \rangle. \quad (6.43)$$

The regularization $\{t^\zeta\}$ is called analytic, if for all functions $f \in \mathcal{D}(\mathbb{R}^n)$ the map

$$\Omega \setminus \{0\} \ni \zeta \mapsto \langle t^\zeta, f \rangle \quad (6.44)$$

is analytic with a pole of finite order at the origin. The regularization $\{t^\zeta\}$ is called finite, if the limit $\lim_{\zeta \rightarrow 0} \langle t^\zeta, f \rangle \in \mathbb{C}$ exists $\forall f \in \mathcal{D}(\mathbb{R}^n)$; in this case $\lim_{\zeta \rightarrow 0} t^\zeta \in \mathcal{D}'(\mathbb{R}^n)$ is called an extension or renormalization of \tilde{t} .

Given the unique extension \bar{t} one can define (in general not unique) extension of t to a distribution in $\mathcal{D}(\mathbb{R}^n)$ by a choice of a projection $W : \mathcal{D}(\mathbb{R}^n) \rightarrow \mathcal{D}_\lambda(\mathbb{R}^n)$ defined as

$$\begin{aligned} W : \mathcal{D}(\mathbb{R}^n) &\rightarrow \mathcal{D}_\lambda(\mathbb{R}^n) \\ f &\mapsto Wf := f - \sum_{|\gamma| \leq \text{sd}(t) - n} w_\gamma \partial^\gamma f(0), \end{aligned} \quad (6.45)$$

given in terms of functions $w_\gamma \in \mathcal{D}(\mathbb{R}^n)$, $|\beta| \leq \text{sd}(t) - n$, fulfilling

$$\partial^\gamma w_\beta(0) = \delta_\beta^\gamma \quad \forall \gamma \in \mathbb{N}_0^n \quad (6.46)$$

In particular, it was shown in [] that for an analytic regularization $\{t^\zeta\}$ we can write (6.43) in the form

$$\langle \bar{t}, Wf \rangle = \lim_{\zeta \rightarrow 0} \left[\langle t^\zeta, f \rangle - \sum_{|\gamma| \leq \text{sd}(t) - n} \langle t^\zeta, w_\gamma \rangle \partial^\gamma f(0) \right]. \quad (6.47)$$

and each of the two expressions appearing inside the square bracket can be expanded in a Laurent series around $\zeta = 0$. This allows to perform the Minimal Subtraction, as done in

Corollary 6.1 (Minimal Subtraction [DFKR14]). The regular part ($\text{rp} = 1 - \text{pp}$) of any analytic regularization $\{t^\zeta\}$ of a distribution $t \in \mathcal{D}'(\mathbb{R}^n \setminus \{0\})$ defines by

$$\langle t^{\text{MS}}, f \rangle := \lim_{\zeta \rightarrow 0} \text{rp}(\langle t^\zeta, f \rangle) \quad (6.48)$$

an extension of t with the same scaling degree, $\text{sd}(t^{\text{MS}}) = \text{sd}(t)$, which we call the Minimal Subtraction.

Let us now come back to the construction of the S-matrix. To relate this to the method of divergent counter terms, we need to find a family of renormalization group elements $Z^\zeta \in \mathcal{R}$ such that

$$\mathcal{S} = \lim_{\zeta \rightarrow \zeta_0} \mathcal{S}^\zeta \circ Z^\zeta . \quad (6.49)$$

It was shown in [DFKR14, BDF09] that for a given \mathcal{S} , such a family (Z^ζ) exists and is uniquely determined up to a sequence which converges to the identity.

Remark 6.4. Note that, by construction, maps Z^ζ obtained this way are local, so the construction provided in [DFKR14] automatically yields local counter terms.

Chapter 7

Gauge theories

In section 4.3 we have already seen that the space of multilocal on-shell functionals $\mathcal{F}_S(\mathcal{M})$ can be characterized as the 0th homology of the differential complex $(\Lambda\mathcal{V}, \delta_S)$ (see (4.7)). The 1-st homology of this complex is interpreted as the space of non-trivial local symmetries. Now we discuss the quantization of theories where this homology class is non-trivial, using the BV framework, in the version proposed in [FR12b, FR12a].

7.1 Classical gauge theory

Recall from section 4.3 that $\Lambda\mathcal{V}$ as the space of multivector fields, is equipped with a natural structure of the Schouten bracket $\{.,.\}$ and the BV differential δ_S is locally generated by the bracket in the sense that

$$\delta_S X = \{S, X\}.$$

The triple $(\Lambda\mathcal{V}, \{.,.\}, \delta_S)$ is an algebraic structure called *differential Gerstenhaber algebra*. In the quantized theory this structure, together with a certain grade 1 operator gives rise to a *BV algebra*. In this chapter we will use a slightly formal notation $X(\varphi) = \int X_x(\varphi) \frac{\delta}{\delta\varphi(x)}$ introduced in section 3.4 for a vector field $X \in \mathcal{V}$. This notation allows to make contact with the standard physics literature on the BV formalism, if one identifies $\frac{\delta}{\delta\varphi(x)}$ with a formal generator φ^\dagger , called the *antifield*.

The structure described above appears also in theories where local symmetries are present, but there the space of multivector fields on an infinite dimensional manifold \mathcal{E} has to be replaced by the space of multivector fields on certain *graded* infinite dimensional manifold, which we denote by $\overline{\mathcal{E}}(\mathcal{M})$.

We show how this space is constructed on the example of Yang-Mills theories and the free electromagnetic field.

7.1.1 Dynamics and symmetries

Let K be a finite dimensional semisimple compact Lie group and $\mathfrak{k} \doteq \text{Lie}(K)$ its Lie algebra. Consider the trivial principal bundle $P = M \times K$ over M . We take the point of view that the QFT model on a given spacetimes should be first constructed from “simple building blocks”, i.e. algebras associated to regions that are topologically simple, and the global structure is recovered from the properties of the net itself. Therefore, it is sufficient for our purposes to restrict ourselves to trivial principal bundles.

Globally the configuration space \mathcal{E} for a Yang-Mills theory should be the space of connection 1-forms, i.e. the space of \mathfrak{k} -valued 1-forms on P which satisfy:

1. $R_\alpha^* \omega = \text{ad}_{\alpha^{-1}} \circ \omega$,
2. $\omega(Z(\xi)) = \xi$,

where $\alpha \in K$, $R_\alpha : K \rightarrow K$ is the right multiplication, ad is the adjoint representation of K on \mathfrak{k} and $Z : \mathfrak{k} \rightarrow \Gamma(TP)$ is the map which assigns to $\xi \in \mathfrak{k}$ its fundamental vector field given by

$$Z_p(\xi) \doteq \left. \frac{d}{dt} \right|_{t=0} p e^{t\xi}.$$

After fixing a background connection A_0 we can characterize all the connection 1-forms in terms of one forms on M with values in the associated bundle $\mathfrak{k}_P \doteq P \times_K \mathfrak{k}$. We denote this space by $\Omega^1(M, \mathfrak{k}_P)$ and this is going to be our configuration space. For simplicity we will assume that A_0 is a trivial connection. For the discussion of the general case see [Zah13].

Definition 7.1. *Define the configuration space \mathcal{E} for a Yang-Mills theory as the space $\Omega^1(M, \mathfrak{k}_P)$ of one forms on M with values in the associated bundle $\mathfrak{k}_P \doteq P \times_K \mathfrak{k}$. Since P is assumed to be trivial, $\mathcal{E} = \Omega^1(M, \mathfrak{k})$.*

Let \mathcal{F} denote the space of multilocal compactly supported functionals on \mathcal{E} and \mathcal{V} the space of multilocal vector fields. The generalized Lagrangian of the Yang-Mills theory is given by

$$L_{YM}(f)(A) = -\frac{1}{2} \int_M f \text{tr}(F \wedge *F), \quad (7.1)$$

where $F = dA + \frac{1}{2}[A, A]$, $*$ is the Hodge operator and tr is the trace on \mathfrak{k} . The equation of motion reads:

$$S'_{YM}(A) = D_A * F = 0,$$

where D_A is the covariant derivative induced by the connection A . To analyse $H^1(\Lambda\mathcal{V}, \delta_S)$, we will construct explicitly non-trivial symmetries of the action corresponding to the Lagrangian (7.1). Let us define the gauge group as the space of vertical G -equivariant compactly supported diffeomorphisms of P :

$$\mathcal{G} := \{\alpha \in \text{Diff}_c(P) \mid \alpha(p \cdot g) = \alpha(p) \cdot g, \pi(\alpha(p)) = \pi(p), \forall g \in K, p \in P\}.$$

We can also characterize \mathcal{G} as $\Gamma_c(M, P \times_K K)$ and for a trivial bundle P this reduces to $\mathcal{C}_c^\infty(M, K)$. It is known ([Nee04, Glö02, KM97], see also [Nee06, Woc06]) that $\mathcal{C}_c^\infty(M, K)$ can be equipped with the structure of an infinite dimensional Lie group modelled on its Lie algebra $\mathfrak{g}_c = \mathcal{C}_c^\infty(M, \mathfrak{k})$. Since the gauge group is just a subgroup of $\text{Diff}(P)$, it has a natural action on $\Omega^1(P, \mathfrak{k})^G$ by the pullback. This induces the action of $\mathcal{G}(\mathcal{M})$ on \mathcal{E} , and the corresponding derived action σ of \mathfrak{g}_c is given by

$$\sigma(c)(A) = dc + [A, c] = D_A c, \quad c \in \mathfrak{g}_c \quad (7.2)$$

The Yang-Mills action is invariant under the transformation (7.2), in the sense that

$$\langle S'_{YM}(A), \sigma(c)(A) \rangle = 0, \quad \forall A \in \mathcal{E},$$

so σ induces a map from \mathfrak{g}_c to \mathcal{V} , whose image is contained in the kernel of δ_S . More generally, we consider $\mathfrak{G} \doteq \mathcal{C}_{\text{ml}}^\infty(\mathcal{E}, \mathfrak{g}_c)$, the space of multilocal functionals on the configuration space with values in the gauge algebra, and a map $\rho : \mathfrak{G} \rightarrow \mathcal{V}$ defined by $\rho(\Xi)(A) \doteq \sigma(\Xi(A))A$, i.e.

$$\partial_{\rho(\Xi)} F(A) \doteq \langle F^{(1)}(A), \sigma(\Xi(A))A \rangle.$$

Remark 7.1. The assignment of $\mathfrak{G}(\mathcal{M})$ to \mathcal{M} induces a functor from **Loc** to **Vec**, which we denote by the same letter, i.e. \mathfrak{G} . The fact that the action of \mathfrak{g}_c on \mathcal{E} is local implies that ρ is a natural transformation between \mathfrak{G} and \mathfrak{V} , both treated as functors from **Loc** to **Vec**.

Remark 7.2. To see a more geometrical interpretation of the map ρ , note that $\mathfrak{G} \subset \Gamma(\mathcal{E} \times \mathfrak{g}_c)$ (the space of sections of a trivial bundle over \mathcal{E}), and we have a morphism of vector bundles $\mathcal{E} \times \mathfrak{g}_c(\mathcal{M}) \rightarrow T\mathcal{E}$ given by $(A, c) \mapsto (A, \rho(c)A)$. This way $\mathcal{E} \times \mathfrak{g}_c$ is made into a Lie algebroid.

7.1.2 The Koszul-Tate complex

The invariance of the Yang-Mills action under σ implies that $\rho(\mathfrak{G}) \subset \ker(\delta_S)$. In fact, one can characterize all non-trivial local symmetries this way, in the sense that for each $X \in \ker \delta_S$ there exists an element $\Xi \in \mathfrak{G}$ and $I \in \delta_S(\Lambda^2 \mathcal{V})$ such that

$$X = I + \rho(\Xi).$$

We can use this fact to kill the homology in degree one of the differential complex (4.7). We need extend the complex by adding \mathfrak{G} in degree 2 and symmetric powers of $\mathfrak{G}(\mathcal{M})$ in higher degrees. This idea is made precise in the following definition.

Definition 7.2. *The underlying algebra of the Koszul-Tate complex is defined as*

$$\mathcal{KT} \doteq \mathcal{O}_{\text{ml}}(\mathcal{E} \oplus \mathcal{E}^*[1] \oplus \mathfrak{g}^*[2]) = \mathcal{C}_{\text{ml}}^\infty(\mathcal{E}, \Lambda \mathcal{E}^{*'} \widehat{\otimes}_\pi S^\bullet \mathfrak{g}^{*'} \otimes \mathbb{C}),$$

where $\mathfrak{g}^* \equiv \mathcal{C}^\infty(M, \mathfrak{k}^*)$.

Remark 7.3. The completion $\Lambda \mathcal{E}^{*'} \widehat{\otimes}_\pi S^\bullet \mathfrak{g}^{*'}$ of the tensor product is identified with $\prod_{\substack{n=0 \\ k+l=n}}^\infty \Gamma'_{\mathfrak{k}|l}(M^n, E^{\boxtimes k} \boxtimes \mathfrak{k}^{\boxtimes l})$, as in the definition 3.14.

We equip \mathcal{KT} with a differential $\delta_{\mathcal{KT}}$ which acts on $\Lambda \mathcal{V}$ as δ_S and on $S^\bullet \mathfrak{G}(\mathcal{M})$ it is given by ρ extended by means of the graded Leibniz rule. The resulting differential complex is called the Koszul-Tate complex.

$$\dots \rightarrow \Lambda^2 \mathcal{V} \oplus \mathfrak{G} \xrightarrow{\delta = \delta_S \oplus \rho} \mathcal{V} \xrightarrow{\delta = \delta_S} \mathcal{F} \rightarrow 0 \quad (7.3)$$

The 0-th homology of this complex is \mathcal{F}_S and higher homologies are trivial, so (\mathcal{KT}, δ) provides a resolution of \mathcal{F}_S .

7.1.3 The Chevalley-Eilenberg complex

We have already seen how to characterize the space of on-shell functionals in Yang-Mills theory, now we want to find a homological interpretation for the space of gauge invariant ones. This can be done with the use of the Chevalley-Eilenberg complex.

Definition 7.3. *The underlying algebra of the Chevalley-Eilenberg complex is defined as $\mathcal{CE} = \mathcal{C}_{\text{ml}}^\infty(\mathcal{E}, \Lambda \mathfrak{g}')$, where $\mathfrak{g} \doteq \mathcal{C}^\infty(M, \mathfrak{k})$ and the tensor product in $\Lambda \mathfrak{g}'$ is the completed projective tensor product.*

Remark 7.4. Note that \mathcal{CE} can be seen as the space of multilocal functionals on the graded manifold $\bar{\mathcal{E}} \doteq \mathcal{E} \oplus \mathfrak{g}[1]$, called the *extended configuration space*.

The Chevalley-Eilenberg differential γ is constructed in such a way that it encodes the action σ of the gauge algebra \mathfrak{g} on \mathcal{F} , induced by (7.2). For $F \in \mathcal{F}$ we define $\gamma F \in \mathcal{C}_{\text{ml}}^\infty(\mathcal{E}, \mathfrak{g}'^c)$ as

$$(\gamma F)(A, c) \doteq (\sigma(c)F)(A) = \left\langle F^{(1)}(A), D_A c \right\rangle, \quad (7.4)$$

where $c \in \mathfrak{g}$. Note that now we have dropped the restriction on the support of gauge parameters. For a form $\omega \in \mathfrak{g}'$, which doesn't depend on A we set

$$\gamma\omega(c_1, c_2) \doteq \omega([c_1, c_2]).$$

Since γ is required to be nilpotent of order 2 and has to satisfy the graded Leibniz rule, for a general $F \in \mathcal{C}_{\text{ml}}^\infty(\mathcal{E}, \Lambda^q \mathfrak{g}')$ we define

$$\begin{aligned} (\gamma F)(A; c_0, \dots, c_q) &\doteq \sum_{i=0}^q (-1)^i \partial_{\rho_{\mathcal{M}}(c_i)} (\iota_{(c_0, \dots, \hat{c}_i, \dots, c_q)} F)(A) + \\ &+ \sum_{i < j} (-1)^{i+j} F(A, [c_i, c_j], \dots, \hat{c}_i, \dots, \hat{c}_j, \dots, c_q), \end{aligned}$$

where the hat over a variable means that this variable is omitted and ι denotes the insertion of n -vector fields into an n -form. The differential complex looks as follows:

$$0 \rightarrow \mathcal{F} \xrightarrow{\gamma} \mathcal{C}_{\text{ml}}^\infty(\mathcal{E}, \mathfrak{g}'^c) \xrightarrow{\gamma} \mathcal{C}_{\text{ml}}^\infty(\mathcal{E}, \Lambda^2 \mathfrak{g}'^c) \rightarrow \dots \quad (7.5)$$

Note that from (7.4) follows that the kernel of γ in degree 0 consists of all the multilocal functionals invariant under the action σ . Hence $H^0(\mathcal{CE}(\mathcal{M}), \gamma) = \mathfrak{F}^{\text{inv}}(\mathcal{M})$, the space of invariants.

Remark 7.5. Note that the assignment of $\mathcal{CE}(\mathcal{M})$ to a spacetime \mathcal{M} induces a covariant functor \mathcal{CE} from **Loc** to **Vec** and the differential γ can be lifted to a natural transformation.

Remark 7.6. Formally we write elements of \mathcal{CE} as sums of functionals of the form

$$\begin{aligned} &F(A)(c_1, \dots, c_n) = \\ &= \sum_{a_1, \dots, a_n} \int f(A)(x_1, \dots, x_n)_{a_1, \dots, a_n} c_1(x_1)^{a_1} \dots c_n(x_n)^{a_n} d\mu(x_1) \dots d\mu(x_n), \end{aligned}$$

where $f(A) \in \Gamma'_a(M^n, \mathfrak{k}^{\otimes n})$. Let us denote by $C^a(x)$ the evaluation functional $C^a(x)(c) \doteq c^a(x)$ (compare with section 3.3). Clearly $C^a(x) \in \mathfrak{g}'$. We call these evaluation functionals *ghosts* and we write

$$\begin{aligned} F(A) &= \\ &= \sum_{a_1 < \dots < a_n} \int f(A)(x_1, \dots, x_n)_{a_1, \dots, a_n} C(x_1)^{a_1} \dots C(x_n)^{a_n} d\mu(x_1) \dots d\mu(x_n). \end{aligned}$$

Analogously to section 3.3 we denote by $\frac{\delta^l}{\delta c}$ the left derivative on $\Lambda \mathfrak{g}'$.

7.1.4 The BV complex

The Chevalley-Eilenberg complex and the Koszul-Tate complex fit together into one structure called the *BV complex*, which encodes information about both the equations of motion and the invariants. To see how it arises in a natural way it is worth to look back at the example of the scalar fields, which we recalled at the beginning of this section. There, in order to characterize the space of on-shell functionals we needed to consider the space of multilocal vector fields on the configuration space. Now, to take the gauge symmetries into account, we have extended the configuration space into a graded manifold $\bar{\mathcal{E}}$. The space of multivector fields on $\bar{\mathcal{E}}$ is formally given as the ring of functions on

$$T^*[-1]\bar{\mathcal{E}} = \mathcal{E} \oplus \mathfrak{g}[1] \oplus \mathcal{E}'[-1] \oplus \mathfrak{g}'[-2], \quad (7.6)$$

the odd cotangent bundle of $\bar{\mathcal{E}}$, with the *negative grading on the fiber*. To give this object a meaning in the infinite dimensional situation we identify $\mathcal{O}(T^*[-1]\bar{\mathcal{E}})$ with

$$\mathcal{BV} \doteq \mathcal{C}_{\text{ml}}^\infty(\mathcal{E}, S^\bullet \mathfrak{g}_c \widehat{\otimes}_\pi \Lambda \mathcal{E}_c \widehat{\otimes}_\pi \Lambda \mathfrak{g}' \otimes \mathbb{C}). \quad (7.7)$$

The geometric interpretation as the space of functions on $T^*[-1]\bar{\mathcal{E}}$ fits very well with the spirit of the functional approach. Constructing the underlying algebra of the BV complex we are working all the time with multilocal functionals but we have to pass from infinite dimensional manifolds to graded infinite dimensional manifolds. Clearly, both \mathcal{CE} and \mathcal{KT} with inverted grading¹ are subalgebras of \mathcal{BV} .

On \mathcal{BV} we introduce two gradings:

¹The grading assigned to the vector fields in the bicomplex \mathcal{BV} is minus the grading of the \mathcal{KT} complex. In the mathematical literature, the resulting structure is called a Beilinson-Drinfeld algebra [BD04], rather than a BV algebra.

- the *pure ghost number* $\#pg$, which is inherited the grading of the Chevalley-Eilenberg complex, i.e. assigns grade $+1$ to elements of \mathfrak{g}' .
- the *antifield number* $\#af$, which is inherited from the grading of the Koszul-Tate complex, i.e. assigns grade $+1$ to elements of \mathcal{E}_c and $+2k$ to elements of \mathfrak{g}_c .

The total grading of the BV complex is called the *ghost number* $\#gh$ and is defined by

$$\#gh = \#pg - \#af$$

and it reflects the grading of the graded manifold (7.6).

We will now extend the differentials δ and γ to the whole of \mathcal{BV} . As in the case of the scalar field, \mathcal{BV} can be equipped with the graded Schouten bracket $\{.,.\}$. We use this structure to extend the Koszul-Tate differential to the whole algebra \mathcal{BV} by setting

$$\delta X \doteq \{S_{YM}, X\}, \quad X \in \mathcal{BV}.$$

The Chevalley-Eilenberg differential can also be written in terms of the bracket in a similar manner. To this end we need to find a natural Lagrangian that implements γ . Firstly, note that the assignments of $\mathcal{BV}(\mathcal{M})$ to spacetimes \mathcal{M} induces a functor $\mathfrak{BV} : \mathbf{Loc} \rightarrow \mathbf{Vec}$. We denote this functor by \mathfrak{BV} .

Proposition 7.1. *There exists a natural transformation $\theta : \mathfrak{D} \rightarrow \mathfrak{BV}$ such that*

$$\gamma X = \{\theta_{\mathcal{M}}(f), X\},$$

where $f \equiv 1$ on $\text{supp } f$ and $X \in \mathfrak{CE}(\mathcal{M})$.

Proof. γ is a derivation of \mathfrak{CE} , so by definition a vector field on $\bar{\mathcal{E}}$. In the antifield notation γ can be written as

$$\gamma(A) = \left\langle \frac{\delta}{\delta A}, \sigma(C)A \right\rangle + \frac{1}{2} \left\langle \frac{\delta^r}{\delta C}, [C, C] \right\rangle$$

In order to obtain an element of \mathcal{BV} , we modify γ by multiplying with a cutoff function. Define

$$\theta_{\mathcal{M}}(f)[A] \doteq \left\langle \frac{\delta}{\delta g}, \sigma(fC)A \right\rangle + \frac{1}{2} \left\langle \frac{\delta_r}{\delta C}, f[C, C] \right\rangle.$$

This definition is local and covariant and it can be easily checked that $\theta \in \text{Nat}(\mathfrak{D}, \mathfrak{BV})$. \square

We will denote the generalized action corresponding to the Lagrangian $\theta_{\mathcal{M}}$ by $\gamma_{\mathcal{M}}$, or simply γ , if we work on a fixed spacetime. We can now write

$$\gamma X \doteq \{\theta_{\mathcal{M}}(f), X\}, \quad X \in \mathfrak{B}\mathfrak{A}(\mathcal{M}),$$

where $f \equiv 1$ on the support of X . On a fixed spacetime we will just use the shorthand notation

$$\gamma X = \{\gamma, X\},$$

since γ can be understood both as a differential on $\mathcal{C}\mathcal{E}$ and a generalized action. We define the BV differential as the sum

$$s_{BV} = \delta + \gamma = \{S_{YM} + \gamma, \cdot\},$$

and call $S^{\text{ext}} \doteq S + \gamma$ the *extended action* of the Yang-Mills theory. In this case $s_{BV}^2 = 0$, since both δ and γ are nilpotent and they anti-commute. The latter is a consequence of the gauge invariance of the equations of motion. For a detailed discussion see [Rej13]. In general, s_{BV} defined above would not be automatically nilpotent and one would need to add to it further terms. To do it systematically, it is convenient to formulate the problem in terms of the natural Lagrangians. First we need some notation. First extend the equivalence relation (4.2) the natural Lagrangians depending on several test functions.

Definition 7.4. We say that $L_1 \sim L_2$, $L_1, L_2 \in \text{Nat}(\mathfrak{D}^k, \mathfrak{B}\mathfrak{A}_{\text{loc}})$ if:

$$\text{supp}((L_1 - L_2)_M(f_1, \dots, f_k)) \subset \text{supp}(df_1) \cup \dots \cup \text{supp}(df_k), \quad \forall f_1, \dots, f_k \in \mathfrak{D}^k(M) \quad (7.8)$$

Next we lift the antibracket to the level of natural transformations.

Definition 7.5. Let L_1, L_2 be natural Lagrangians such that $L_1 \in \text{Nat}(\mathfrak{D}^p, \mathfrak{B}\mathfrak{A}_{\text{loc}})$, $L_2 \in \text{Nat}(\mathfrak{D}^q, \mathfrak{B}\mathfrak{A}_{\text{loc}})$. We define

$$\{L_1, L_2\}_M(f_1, \dots, f_{p+q}) = \frac{1}{p!q!} \sum_{\pi \in P_{p+q}} \{L_{1M}(f_{\pi(1)}, \dots, f_{\pi(p)}), L_{2M}(f_{\pi(p+1)}, \dots, f_{\pi(p+q)})\}, \quad (7.9)$$

where P_{p+q} denotes the permutation group.

It was shown in [FR12b] that the nilpotency of s_{BV} is equivalent to the condition that

$$\{L^{\text{ext}}, L^{\text{ext}}\} \sim 0, \quad (7.10)$$

formulated for the natural Lagrangian L^{ext} . This condition is called the *classical master equation*. One can now formulate the problem of finding the BV differential that extends $\delta + \gamma$ to the problem of finding a natural Lagrangian that satisfies (7.10), with fixed initial terms in degree $\#\text{af} = -1$ and $\#\text{af} = 0$.

Using the two gradings of \mathcal{BV} we construct a bicomplex, whose columns are numbered by $\#\text{af}$ and rows by $\#\text{gh}$. We obtain

$$\begin{array}{ccccccc} \dots & \xrightarrow{\delta} & (\Lambda^2\mathcal{V} \oplus \mathfrak{G}) & \xrightarrow{\delta} & \mathcal{V} & \xrightarrow{\delta} & \mathcal{F} & \xrightarrow{\delta} & 0 \\ & & \downarrow \gamma & & \downarrow \gamma & & \downarrow \gamma & & \\ \dots & \xrightarrow{\delta} & \mathcal{C}_{\text{ml}}^\infty(\mathcal{E}, (\Lambda^2\mathfrak{E}_c \oplus \mathfrak{g}_c) \widehat{\otimes} \mathfrak{g}'^c) & \xrightarrow{\delta} & \mathcal{C}_{\text{ml}}^\infty(\mathcal{E}, \mathfrak{E}_c \widehat{\otimes} \mathfrak{g}'^c) & \xrightarrow{\delta} & \mathcal{C}_{\text{ml}}^\infty(\mathcal{E}, \mathfrak{g}'^c) & \xrightarrow{\delta} & 0 \end{array}$$

The grading of the total complex is $\#\text{gh}$. Note that the first row is just the Koszul-Tate complex \mathcal{KT} with inverted grading. Since it is by construction a resolution, and a standard result in homological algebra tells us that the cohomology of the total complex is given by

$$H^k(\mathcal{BV}, s_{BV}) = H^k(H_0(\mathcal{BV}, \delta), \gamma).$$

Note that taking the 0th homology of δ amounts to going on-shell, while taking the 0th cohomology of γ characterizes gauge invariants. Hence,

$$H^0(\mathcal{BV}, s_{BV}) = \mathcal{F}_S^{\text{inv}}(\mathcal{M}),$$

is the space of gauge invariant on-shell functionals.

7.2 Gauge-fixing

In the next step we use the differential Gerstenhaber algebra (\mathcal{BV}, s_{BV}) to implement the gauge fixing by modifying the extended action. Note that the $\#\text{af} = 0$ term of S^{ext} is still the original Yang-Mills action, which doesn't induce normally hyperbolic equations of motion, so we cannot construct retarded and advanced Green's operators. The idea now is to find an automorphism α of $(\mathcal{BV}, \{.,.\})$ such that $S \doteq \alpha(S^{\text{ext}})$ at $\#\text{af} = 0$ induces normally hyperbolic equations of motion. Such an automorphism can be defined by means of a *gauge fixing fermion*.

Definition 7.6. *Let $\Psi \in \text{Nat}(\mathfrak{D}, \mathfrak{B}\mathfrak{V})$ be a natural Lagrangian of degree $\#\text{gh} = -1$. We call it the gauge fixing fermion and define*

$$\alpha_\Psi(X) := \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\{\Psi_{\mathcal{M}}(f), \dots, \{\Psi_{\mathcal{M}}(f), X\} \dots\}}_n, \quad (7.11)$$

where $X \in \mathfrak{B}\mathfrak{V}(\mathcal{M})$ and $f \equiv 1$ on $\text{supp } X$.

A concrete form of Ψ depends on the choice of gauge fixing and for particular choices one might need to extend $\mathfrak{B}\mathfrak{V}$ with some further generators. This is the case for the Lorentz gauge, which is commonly used in the context of Yang-Mills theory.

Therefore we extend the BV-complex by adding to the configuration space the so called *non-minimal sector*. It consists of the Nakanishi-Lautrup fields $b \in S^{\mathfrak{g}}[0]$, which we add in degree 0 and the antighosts $\bar{c} \in \mathfrak{g}[-1]$, added in degree -1. The new extended configuration space is written explicitly as

$$\bar{\mathcal{E}} = \mathcal{E} \oplus \mathfrak{g}[1] \oplus \mathfrak{g}[0] \oplus \mathfrak{g}[-1].$$

The underlying algebra of the BV complex is defined as the space of multi-local functionals on $T^*[-1]\bar{\mathcal{E}}$, analogously to (7.7).

The BV differential is extended to the non-minimal sector in such a way that the cohomology of the BV complex remains unchanged. We define: $sF = 0$, and $sG = \Pi G \circ m_i$ for $F \in S^1\mathfrak{g}'(M)$, $G \in \Lambda^1\mathfrak{g}'(M)$, where Π denotes the grade shift by +1 and m_i the multiplication of the argument by i . The last operation is just a convention used in physics to make antighosts hermitian. We use it to stay consistent with the literature. The extended Lagrangian is now:

$$\begin{aligned} L^{\text{ext}}(f)[A] = & -\frac{1}{2} \int_M f \text{tr}(F \wedge *F) + \left\langle \frac{\delta}{\delta A}, \sigma(fC)A \right\rangle + \frac{1}{2} \left\langle \frac{\delta_r}{\delta \bar{C}}, f[C, C] \right\rangle + \\ & - i \left\langle \frac{\delta_r}{\delta \bar{C}}, fB \right\rangle. \end{aligned} \quad (7.12)$$

The last term corresponds to the action of s on the non-minimal sector and we used the traditional notation B for evaluation functionals on the space of the Nakanishi-Lautrup fields and \bar{C} for the evaluation functional in the antighosts.

Let us define $\tilde{S}^{\text{ext}} \doteq \alpha_{\Psi}(S^{\text{ext}})$ and $s \doteq \alpha_{\Psi} \circ s_{BV} \circ \alpha_{\Psi}^{-1} = \{\tilde{S}^{\text{ext}}, \cdot, \cdot\}$. Clearly, we have $H^0(\mathfrak{B}\mathfrak{V}, s) = H^0(\mathfrak{B}\mathfrak{V}, s_{BV}) = \mathcal{F}_S^{\text{inv}}(\mathcal{M})$, so we didn't lose the information about the gauge invariant on-shell observables. We want to choose the gauge fixing fermion in such a way that the $\#_{\text{af}} = 0$ term of the new extended action \tilde{S} induces normally hyperbolic EOM's. This is achieved in the Lorentz gauge, which is implemented by

$$\Psi_{\mathcal{M}}(f)[A] = i \int_M f \left(\frac{\alpha}{2} \kappa(\bar{C}, B) + \langle \bar{C}, *d *A \rangle_{\mathfrak{t}} \right) d\mu, \quad (7.13)$$

where κ is the Killing form on the Lie algebra \mathfrak{k} and $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ is the pairing between \mathfrak{k} and its dual \mathfrak{k}^* . The transformed extended Lagrangian takes the form

$$\begin{aligned} \tilde{L}^{\text{ext}}(f) = & -\frac{1}{2} \int_M f \text{tr}(F \wedge *F) + \\ & -i \int_M f \text{tr}[d\bar{C}, *DC] - \int_M f \left(\frac{\alpha}{2} \kappa(B, B) + \langle B, *^{-1}d *A \rangle_{\mathfrak{k}} \right) d\mu + L_{AF}(f), \end{aligned} \quad (7.14)$$

where $L_{AF}(f)$ is the term with $\#af > 0$.

Now it is convenient to redefine the gradings again. Let $\#ta$ denote the *total antifield number*, which is 1 for all the vector fields on $\bar{\mathcal{E}}$ and 0 for functions. We expand s with respect to this grading and we obtain two terms

$$s = \tilde{\delta} + \tilde{\gamma},$$

where the first term has $\#ta = -1$ and the second $\#ta = 0$. Both $\tilde{\delta}$ and $\tilde{\gamma}$ are nilpotent and they anti-commute with each other.

$\tilde{\delta}$ is the Koszul differential for the Lagrangian L defined as the $\#ta = 0$ term in (7.14). L is a graded functional depending on the multiplet of variables $\varphi = (A, c, \bar{c}, b)$ and we label the components in this multiplet by φ^α . We define S'' as a map from the extended configuration space to the space of vector-valued distributions (see section 3.3 and [Rej11b]) given by

$$\left\langle (S'')_{\beta\alpha}, \psi_1^\alpha \otimes \psi_2^\beta \right\rangle \doteq \left\langle \frac{\delta^t}{\delta\varphi^\beta} \frac{\delta^r}{\delta\varphi^\alpha} L(f), \psi_1^\alpha \otimes \psi_2^\beta \right\rangle,$$

where $\psi_1 \in \bar{\mathcal{E}}$, $\psi_2 \in \bar{\mathcal{E}}'_c$ are field configuration multiplets and $f \equiv 1$ on the support of ψ_2 . Note that S'' induces normally hyperbolic equations of motion, so using 4.3 we conclude that there are no non-trivial local symmetries and hence $(\mathcal{BV}, \tilde{\delta})$ is a resolution.

The differential $\tilde{\gamma}$ is called the gauge-fixed BV differential, or just the BRST differential. The action of $\tilde{\gamma}$ on the elements of \mathcal{BV} is summarized in the table below.

	$\tilde{\gamma}$
$F \in \mathcal{F}$	$\langle F^{(1)}, dC + [\cdot, C] \rangle$
C	$-\frac{1}{2}[C, C]$
B	0
\bar{C}	iB

Because $(\mathcal{BV}, \tilde{\delta})$ is a resolution, we can characterize the space of gauge invariant on-shell functionals as

$$\mathcal{F}_S^{\text{inv}} = H^0(\mathcal{BV}, s) = H^0(H_0(\mathcal{BV}, \tilde{\delta}), \tilde{\gamma}).$$

The advantage of this reformulation is that now we are working with field equations that normally hyperbolic and the abstract homological argument tells us that we still recover the correct space of functionals at the end.

For the action S we can find Δ_S^A and Δ_S^R and introduce the Peierls bracket $\{.,.\}_S$ on \mathcal{BV} , analogously to (4.8):

$$[A, B](g, b_\mu, c, \bar{c}_\mu) \doteq \sum_{\alpha, \beta} \left\langle \frac{\delta^l A}{\delta \varphi^\alpha}, \Delta_g^{\alpha\beta} \frac{\delta^r B}{\delta \varphi^\beta} \right\rangle (g, b_\mu, c, \bar{c}_\mu), \quad \Delta_g = \Delta_g^A - \Delta_g^R.$$

Note that L can be expressed as

$$L(f) = L_{YM}(f) + \tilde{\gamma}(\Psi(f)),$$

and S_{YM} is invariant under $\tilde{\gamma}$, so S is also $\tilde{\gamma}$ -invariant. The latter can be expressed as

$$\{L, \tilde{\theta}\} \sim 0, \quad (7.15)$$

where $\tilde{\theta}$ is the natural Lagrangian implementing $\tilde{\gamma}$. This identity allows one to prove the following result, stated in [FR12a] generalized in [BFR13].

Proposition 7.2. *The BRST differential $\tilde{\gamma}$ is a derivation with respect to the Peierls bracket induced by the gauge-fixed action S , modulo the image of $\tilde{\delta}$.*

after [BFR13]. To prove the result we need to show that

$$m \circ (\tilde{\gamma} \otimes 1 + 1 \otimes \tilde{\gamma}) \circ \Gamma'_{\Delta_S} = m \circ \Gamma'_{\Delta_S} \circ (\tilde{\gamma} \otimes 1 + 1 \otimes \tilde{\gamma}),$$

modulo the image of $\tilde{\delta}$. Here we use the notation where

$$\Gamma'_{\Delta_S} \doteq \sum_{\alpha, \beta} \left\langle \Delta_S^{\alpha\beta}, \frac{\delta^l}{\delta \varphi^\alpha} \otimes \frac{\delta^r}{\varphi^\beta} \right\rangle,$$

After a short calculation, we obtain the following condition (compare with Prop. 2.3. of [Rej13]):

$$(-1)^{|\sigma|} K_\varphi^\sigma(x) \Delta_S(\varphi)^{\beta\alpha}(x, y) + K_\varphi^\alpha(y) \Delta_S(\varphi)^{\sigma\beta}(x, y) = \tilde{\gamma}(\Delta_S(\varphi)^{\sigma\alpha}(x, y)), \quad (7.16)$$

where $|\sigma|$ denotes $\#\text{gh}(\varphi^\sigma)$, while K_φ is defined with the use of the evaluation functionals Φ_x^α as

$$\gamma_{0\varphi} \Phi_x^\alpha = \sum_\sigma K_\varphi^\alpha{}_\sigma(x) \Phi_x^\sigma \equiv (K_\varphi \Phi)^\alpha,$$

and $\gamma_{0\varphi}$ is the linearization of $\tilde{\gamma}$ around φ . In a more compact notation we can write this condition as

$$(-1)^{|\sigma|} (K_\varphi \circ \Delta_S(\varphi))^{\sigma\alpha} + (\Delta_S(\varphi) \circ K_\varphi^\dagger)^{\sigma\alpha} = \tilde{\gamma}(\Delta_S(\varphi)^{\sigma\alpha}),$$

where K_φ^\dagger means taking the transpose of the operator-valued matrix and adjoints of its entries. Now we use (7.15) to conclude that

$$\left\langle \frac{\delta^l L(f')}{\delta \varphi^\alpha}, \tilde{\theta}^\alpha(f) \right\rangle = F(f, f'),$$

where $\tilde{\theta}^\alpha(f)$ is the term in $\tilde{\theta}(f)$ which is contracted with $\frac{\delta^l}{\delta \varphi^\alpha}$ and F is a generalized Lagrangian, with the support contained in $\text{supp}(f) \sum \text{supp}(f')$. We can now apply on the both sides the differential operator $\left\langle f_1 \Delta_S^R(\varphi)^{\mu\beta} \circ \frac{\delta^l}{\delta \varphi^\beta} \frac{\delta^r}{\delta \varphi^\kappa}, \Delta_S^R(\varphi)^{\kappa\nu} f_2 \right\rangle$, where $f_1, f_2 \in \mathcal{D}(\mathcal{M})$. We obtain

$$\begin{aligned} & \left\langle f_1 \Delta_S^R(\varphi)^{\mu\beta} \circ \left\langle \frac{\delta^l}{\delta \varphi^\beta} \frac{\delta^l}{\delta \varphi^\alpha} \frac{\delta^r}{\delta \varphi^\kappa} L(f'), \tilde{\theta}^\alpha(f) \right\rangle, \Delta_S^R(\varphi)^{\kappa\nu} f_2 \right\rangle + \\ & + \left\langle f_1 \Delta_S^R(\varphi)^{\mu\beta} \circ \left\langle \frac{\delta^l}{\delta \varphi^\beta} \frac{\delta^l}{\delta \varphi^\alpha} L(f'), \frac{\delta \tilde{\theta}^\alpha(f)}{\delta \varphi^\kappa} \right\rangle, \Delta_S^R(\varphi)^{\kappa\nu} f_2 \right\rangle + \\ & + \left\langle f_1 \Delta_S^R(\varphi)^{\mu\beta} \circ \left\langle \frac{\delta^l}{\delta \varphi^\alpha} \frac{\delta^r}{\delta \varphi^\kappa} L(f'), \frac{\delta \tilde{\theta}^\alpha(f)}{\delta \varphi^\beta} \right\rangle, \Delta_S^R(\varphi)^{\kappa\nu} f_2 \right\rangle + \\ & + \left\langle f_1 \Delta_S^R(\varphi)^{\mu\beta} \circ \left\langle \frac{\delta L(f')}{\delta \varphi^\alpha}, \frac{\delta^r}{\delta \varphi^\kappa} \frac{\delta^l}{\delta \varphi^\beta} \tilde{\theta}^\alpha(f) \right\rangle, \Delta_S^R(\varphi)^{\kappa\nu} f_2 \right\rangle = \\ & = \left\langle \frac{\delta^l}{\delta \varphi^\beta} \frac{\delta^r}{\delta \varphi^\kappa} F(f, f'), \Delta_S^A(\varphi)^{\beta\mu} f_1 \otimes (\Delta_S^R)^{\kappa\nu} f_2 \right\rangle \end{aligned}$$

Setting $\mathbf{f}' \equiv 1$ on the support of \mathbf{f} we see that the last term is proportional to equations of motion, so we can ignore it. In the remaining terms on the left-hand side we can make use of the fact that $\Delta_S^R(\varphi)$ is the Green's function for $S''_{\mathcal{M}}$. As for the right-hand side, we choose f and f' such that they are equal to 1 on $J^- \text{supp}(f_2) \cap J^+(\text{supp } f_1)$. Now use the fact that F is local, depends locally on both f and f' and the support of $F(f, f')$ is contained within the support of $df \cup df'$. It follows that the term on the right-hand side vanishes and because f_1, f_2 were chosen arbitrarily, we obtain the identity

$$\gamma(\Delta_S^R) \stackrel{\text{o.s.}}{=} (-1)^{|\sigma|} (K_\varphi \circ \Delta_S(\varphi))^{\sigma\alpha} + (\Delta_S(\varphi) \circ K_\varphi^\dagger)^{\sigma\alpha}.$$

The same argument can be applied to $\Delta_S^A(\varphi)$, so the identity (7.16) follows. This concludes the proof. \square

Since $\tilde{\gamma}$ is a derivation with respect to the Peierls bracket modulo the image of $\tilde{\delta}$, $[\cdot, \cdot]_{\tilde{S}}$ is well defined on $H_0(\mathcal{BV}, \tilde{\delta})$ and on $\mathfrak{F}_{\tilde{S}}^{\text{inv}}(\mathcal{M})$. In order to obtain a space which is closed under the Poisson bracket, we extend \mathcal{BV} to the space $\mathcal{BV}_{\mu c}$ of microcausal functionals on $T^*[-1]\bar{\mathcal{E}}$.

The classical net of local algebras on a spacetimes \mathcal{M} is then defined by assignments

$$\mathcal{O} \mapsto (\mathfrak{F}_{S, \mu c}^{\text{inv}}(\mathcal{O}), [\cdot, \cdot]),$$

where $\mathcal{O} \subset \mathcal{M}$.

Remark 7.7. Although the construction of the BV complex has been done covariantly, it is injective only with \mathbf{Vec} as a target category, so might not extend to an injective functor into the category of differential graded algebras. Consequently, the assignment of $(\mathfrak{F}_{S, \mu c}^{\text{inv}}(\mathcal{M}), [\cdot, \cdot])$ does not, in general, give a locally covariant field theory model in the sense of 2.47 on the full category \mathbf{Loc} . This is due to the existence of topological charges, as shown in [BSS14]

Example 7.1 (Electromagnetic field). *Let us illustrate the general construction described above on the example of the electromagnetic field. The gauge group is $G = U(1)$, so $g = \mathbb{R}$ and the Lagrangian takes the form*

$$L_M(f)(A) = -\frac{1}{2} \int_M f(F \wedge *F).$$

\mathcal{E} is the space of principal connections on $M \times U(1)$ and it is an affine space modeled on $\mathfrak{E}_c(\mathcal{M}) = \Omega_c^1(M)$. As in the case of the free field we can consider the space $\mathfrak{F}_{\text{lin}}(\mathcal{M})$ of linear functionals on \mathcal{E} . They are of the form

$$F_\beta(A) = \int_M A \wedge *\beta,$$

We can now apply to $\mathfrak{F}_{\text{lin}}(\mathcal{M})$ the general BV formalism and compare the construction of [Dim92]. The equation of motion is given by

$$\delta dA = 0,$$

so the image of δ_S consists of functionals F_β , where $\beta = \delta d\eta$ for some $\eta \in \Omega_c^1(M)$. We can realize $\mathfrak{F}_{\text{lin},S}(\mathcal{M})$ as the space of equivalence classes of forms

$$\mathfrak{F}_{\text{lin},S}(\mathcal{M}) \cong \frac{\Omega_c^1(M)}{\delta d\Omega_c^1(M)}.$$

Now we have to characterize the kernel of γ . It consists of linear functionals that satisfy

$$0 = (\gamma F_\beta)(c) = \int_M dc \wedge * \beta = \int_M c \wedge * \delta \beta.$$

It follows that $\delta \beta = 0$. Let us denote $\Omega_{c,\delta}^1(M) \doteq \{\omega \in \Omega_c^1(M) \mid \delta \omega = 0\}$. The space of gauge invariant on-shell linear functionals is isomorphic to (compare with [SDH14, DS13, DL12])

$$\mathfrak{F}_{\text{lin},S}^{\text{inv}}(\mathcal{M}) \cong \frac{\Omega_{c,\delta}^1(M)}{\delta d\Omega_c^1(M)}.$$

Among these functionals we can distinguish the ones which are constructed from the field strength, i.e. those of the form

$$\int_M dA \wedge * \eta = \int_M A \wedge * \delta \eta = F_{\delta \eta}(A).$$

If $H^1(M)$ is trivial, then all elements of $\mathfrak{F}_{\text{lin},S}^{\text{inv}}$ arise from field strength functionals, since all co-closed forms are also co-exact.

7.3 Quantization in the Batalin-Vilkoviski formalism

In this section we discuss quantization along the lines of [FR12a]. We start with the discussion of the free scalar field. We consider the deformation of δ_{S_0} under the time-ordering operator \mathcal{T} . This deformation corresponds to the difference between the ideal generated by EOM's in the classical theory (i.e. with respect to “ \cdot ”) and the ideal generated by EOM's with respect to $\cdot_{\mathcal{T}}$. We define

$$\delta_{S_0}^{\mathcal{T}} = \mathcal{T} \circ \delta_{S_0} \circ \mathcal{T}^{-1}, \quad (7.17)$$

Let us first consider regular functionals. Explicit computation shows that, on \mathcal{F}_{reg} ,

$$\delta_{S_0}^{\mathcal{T}} := \mathcal{T} \circ \delta \circ \mathcal{T}^{-1} = \delta + i\hbar\Delta,$$

where Δ acts on vector fields $X \in \mathcal{V}_{\text{reg}}$ as

$$\Delta X(\varphi) = \int \frac{\delta X_x}{\delta \varphi(x)}(\varphi), \quad \text{where } X(\varphi) = \int X_x(\varphi) \frac{\delta}{\delta \varphi(x)},$$

It's remarkable that the operator Δ is “almost” a derivation of $\Lambda\mathcal{V}_{\text{reg}}$ and the failure is characterized by $\{.,.\}$, i.e:

$$\Delta(X \wedge Y) - \Delta(X) \wedge Y - (-1)^{|X|} X \wedge \Delta(Y) = \{X, Y\},$$

The triple $(\Lambda\mathfrak{V}_{\text{reg}}(\mathcal{M}), \{.,.\}, \Delta)$ is an example of a mathematical structure called Beilinson-Drinfeld (BD) algebra [BD04], however in the physics literature it is called a *BV algebra*. The difference between the usage of these two terms lies in the choice of gradings. In the quantization of gauge theories, one simply replaces $\Lambda\mathcal{V}_{\text{reg}}$ with $\mathcal{BV}_{\text{reg}}$ and S_0 is the $\#af = 0$ quadratic term of the extended action. The resulting structure is also a BD algebra.

Remark 7.8. Physically the relation between $\delta_{S_0}^{\mathcal{T}}$ and δ_{S_0} corresponds to the *Schwinger-Dyson equation*. Let $X(\varphi) = \int X_x(\varphi) \frac{\delta}{\delta \varphi(x)}$. We obtain

$$(\delta_{S_0}^{\mathcal{T}} X)(\varphi) = \int \left(X_x \cdot_{\mathcal{T}} \frac{\delta S_0(\varphi)}{\delta \varphi(x)} \right) (\varphi) = \int X_x(\varphi) \frac{\delta S_0(\varphi)}{\delta \varphi(x)} + i\hbar \int \frac{\delta X_x}{\delta \varphi(x)}(\varphi),$$

where $\frac{\delta S_0(\varphi)}{\delta \varphi(x)}$ is a shorthand notation for $\frac{\delta L_0(f)(\varphi)}{\delta \varphi(x)}$, where we take the limit $f \rightarrow 1$. Note that

$$\int X_x(\varphi) \frac{\delta S_0(\varphi)}{\delta \varphi(x)} = \int \left(X_x \star \frac{\delta S_0}{\delta \varphi(x)} \right) (\varphi).$$

Hence,

$$\int \left(X_x \cdot_{\mathcal{T}} \frac{\delta S_0(\varphi)}{\delta \varphi(x)} \right) (\varphi) = i\hbar \int \frac{\delta X_x}{\delta \varphi(x)}(\varphi),$$

modulo the \star -ideal generated by the EOM's, which is exactly the algebraic Schwinger-Dyson equation.

Let us now consider a deformation of δ_{S_0} which corresponds to introducing the interaction. Here we treat the scalar field and gauge theories

together, but still restrict to regular functionals, i.e. $V \in \mathcal{BV}_{\text{reg}}$ is a regular interaction term. We define the *quantum BV operator* \hat{s} as

$$\hat{s} \doteq R_V^{-1} \circ \delta_{S_0} \circ R_V. \quad (7.18)$$

Analogously to section 6.2.3, R_V is a map from the interacting quantum theory to the free quantum theory, so δ_{S_0} acts on \mathfrak{A}_0 , as explained in detail in section 5.2. Note that $V \in \mathcal{BV}$ is physical if R_V is in $\ker(\delta_{S_0})$ and is identified with elements that differ by the image of \hat{s} , i.e. $R_V(F)$ is in the same equivalence class as $R_V(F) + R_V(I)$, where $I \in \text{Im}(\delta_0)$. This is the same equivalence relation as the one used in section 5.2 to define the space of on-shell observables. The difference is that now the kernel of δ_{S_0} is not the full space of functionals, but only a subspace, since δ_{S_0} has non-trivial action on antifields. In fact, we will see later on that in the renormalized theory V (and hence also $R_V(F)$) necessarily contains antifields.

Let us assume that

$$\delta_{S_0}(e_{\mathcal{T}}^{iV/\hbar}) = 0. \quad (7.19)$$

This condition reduces to the known *quantum master equation* QME, since

$$\delta_{S_0}(e_{\mathcal{T}}^{iV/\hbar}) = \frac{1}{2}\{S_0 + V, S_0 + V\} - i\hbar \Delta V, \quad (7.20)$$

where we set $\{S_0, S_0\} \equiv 0$, since S_0 doesn't contain antifields. If (7.19) holds, then

$$\hat{s}F = \{F, S_0 + V\} - i\hbar \Delta F, \quad (7.21)$$

for $F \in \mathfrak{F}_{\text{reg}}(\mathcal{M})$. If QME is fulfilled, then the cohomology of \hat{s} characterizes the space of *quantum gauge invariant on-shell observables*.

Now we want to extend the QME and \hat{s} to local functionals. This is done by renormalizing the time-ordered products present in (7.18) and the (7.19) using the Epstein-Glaser framework. Clearly, formulas (7.20) and (7.25) are not well defined for local arguments, since Δ is singular. Nevertheless, very similar results can be obtained using the anomalous Master Ward Identity ([BD08, Hol08]), which states that there exists a family of maps

$$\tilde{\Delta}^n : \mathcal{T}(\mathcal{BV}_{\text{loc}})^{n+1} \rightarrow \mathfrak{A}_{\text{loc}}(M), \quad (7.22)$$

which depend locally on their arguments and the formal power series

$$\tilde{\Delta}(V) \doteq \sum_{n=0}^{\infty} \tilde{\Delta}^n(V^{\otimes n}; V),$$

fulfills the identity

$$\int \left(e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} \frac{\delta V}{\delta \varphi(x)} \right) \star \frac{\delta S_0}{\delta \varphi(x)} = e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} \left(\frac{1}{2} \{V + S_0, V + S_0\}_{\mathcal{T}} - \tilde{\Delta}(V) \right), \quad (7.23)$$

The maps $\tilde{\Delta}^n$ can be determined recursively. For an explicit formula, see [BD08, Rej13]. We can now see that the renormalized QME reduces to

$$\delta_{S_0}(e_{\mathcal{T}}^{iV/\hbar}) = \frac{1}{2} \{S_0 + V, S_0 + V\} - i\hbar \tilde{\Delta}(V). \quad (7.24)$$

and the renormalized quantum BV operator takes the form

$$\hat{s}F = \{F, S_0 + V\} - i\hbar \Delta_V F, \quad (7.25)$$

where $\Delta_V(F) \doteq \frac{d}{d\lambda} \Big|_{\lambda=0} \tilde{\Delta}(V + \lambda F)$. Note that the renormalized operator Δ_V depends on V , in contrast to the non-renormalized one Δ . It has no longer the interpretation of a graded Laplacian, but is still a functional differential operator.

The construction outlined above allows us to choose V which is a local functional. In the Yang-Mills theory example, which we are focusing on, we can define the local interacting net corounded region $\mathcal{O} \subset \mathcal{M}$ by taking $V = \tilde{L}_{\mathcal{M}}^{\text{ext}}(f) - L_{0\mathcal{M}}(f)$ and choosing $f \equiv 1$ on \mathcal{O} . The details of the construction, in particular the algebraic adiabatic limit, can be found in [FR12a].

Remark 7.9. The remarkable aspect of the approach presented above is that using the BV formalism in the sense of [FR12a] we can construct Møller maps R_V that intertwine between theories with *different gauge symmetries*. Indeed, the symmetries are encoded in the Chevalley-Eilenberg differential γ , which then enters into the extended action S^{ext} and consequently into V .

Chapter 8

Effective quantum gravity

The functional approach to pQFT together with the BV framework introduced in chapter 7 has been successfully applied to gauge theories [FR12b, FR12a] and can also be used in quantization of theories where the local symmetries involve transformation of spacetime points. The first model, where this has been achieved was the quantization of a bosonic string in arbitrary dimension, where the local symmetries are compactly supported diffeomorphisms of the string world-sheet. Recently these methods have also been applied to construct the effective theory of quantum gravity [BFR13]. By this we mean a quantum theory understood in terms of formal power series in the both \hbar and the coupling constant, where at each order of the perturbative construction new types of contributions appear, but they are always finite. The theory is not UV complete, so the the relevant formal power series do not converge, but one can, nevertheless apply it to model physical situations where the quantum gravity (QG) effects are not very strong. This seems to be a sensible Ansatz for the start, as the QG effects which we expect to observe in the nearest future should be relatively small. Example physical applications include cosmology and black hole physics.

8.1 From LCQFT to quantum gravity

The road to quantum gravity is paved with numerous technical and conceptual problems. In contrast to QFT on curved spacetimes, in QG the spacetime structure is dynamical. This means that we cannot treat the metric as a fixed structure, but it interacts with the matter field. One can partially model this situation using the framework involving *backreaction*. In this formalism one treats matter fields as quantum objects and studies

their effect of the metric by inserting the expectation value of the quantum stress-energy tensor in a given state ω into Einstein's equations:

$$\langle T_{\mu\nu} \rangle_\omega = G_{\mu\nu},$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor. In the pAQFT framework this approach has been applied in cosmology and in the study of QFT in black-hole spacetimes, see for example [Hac10, Hac14, DMP09b, DMP09a, DMP11] and a recent book [Hac15].

On the next level of approximation one can split the metric g into the background metric g_0 and a perturbation h and quantize the the perturbation as a quantum field on the background g_0 . This is the approach which has been taken in [BFR13]. Since this tentative split into background and perturbation metric is not physical, one needs to show that the predictions of the theory do not depend on the way g is being split. Such consistency condition is called *background independence* and we will come back to it later in this section. In the pAQFT approach, the background independence of effective QG has been proven in [BFR13] in the sense that a localized change in the background which formally yields an automorphism on the algebra of observables (called relative Cauchy evolution in [BFV03]) is actually trivial, in agreement with the proposal made in [BF07] (see also [FR11]).

Another conceptual difficulty in quantizing gravity is that the Einstein-Hilbert action is reparametrization invariant, hence the theory has a huge symmetry group, the diffeomorphism group. This means that labeling of spacetime points doesn't have a physical meaning. As a consequence, physical observables have to be diffeomorphism invariant. In the framework of [BFR13] the characterization of diffeomorphism invariant observables is given by means of the BV formalism. The abstract setting looks very similar to the one discussed in 7, but there remains the difficulty in finding non-trivial elements of the 0-th cohomology of the BV differential. In [FR12b, Rej11a] it has been proposed to use locally covariant quantum fields in the sense of definition 2.51 as diffeomorphism invariant physical quantities in effective classical and quantum gravity. In [BFR13] this idea has been refined and a more explicit characterisation of these objects has been given in terms of *relational observables*. The latter are conceptually similar to the notion of observables introduced by Rovelli in the framework of loop quantum gravity [Rov02] and later used and further developed in [Dit06, Thi06].

Finally, there is a known difficulty that quantum gravity, as a QFT, is power counting non-renormalizable. We deal with this problem by using the Epstein-Glaser renormalization scheme, which allows to calculate finite contributions to renormalized time-ordered products in every order in \hbar and the

coupling constant. The theory is then interpreted as an effective theory with the property that only finitely many parameters have to be considered below a fixed energy scale [GW96]. Another possible direction would be to make contact with the *asymptotic safety* approach. A theory is called asymptotically safe if there exists an ultraviolet fixed point of the renormalisation group flow with only finitely many relevant directions [Wei79]. Results supporting this perspective have been obtained by Reuter et al. [Reu98, RS02].

8.2 Dynamics and symmetries

We consider a formal metric g on a M given by $g = g_0 + \lambda h$, where λ is a formal parameter. In this setting $\mathcal{M} \equiv (M, g_0)$ is the background manifold and h is the metric perturbation. For the effective theory of gravity the configuration space is $\mathcal{E}(\mathcal{M}) = \Gamma((T^*M)^{\otimes 2})$.

Definition 8.1. *In this section $\mathcal{F}(\mathcal{M})$ denotes the space of multilocal functionals on $\mathcal{E}(\mathcal{M})$ that are Laurent series in λ .*

Note that for physical interpretation to make sense, the observables we obtain at the end cannot depend on negative powers of λ .

Definition 8.2. *A functional derivative of $F \in \mathcal{F}(\mathcal{M})$ is defined by*

$$\langle F^{(1)}(h), h_1 \rangle \doteq \frac{1}{\lambda} \left\langle \frac{\delta}{\delta h} F(h), h_1 \right\rangle,$$

where $\frac{\delta}{\delta h} F$ means that we take the functional derivative of coefficients of F at each order in λ separately.

It is convenient to use the natural units, where λ (identified with the square of the coupling constant) has a dimension of length, so h has a dimension of 1/length. The action used in quantization must be dimensionless, so we use L^{EH}/λ^2 , where L^{EH} is the Einstein-Hilbert Lagrangian

$$L_{(M,g)}^{EH}(f)[h] \doteq \int R[g] f \, d\mu_g, \quad g = g_0 + \lambda h, \quad h \in \mathcal{E}(\mathcal{M}), \quad (8.1)$$

The Euler-Lagrange derivative of L^{EH} is defined as in 4.2, i.e.

$$\langle S_M^{EH'}(h), h_1 \rangle \doteq \left\langle L_M^{EH}(f)^{(1)}(h), h_1 \right\rangle.$$

For general relativity local symmetries arise from infinitesimal diffeomorphism symmetries. The compactly supported diffeomorphism group acts on

$\mathcal{E}(\mathcal{M})$ via the pullback: $\rho_{\mathcal{M}}(\alpha)h = (\alpha^{-1})^*h$, where $\alpha \in \text{Diff}_c(\mathcal{M})$, $t \in \mathcal{E}(\mathcal{M})$. This induces the action of $\mathfrak{X}(\mathcal{M}) \equiv \Gamma(TM)$ via the Lie derivative:

$$\rho_{\mathcal{M}}(\xi)\lambda h \doteq \left. \frac{d}{dt} \right|_{t=0} (\exp(-t\xi))^*g = \mathcal{L}_{\xi}(g_0 + \lambda h),$$

where $\xi \in \mathfrak{X}(\mathcal{M})$. On the level of functionals $F \in \mathcal{F}$ we obtain

$$\rho_{\mathcal{M}}(\xi)F[h] \doteq \left\langle F^{(1)}(h), \mathcal{L}_{\xi}(g + \lambda h) \right\rangle.$$

Definition 8.3. $\mathcal{CE}(\mathcal{M})$, the underlying algebra of the Chevalley-Eilenberg complex, is defined the space of Laurent series in λ with coefficients in multilocal functionals on the extended configuration space $\bar{\mathcal{E}}(\mathcal{M}) \doteq \mathcal{E} \oplus \mathfrak{X}[1](\mathcal{M})$. The Chevalley-Eilenberg differential γ_{CE} is defined as in section 7.1.3.

Definition 8.4. $\mathcal{BV}(\mathcal{M})$, the underlying algebra of the BV complex is defined as $T^*[-1]\bar{\mathcal{E}}(\mathcal{M})$ in the sense of (7.7).

As in gauge theories, $\mathcal{BV}(\mathcal{M})$ is equipped with the natural structure of the Schouten bracket $\{.,.\}$ and the BV differential s_{BV} can be expressed as

$$s_{BV} = \{., S^{EH} + \gamma\},$$

where we choose the natural Lagrangian θ , which represents γ as

$$\theta_{(M,g_0)}^{CE}(f)[h, c] = \left\langle \frac{\delta}{\delta g}, \mathcal{L}_{fc}g \right\rangle + \frac{1}{2} \left\langle \frac{\delta^r}{\delta c}, f[c, c] \right\rangle, \quad (8.2)$$

The space of ‘‘gauge-invariant’’ on-shell observables, $\mathcal{F}_S^{\text{inv}}$ is characterized by

$$\mathcal{F}_S^{\text{inv}}(\mathcal{M}) = H^0(\mathcal{BV}(\mathcal{M}), s_{BV}).$$

In the next step we perform the gauge-fixing. In general relativity fixing the gauge means essentially fixing the coordinate system. For the specific choice of gauge we need, we have to extend the BV complex by adding auxiliary scalar fields: 4 scalar antighosts \bar{c}_{μ} in degree -1 and 4 scalar Nakanishi-Lautrup fields b_{μ} , $\mu = 0, \dots, 3$ in degree 0. The new extended configuration space is again denoted by $\bar{\mathcal{E}}(\mathcal{M})$. We define s on functionals of antighosts and Nakanishi-Lautrup fields by fixing the action of s on evaluation functionals $\bar{C}_{\mu}(x)$, $B_{\mu}(x)$:

$$\begin{aligned} s(\bar{C}_{\mu}) &= iB_{\mu} - \mathcal{L}_C \bar{C}_{\mu}, \\ s(B_{\mu}) &= \mathcal{L}_C B_{\mu}. \end{aligned}$$

To implement these new transformation laws we need to add to the Lagrangian a term

$$\left\langle \frac{\delta^r}{\delta \bar{c}_\mu}, i f b_\mu - \mathcal{L}_{f c} \bar{c}_\mu, \right\rangle + \left\langle \frac{\delta^r}{\delta b_\mu}, \mathcal{L}_{f c} b_\mu, \right\rangle.$$

Next, we perform an automorphism α_Ψ of $(\mathcal{BV}(\mathcal{M}), \{.,.\})$ such that the part of the transformed action which doesn't contain antifields has a well posed Cauchy problem. We choose the gauge-fixing Fermion as

$$\Psi_{(M, g_0)}(f)[h, c, \bar{c}_\mu, b_\mu] = i \sum_{\mu, \nu} \int (\partial_\mu \bar{c}_\nu g^{\mu\nu} - \frac{1}{2} b_\mu \bar{c}_\nu \kappa^{\mu\nu}) f d\mu_g, \quad (8.3)$$

where $g = g_0 + \lambda h$ and κ is a non-degenerate 2-form on \mathbb{R}^4 . The explicit appearance of this form in the gauge fixing Fermion is related to the choice of a dual pairing for Nakanishi-Lautrup fields. Note that expression (8.3) is explicitly coordinate-dependent. This is necessary, because we need to brake the reparametrization invariance of the action. The new terms appearing in the α_Ψ -transformed action arise from

$$\begin{aligned} \{\Psi(f'), L^{\text{ext}}(f)\}(h, c, \bar{c}_\mu, b_\mu) &= - \int (\partial_\mu (f b_\nu) g^{\mu\nu} - \frac{1}{2} f_2 b_\mu b_\nu \kappa^{\mu\nu}) \sqrt{-\det g} d^4 x + \\ &+ i \int (\partial_\mu \bar{c}_\nu \sqrt{-\det g} g^{\mu\alpha} \partial_\alpha (f c^\nu)) d^4 x, \end{aligned}$$

where $f' \equiv 1$ on $\text{supp } f$. We rewrite the above expression as

$$- \int \partial_\mu (f b_\nu) g^{\mu\nu} d\mu_g + \int (\frac{1}{2} f b_\mu b_\nu) \kappa^{\mu\nu} d\mu_g + i \int c^\nu f \square_g \bar{c}_\nu d\mu_g,$$

where \square_g is the d'Alembertian constructed from the formal metric $g = g_0 + \lambda h$. We denote the first term in the above formula by $L^{GF}(f)$ and the second by $L^{FP}(f)$ (gauge-fixing and Fadeev-Popov terms, respectively). The full transformed Lagrangian is given by:

$$L^{\text{ext}} = L^{EH} + L^{GF} + L^{FP} + L^{AF}, \quad (8.4)$$

where L^{AF} is the term containing antifields.

The variables of the theory (i.e. the components φ^α of the multiplet $\varphi \in \bar{\mathcal{E}}(\mathcal{M})$) are now: the metric $g \in \mathfrak{E}(\mathcal{M})$, the Nakanishi-Lautrup fields b_μ and the antighosts \bar{c}_μ , $\mu = 0, \dots, 3$ (scalar fields), ghosts $c \in \mathfrak{X}(\mathcal{M})$. As in 7.2 we introduce a new grading $\#ta$, which is equal to 0 for functions on $\bar{\mathcal{E}}(\mathcal{M})$ and equal to 1 for all the vector fields on $\bar{\mathcal{E}}(\mathcal{M})$.

The new field equations are now equations for the full multiplet $\varphi = (g, b_\mu, c, \bar{c}_\mu)$, $\mu = 0, \dots, 3$ and are derived from the $\#ta = 0$ term of L^{ext} , denoted by L . The α_Ψ -transformed BV differential $s = \alpha_\Psi \circ s_{BV} \circ \alpha_\Psi^{-1}$ is given by:

$$s = \{., S^{\text{ext}}\} = \gamma + \delta.$$

The action of γ on $\mathcal{F}(\mathcal{M})$ and the evaluation functionals B_μ, C, \bar{C}_μ is summarized in the table below:

	γ
$F \in \mathcal{F}$	$\langle F^{(1)}, \mathcal{L}_C g \rangle$
C	$-\frac{1}{2}[C, C]$
B_μ	$\mathcal{L}_C B_\mu$
\bar{C}_μ	$iB - \mathcal{L}_C \bar{C}_\mu$

The equations of motion are:

$$R_{\lambda\nu}[g] = -i\partial_\lambda \bar{C}_\alpha \partial_\nu C^\alpha - \partial_{(\lambda} B_{\nu)} \quad (8.5)$$

$$\square_g C^\mu = 0 \quad (8.6)$$

$$\square_g \bar{C}_\mu = 0 \quad (8.7)$$

$$\frac{1}{\sqrt{-\det g}} \partial_\mu (\sqrt{-\det g} g^{\mu\nu}) = B_\mu \kappa^{\mu\nu} \quad (8.8)$$

where $B_\mu, C^\mu, \bar{C}_\mu$ are evaluation functionals. The equation for B^μ is obtained by using the Bianchi identity satisfied by $R_{\lambda\nu}[g]$ in equation (8.5) and takes the form

$$\square_g B_\mu = 0. \quad (8.9)$$

8.3 Linearized theory

Definition 8.5. *The linearized Lagrangian L_0 is defined as*

$$L_0 \doteq \frac{\lambda^2}{2} L^{(2)}_{(M, g_0)}(g_0, 0, 0, 0),$$

where L is the $\#af = 0$ term in L^{ext} . We introduce the notation $S^{\text{ext}} = \lambda^2(S_0 + S_I)$.

Remark 8.1. The dimensionless action we use in the quantization is S^{ext}/λ^2 , so to quadratic part is of order 0 in λ . If g_0 is not a solution to Einstein's equations, the λ -linear term in S^{ext} doesn't vanish and negative

powers of λ appear in the interaction S_I . Formally, we solve this problem by introducing another parameter μ , so that $\frac{1}{\lambda}L_{(M,g_0)}^{(1)}(g_0, 0, 0, 0) \equiv \mu J_{g_0}$, where J_{g_0} is the source term, linear in h . Our observables are now formal power series in both λ and μ . For the physical interpretation we restrict ourselves to spacetimes where g_0 is a solution and put $\mu = 0$, but algebraically we can perform our construction of quantum theory on arbitrary backgrounds.

We choose from now on the gauge with $\kappa = g_0$. Let us introduce some notation.

Definition 8.6. *The divergence operator $\text{div} : \Gamma(S^2T^*M) \rightarrow \Gamma(T^*M)$ is defined by*

$$(\text{div}t)_\alpha \doteq \frac{1}{\sqrt{-\det g_0}} g_0^{\beta\mu} \partial_\mu (t_{\beta\alpha} \sqrt{-\det g_0}).$$

Definition 8.7. *We define a product*

$$\langle u, v \rangle_{g_0} = \int_M \langle u^\#, v \rangle d\mu_{g_0},$$

where u, v are tensors of the same rank and $\#$ is the isomorphism between T^*M and TM induced by g_0 .

Definition 8.8. *The formal adjoint of div with respect to the product $\langle \cdot, \cdot \rangle_{g_0}$ is denoted by $\text{div}^* : \Gamma(T^*M) \rightarrow \Gamma(S^2T^*M)$.*

Definition 8.9. *The trace reversal operator $G : (TM)^{\otimes 2} \rightarrow (TM)^{\otimes 2}$ is defined by*

$$Gt = t - \frac{1}{2}(\text{tr}t)g_0. \quad (8.10)$$

We have $\text{tr}(Gt) = -\text{tr}t$ and $G^2 = \text{id}$. Using the notation above we write $L_{0\mathcal{M}}$ in the form:

$$\begin{aligned} L_{0\mathcal{M}}(f)[h, c, \bar{c}_\mu, b_\mu] &= \\ &= \int_M \frac{\delta}{\delta g} (Rfd\mu) \Big|_{g_0} (h) + 2i \sum_{\nu=0}^3 \langle d\bar{c}_\nu, d(fc^\nu) \rangle_{g_0} + \langle fb, \text{div}(Gh) - \frac{1}{2}b \rangle_{g_0}, \end{aligned}$$

where $\frac{\delta}{\delta g} (Rd\mu) \Big|_{g_0} (h)$ denotes the linearization of the Einstein-Hilbert Lagrangian density around the background g_0 and b is a 1-form on M con-

structed from b_ν 's in the fixed coordinate system. The linearized EOM's are

$$S''_{\mathcal{M}}(z, x) = \delta(z, x) \begin{pmatrix} -\frac{1}{2}(\square_L G + 2G \operatorname{div}^* \circ \operatorname{div} \circ G) & G \circ \operatorname{div}^* & 0 & 0 \\ \operatorname{div} \circ G & -1 & 0 & 0 \\ 0 & 0 & 0 & -i\square_H \\ 0 & 0 & 0 & i\square_H \end{pmatrix} (x), \quad (8.11)$$

where the variables are $(h, b, c^0, \dots, c^3, \bar{c}_0, \dots, \bar{c}_3)$; $\square_H = \delta d$ is the Hodge Laplacian, $\delta \doteq *^{-1} d *$ is the codifferential and \square_L is given in local coordinates by

$$(\square_L h)_{\alpha\beta} = \nabla^\mu \nabla_\mu h_{\alpha\beta} - 2(R_{(\alpha}{}^\mu h_{\beta)\mu} + R_{(\alpha}{}^{\mu\nu} h_{\mu\nu)}). \quad (8.12)$$

The retarded and advanced propagators for S_0 are given by:

$$\Delta^{A/R}(x, y) = -2 \begin{pmatrix} G\Delta_t^{A/R} & G\Delta_t^{A/R} G \circ \operatorname{div}_y^* & 0 & 0 \\ \operatorname{div}_x \circ \Delta_t^{A/R} & \operatorname{div}_x \circ \Delta_t^{A/R} G \circ \operatorname{div}_y^* + \frac{1}{2}\delta_4 & 0 & 0 \\ 0 & 0 & 0 & -i\Delta_s^{A/R} \\ 0 & 0 & i\Delta_s^{A/R} & 0 \end{pmatrix},$$

where δ_4 denotes the Dirac delta in 4 dimensions and subscript y in div_y^* means that the operator should be applied on the second variable. We introduce the Peierls bracket on $\mathcal{BV}(\mathcal{M})$:

$$[F, G]_{g_0} = \sum_{\alpha, \beta} \left\langle \frac{\delta^l F}{\delta \varphi^\alpha}, \Delta^{\alpha\beta} \frac{\delta^r G}{\delta \varphi^\beta} \right\rangle,$$

where $\Delta = \Delta^R - \Delta^A$. As in section 7 we extend $\mathcal{BV}(\mathcal{M})$ to the space $\mathcal{BV}_{\mu c}(\mathcal{M})$ of microcausal functionals, which is closed under the Peierls bracket.

8.4 Quantization

For the definition of the \star -product we need a 2-point function Δ_S^+ . Assume that Δ_S^+ is of the form:

$$\Delta_S^+ = -2 \begin{pmatrix} G\omega_t & \omega_t^T \operatorname{div}_y^* & 0 & 0 \\ \operatorname{div}_x \omega_t & \operatorname{div}_x G \omega_t^T \operatorname{div}_y^* & 0 & 0 \\ 0 & 0 & 0 & -i\omega_\nu \\ 0 & 0 & i\omega_\nu & 0 \end{pmatrix}, \quad (8.13)$$

In this case, the conditions for Δ_S^+ to be a Hadamard 2-point function reduce to:

$$\omega_{v/t}(x, y) - \omega_{v/t}(y, x) = i\Delta_{v/t}(x, y), \quad (8.13a)$$

$$\square_L \omega_t = 0, \quad \square_H \omega_v = 0, \quad (8.13b)$$

$$\text{WF}(\omega_{v/t}) \subset C_+, \quad (8.13c)$$

$$\overline{\omega_{v/t}(x, y)} = \omega_{v/t}(y, x). \quad (8.13d)$$

We choose arbitrary parametrices ω_t, ω_v of \square_L and \square_H respectively. Their existence was already proven in [?] (the paper actually discusses general wave operators acting on vector-valued field configurations). Now, from a parametrix, one can construct a bisolution using a following argument: let ω be a Hadamard parametrix and by O we denote the hyperbolic operator from (??), so $O_x \omega = h$, $O_y \omega = k$, hold for some smooth functions h and k . Let χ be a smooth function such that $\text{supp } \chi$ is past-compact and $\text{supp}(1 - \chi)$ is future-compact. Define

$$G_\chi \doteq \Delta^R \chi + \Delta^A(1 - \chi).$$

Clearly G_χ is a right inverse for O . A Hadamard bisolution ω_χ can be now obtained as

$$\omega_\chi \doteq (1 - G_\chi O) \circ \omega \circ (1 - O G_\chi^T).$$

From Hadamard solutions for \square_L and \square_H we can then construct Δ_S^+ using (8.13).

We define $\mathfrak{A}(\mathcal{M})$ as in section 5.1 and introduce the interaction using the Epstein-Glaser renormalization. The rest of the construction follows exactly the scheme described in section 6, so the abstract net of algebras can be defined without problems on arbitrary backgrounds. There are two questions that remain. First is the existence of non-trivial gauge invariant observable and the other is the background independence of the resulting theory. We will address these problems in two following subsections, referring to the results of [BFR13].

8.5 Relational observables

Abstractly we have characterized the classical gauge-invariant observables as

$$\mathcal{F}_S^{\text{inv}}(\mathcal{M}) = H^0(\mathcal{BV}(\mathcal{M}), s),$$

but there is no a priori reason for this space to be non-empty. To prove that non-trivial observables exist, we will construct them explicitly. We start with

a heuristic reasoning. If we think about an experiment that locally probes the geometric structure of spacetime, we can associate to our setup a causally convex spacetime region \mathcal{O} of spacetime \mathcal{M} and an observable Φ localised in \mathcal{O} , which we measure. Since the experiment has a finite resolution, we don't really measure values of the geometric data at a point. There is always some smearing involved. For example, in case of the Ricci curvature R we can model it by defining our observable quantity as $\Phi(f) = \int f(x)R(x)$, where f is the smearing function with $\text{supp}(f) \subset \mathcal{O}$. In certain situations, we can think of the measured observable as a perturbation of the fixed background metric. This is for example the case if we want to observe gravitational waves. We make a tentative split: $g = g_0 + \lambda h$. The situation is pictured on the figure 8.1. To formulate what the diffeomorphism invariance means, we first have to answer the question: what happens if we move our experimental setup to a different region \mathcal{O}' ? Now to compare $\Phi_{(\mathcal{O},g)}(f)$ and $\Phi_{(\mathcal{O}',\alpha_*g)}(\alpha_*f)$ we

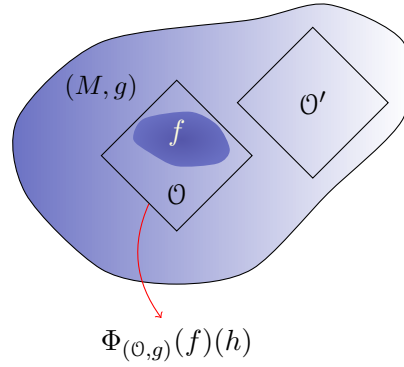


Figure 8.1: Experimental situation while probing the spacetime geometry.

need to know what does it mean to have “the same observable in a different region”. We can give sense to this statement using the notion of locally covariant quantum fields, as defined in 2.51.

Let Φ be a natural transformation from \mathfrak{D} to $\mathfrak{F}_{\text{loc}}$. The condition for Φ to be a natural transformation reads

$$\Phi_{\mathcal{O}}(f)(\chi^*h) = \Phi_M(\chi_*f)(h). \quad (8.14)$$

For a fixed spacetime \mathcal{M} we define the action of infinitesimal diffeomorphism algebra $\mathfrak{X}(\mathcal{M})$ on maps $\Phi_{\mathcal{M}}$ as

$$(\rho(\xi)\Phi_{(M,g_0)})(f)[h] \doteq \left\langle (\Phi_{(M,g_0)}(f))^{(1)}(h), \mathcal{L}_\xi g \right\rangle + \Phi_{(M,g_0)}(\mathcal{L}_\xi f)[h],$$

where $\xi \in \mathfrak{X}(\mathcal{M})$.

Definition 8.10. *We say that a locally covariant quantum field $\Phi_{\mathcal{M}}$ is diffeomorphism invariant if*

$$\rho(\xi)\Phi_{\mathcal{M}} \equiv 0, \quad \forall \mathcal{M} \in \text{Obj}(\mathbf{Loc}), \xi \in \mathfrak{X}(\mathcal{M}).$$

Example 8.1. *As an example of a diffeomorphism invariant field we can take*

$$\Phi_{1(M,g_0)}(f)[h] = \int R[g]f \, d\mu_g, \quad \text{where } g = g_0 + \lambda h.$$

Note that both the scalar curvature and the volume form depend on the full metric \tilde{g} . However if we take a field defined as

$$\Phi_{2(M,g_0)}(f)[h] = \int R[g]f \, d\mu_{g_0}, \quad \text{where } g = g_0 + \lambda h,$$

it is still a locally covariant quantum field, but it is no longer diffeomorphism invariant.

The reasoning presented above suggests that locally covariant quantum fields are good candidates for diffeomorphism invariant quantities. The question remains, how to relate these with non-trivial elements of $H^0(\mathcal{BV}(\mathcal{M}), s)$.

For a fixed spacetime \mathcal{M} and a locally covariant quantum field Φ , a test function specifies the geometrical setup for an experiment, and the concrete choice of $f \in \mathfrak{D}(\mathcal{M})$ can be made only if we fix a coordinate system. In our framework, following [BFR13], we realize the choice of a coordinate system by introducing four scalar fields X^μ , which will parametrize points of spacetime. We can write any test function $f \in \mathfrak{D}(\mathcal{M})$ in the coordinate basis induced by $X \doteq (X^\mu | \mu = 0, \dots, 3)$, so if we fix $\mathbf{f} : \mathbb{R}^4 \rightarrow \mathbb{R}$, then the change of $f = X^* \mathbf{f}$ due to the change of the coordinate system is realized through the change of scalar fields X^μ .

Definition 8.11. *For a natural transformation $\Phi \in \text{Nat}(\mathfrak{D}, \mathfrak{F})$ we obtain a map*

$$\Phi_{\mathcal{M}, \mathbf{f}}(h, X) \doteq \Phi_{\mathcal{M}}(X^* \mathbf{f})(h).$$

As long as we keep \mathcal{M} fixed, we drop \mathcal{M} in $\Phi_{\mathcal{M}, \mathbf{f}}$ and use the notation $\Phi_{\mathbf{f}}$ instead. $\Phi_{\mathbf{f}}$ is a function of the metric and the coordinate system and transforms under the infinitesimal diffeomorphisms according to

$$(\rho(\xi)\Phi_{\mathbf{f}}) = \left\langle \frac{\delta \Phi_{\mathbf{f}}}{\delta g} \Big|_X, \mathcal{L}_\xi g \right\rangle + \sum_{\mu=0}^3 \left\langle \frac{\delta \Phi_{\mathbf{f}}}{\delta X^\mu} \Big|_g, \mathcal{L}_\xi X^\mu \right\rangle. \quad (8.15)$$

This is still not satisfactory, since X^μ 's are not dynamical variables, so there are no vector fields in $\mathcal{BV}(\mathcal{M})$ which would implement the second term in the above transformation. To solve this problem, we can replace X^μ with some scalars X_g^μ , $\mu = 0, \dots, 3$, which depend locally on the metric. They could be, for example, scalars constructed from the Riemann curvature tensor and its covariant derivatives. The caveat is that some particularly symmetric spacetimes do not admit such metric dependent coordinates, since in such cases the curvature might vanish (for a detailed discussion see [CHP09, HC10]). If matter fields are present, one can construct X^μ 's using the matter fields. A known example is the Brown-Kuchař model [?], which uses dust fields.

Let us denote by β the map $g \mapsto (X_g^0, \dots, X_g^3)$ and we define

$$\Phi_{\mathbf{f}}^\beta(h) \doteq \Phi_{\mathbf{f}}(g, X_g), \quad (8.16)$$

where $g = g_0 + \lambda h$. Note that for (8.16) to be well defined we need to choose \mathbf{f} and β in such a way that the support of \mathbf{f} is contained in the interior of the image of M inside \mathbb{M} under the quadruple of maps $X_{g_0}^\mu$. If this can be done, then a functional of the form (8.16) is an element of $H^0(\mathcal{BV}(\mathcal{M}), s)$ if and only if Φ is a diffeomorphism invariant locally covariant quantum field. Observables of this type are interpreted as relational observables, since they capture the relations between different quantities constructed from the metric (and possibly also matter fields) and they do not rely on absolute labeling of spacetime points. Instead, the map β provides relative labels X_g , which change with g .

Example 8.2. *Assume that for a fixed background $\mathcal{M} = (M, g_0)$ we can choose \mathbf{f} and β in such a way that $f \doteq X_{g_0}^* \mathbf{f}$ is compactly supported. Then an example $\mathfrak{X}(\mathcal{M})$ -invariant functional can be obtained from the scalar curvature*

$$\begin{aligned} \Phi_{\mathbf{f}}^\beta(h) &= \int_M R[g_0] \mathbf{f}(X_{g_0}) d\mu_{g_0} + \\ &+ \lambda \left(\int_M \mathbf{f}(X_{g_0}) \frac{\delta}{\delta g} (Rd\mu) \Big|_{g_0} (h) + \int_M R[g_0] \partial_\mu \mathbf{f}(X_{g_0}) \frac{\delta X_g^\mu}{\delta g} \Big|_{g_0} (h) \right) + \mathcal{O}(\lambda^2). \end{aligned}$$

To summarize, we have three ways to realize diffeomorphism invariant quantities in classical gravity:

- as locally covariant fields $\Phi_{\mathcal{M}} : \mathfrak{D}(\mathcal{M}) \rightarrow \mathfrak{F}(\mathcal{M})$,
- as functionals of the metric and the coordinates $\Phi_{\mathbf{f}}(h, X)$,
- as relational observables $\Phi_{\mathbf{f}}(\cdot, X_g)$.

8.6 Background independence

The last issue which we have to discuss in the background independence. We have made a tentative split into the free and interacting Lagrangian, relying on the Taylor expansion around the background metric g_0 . Now we want to see what will happen if we slightly perturb the background. If the theory is background independent then physical quantities do not change under such a perturbation. Following [BFR13] we sketch the argument that this is true in effective QG constructed by the methods of pAQFT.

In [?] it was proposed that a condition of background independence can be formulated by means of the relative Cauchy evolution. Let us fix a spacetime $\mathcal{M}_1 = (M, g_1) \in \text{Obj}(\mathbf{Loc})$ and choose Σ_- and Σ_+ , two Cauchy surfaces in \mathcal{M}_1 , such that Σ_+ is in the future of Σ_- . Consider another globally hyperbolic metric g_2 on M , such that $k \doteq g_2 - g_1$ is compactly supported and its support K lies between Σ_- and Σ_+ . Let us take $\mathcal{N}_\pm \in \text{Obj}(\mathbf{Loc})$ that embed into $\mathcal{M}_1, \mathcal{M}_2$, via $\chi_{1\pm}, \chi_{2\pm}$ and $\chi_{i\pm}(\mathcal{N}_\pm)$, $i = 1, 2$ are causally convex neighborhoods of Σ_\pm in \mathcal{M}_i . We use the time-slice axiom to define

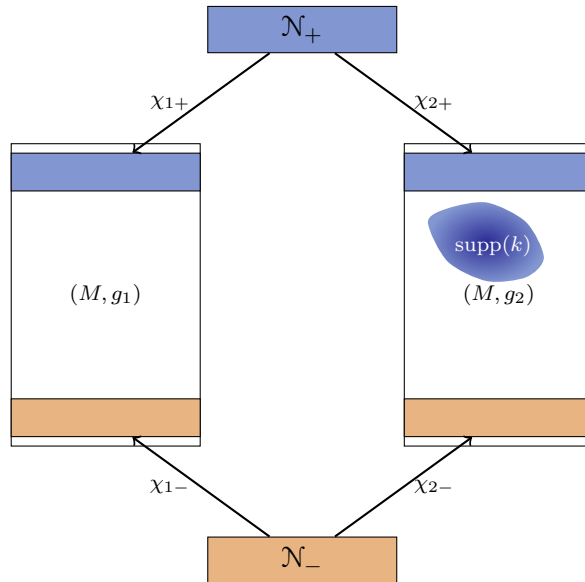


Figure 8.2: Embeddings of neighborhoods of Cauchy surfaces into spacetimes $\mathcal{M}_1 = (M, g_1)$ and $\mathcal{M}_2 = (M, g_2)$.

isomorphisms $\alpha_{\chi_{i\pm}} \doteq \mathfrak{A}_{\chi_{i\pm}}$ and the free relative Cauchy evolution is an

automorphism of $\mathfrak{A}(\mathcal{M}_1)$ given by $\beta_{0g} = \alpha_{0\chi_{1-}} \circ \alpha_{0\chi_{2-}}^{-1} \circ \alpha_{0\chi_{2+}} \circ \alpha_{0\chi_{1+}}^{-1}$. It was shown in [BFV03] that the functional derivative of β_{0g} with respect to g is the commutator with the free stress-energy tensor. A different proof of this result has been given in [BFR13] with the use of the *principle of perturbative agreement*, which is a condition introduced by Hollands and Wald in [?] and recently proven in a more general context in [?]. Following these ideas, we introduce a map $\tau^{\text{ret}} : \mathfrak{A}(\mathcal{M}_2) \rightarrow \mathfrak{A}(\mathcal{M}_1)$, such that τ^{ret} maps $\Phi_{\mathcal{M}_2}(f)$ to $\Phi_{\mathcal{M}_1}(f)$ (modulo the image of δ_0), if the support of f lies outside the causal future of K . Physically it means that free algebras $\mathfrak{A}(\mathcal{M}_1)$ and $\mathfrak{A}(\mathcal{M}_2)$ are identified in the past of K . Analogously, we introduce a map τ^{adv} , which identifies the free algebras in the future. The free relative Cauchy evolution is then expressed as

$$\beta_{0g} \doteq \tau_{g_1 g_2}^{\text{ret}} \circ (\tau_{g_1 g_2}^{\text{adv}})^{-1}, \quad (8.17)$$

As we choose to work off-shell, we define τ^{ret} as the classical retarded Møller operator constructed in [DF02]. The perturbative agreement is a condition that, on shell,

$$\tau_{g_1 g_2}^{\text{ret}} \circ \mathcal{S}_2 = \mathcal{S}_{S_{0\mathcal{M}_2} - S_{0\mathcal{M}_1}} \quad \text{holds.} \quad (8.18)$$

Here $\mathcal{S}_{S_{0\mathcal{M}_1} - S_{0\mathcal{M}_2}}$ denotes the relative S-matrix constructed with the interaction $S_{0\mathcal{M}_1} - S_{0\mathcal{M}_2}$ and the background metric g_1 , while \mathcal{S}_2 is the S-matrix constructed on \mathcal{M}_2 with the $\mathcal{T}_{\mathcal{M}_2}$ product. The perturbative agreement condition for $\tau_{g_1 g_2}^{\text{adv}}$ is analogous to (??). A straightforward calculation shows that The functional derivative of β_{0g} with respect to $k \doteq g_2 - g_1$ can now be easily calculated, yielding

$$\frac{\delta}{\delta k_{\mu\nu}} \beta_{0g} \left(e_{\mathcal{T}_{\mathcal{M}_1}}^{i\Phi_{\mathcal{M}_1} f' / \hbar} \right) \Big|_{g_1} = \frac{i}{\hbar} \left[T_{0\mu\nu}, e_{\mathcal{T}_{\mathcal{M}_1}}^{i\Phi_{\mathcal{M}_1} f' / \hbar} \right]_{\star},$$

where $T_{0\mu\nu}$ is the stress-energy tensor of the linearized theory.

To obtain the relative Cauchy evolution for the full interacting theory, we use the quantum Møller maps introduced in (6.12). The following theorem has been proven in [BFR13]

Theorem 8.1. *The functional derivative $\Theta_{\mu\nu}$ of the relative Cauchy evolution can be expressed, on-shell, as*

$$\Theta_{\mu\nu}(\Phi_{\mathcal{M}_1}(f)) \stackrel{\text{o.s.}}{=} [R_{V_1}(\Phi_{\mathcal{M}_1}(f)), R_{V_1}(T_{\mu\nu})]_{\star},$$

where $T_{\mu\nu}$ is the stress-energy tensor of the extended action and one can define the time-ordered products in such a way that $T_{\mu\nu} = 0$ holds, so the interacting theory is background independent.

Proof. We write the interacting relative Cauchy evolution as:

$$\beta = R_{V_1}^{-1} \circ \tau_{g_1 g_2}^{\text{ret}} \circ R_{V_2} \circ A_{V_2}^{-1} \circ (\tau_{g_1 g_2}^{\text{adv}})^{-1} \circ A_{V_1}.$$

The condition of background independence is

$$R_{V_1}^{-1} \circ \tau_{g_1 g_2}^{\text{ret}} \circ R_{V_2} = A_{V_1}^{-1} \circ \tau_{g_1 g_2}^{\text{adv}} \circ A_{V_2}.$$

Differentiating with respect to $k_{\mu\nu}$ yields a condition

$$[R_{V_1}(\Phi_{\mathcal{M}_1, \mathbf{f}'}) , R_{V_1}(T(\eta))]_{\star} \stackrel{\text{o.s.}}{=} 0,$$

where

$$T(\eta) \doteq \left\langle T_{\mu\nu, \mathbf{f}}, \eta^{\mu\nu} \right\rangle = \left\langle \frac{\delta L_{\mathcal{M}_2, \mathbf{f}}^{\text{ext}}}{\delta k^{\mu\nu}} \Big|_{g_1}, \eta^{\mu\nu} \right\rangle$$

To prove that the infinitesimal background independence is fulfilled, we have to show that $T(\eta) = 0$ in the cohomology of \hat{s} . This is easily done, as

$$T(\eta) = \left\langle \frac{\delta S_{\mathcal{M}_2}^{\text{ext}}}{\delta k_{\mu\nu}} \Big|_{g_1}, \eta^{\mu\nu} \right\rangle = \left\langle \frac{\delta S_{\mathcal{M}_2}^{\text{ext}}}{\delta h_{\mu\nu}} \Big|_{g_1}, \eta^{\mu\nu} \right\rangle = s \left\langle h^{\ddagger}, \eta \right\rangle = \hat{s} \left\langle h^{\ddagger}, \eta \right\rangle,$$

where h is the perturbation metric. The last equality follows from the fact that the anomaly can always be removed for linear functionals [BD08]. This concludes the argument, so the theory is perturbatively background independent. \square

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