Math 2 - Contrôle continu n. 5 - 22 janvier 2010 - corrigé A. Frabetti (1)

Ex. 1 $\vec{V}(x,y) = (2xy^2-y)\,\vec{\imath} + (2x^2y-x)\,\vec{\jmath} + 0.\vec{k}$

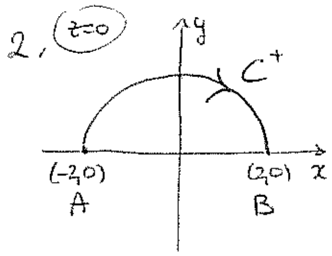
1. $\vec{\mathrm{rot}}\,\vec{V} = \vec{k}\left[\frac{\partial}{\partial x}(2x^2y-x) - \frac{\partial}{\partial y}(2xy^2-y)\right] = \vec{k}\cdot(4xy-1-4xy+1) = \vec{0}.$

Comme le domaine de définition de $\vec{V}$ est $\mathbb{R}^3$ qui est simplement connexe, et comme $\vec{\mathrm{rot}}\,\vec{V} = \vec{0}$, $\vec{V}$ est bien le gradient d'une fct $f : \mathbb{R}^3 \to \mathbb{R}$. Cherchons f t.q. $\vec{V} = \vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$:

$$\begin{cases} \frac{\partial f}{\partial x} = 2xy^2-y & \longrightarrow f(x,y) = \int (2xy^2-y)\,dx + g(y) = x^2y^2 - xy + g(y) \\ \frac{\partial f}{\partial y} = 2x^2y - x & \Uparrow \; \frac{\partial f}{\partial y} = 2x^2y - x + g'(y) = 2x^2y - x \iff g'(y) = 0 \iff g(y) = c \\ \frac{\partial f}{\partial z} = 0 & \Rightarrow f \text{ ne dépend pas de } z \end{cases}$$

$$\Longrightarrow f(x,y,z) = x^2y^2 - xy + c.$$

2. $(z=0)$

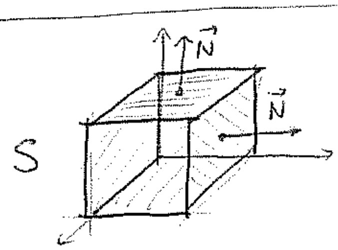


$$\int_{C^+} \vec{V}\cdot\vec{dl} = \int_{C^+} \vec{\nabla} f\cdot\vec{dl} = f(B) - f(A) = f(2,0,0) - f(-2,0,0) = 0.$$

3. $\mathrm{div}\,\vec{V} = \frac{\partial(2xy^2-y)}{\partial x} + \frac{\partial(2x^2y-x)}{\partial y} = 2y^2 + 2x^2.$

Sur $\mathbb{R}^3$, $\vec{V} = \vec{\mathrm{rot}}\,\vec{U} \iff \mathrm{div}\,\vec{V} = 0$. Comme $\mathrm{div}\,\vec{V} \neq 0$, sauf en $(0,0)$, $\vec{V}$ n'est pas le rotationnel d'un champ $\vec{U}$.

4.



Le cube S est une surface fermée de $\mathbb{R}^3$, donc $S = \partial D$ où D est le cube plein de $\mathbb{R}^3$ qui se trouve dedans S :

$$D = \{(x,y,z) \in \mathbb{R}^3,\ 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}.$$

Par le théorème de Ostrogradski, on a alors :

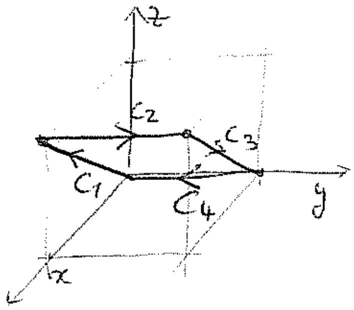
$$\oiint_S \vec{V}\cdot\vec{dS} = \iiint_D \mathrm{div}\,\vec{V}\cdot dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 (2x^2+2y^2)\,dx\,dy\,dz =$$

$$= 2\int_0^1\int_0^1 (x^2+y^2)\,[z]_0^1\,dx\,dy = 2\int_0^1\int_0^1 (x^2+y^2)\,dx\,dy = 2\int_0^1 \left[x^2y + \tfrac{1}{3}y^3\right]_0^1 dx$$

$$= 2\int_0^1 \left(x^2 + \tfrac{1}{3}\right) dx = 2\left[\tfrac{1}{3}x^3 + \tfrac{1}{3}x\right]_0^1 = 2\left(\tfrac{1}{3} + \tfrac{1}{3}\right) = 2\cdot\tfrac{2}{3} = \tfrac{4}{3}.$$

2) Ex. 2 : $\vec{U}(x,y,z) = xz\,\vec{i} + y^2\,\vec{j} + yz\,\vec{k}$

1.



C_1 : $0 \le x \le 1$, $x : 0 \to 1$, $y = 0$, $dy = 0$, $z = x$, $dz = dx$

C_2 : $x = 1$, $dx = 0$, $y : 0 \to 1$, $z = 1$, $dz = 0$

C_3 : $x : 1 \to 0$, $y = 1$, $dy = 0$, $z = x$, $dz = dx$

$\vec{dl} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$

C_4 : $x = 0$, $dx = 0$, $y : 1 \to 0$, $z = 0$, $dz = 0$

$$\int_{C_1} \vec{U}\cdot\vec{dl} = \int_{C_1} \Big(\underset{x}{\underset{\shortparallel}{xz}}\,dx + y^2 \underset{0}{\underset{\shortparallel}{\cancel{dy}}} + \underset{0}{\underset{\shortparallel}{yz}}\,dz\Big) = \int_0^1 x^2 dx = \Big[\tfrac{1}{3}x^3\Big]_0^1 = \frac{1}{3}.$$

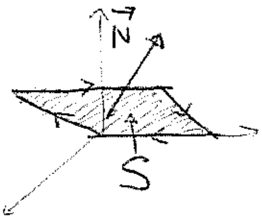
$$\int_{C_2} \vec{U}\cdot\vec{dl} = \int_{C_2} \Big(xz\,\underset{0}{\underset{\shortparallel}{dx}} + y^2 dy + yz\,\underset{0}{\underset{\shortparallel}{dz}}\Big) = \int_0^1 y^2 dy = \Big[\tfrac{1}{3}y^3\Big]_0^1 = \frac{1}{3}.$$

$$\int_{C_3} \vec{U}\cdot\vec{dl} = \int_{C_3} \Big(\underset{x}{\underset{\shortparallel}{xz}}\,dx + y^2 \underset{0}{\underset{\shortparallel}{dy}} + \underset{1.x}{\underset{\shortparallel}{yz}}\,\underset{dx}{\underset{\shortparallel}{dz}}\Big) = \int_1^0 (x^2+x)\,dx = \Big[\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\Big]_1^0 = -\frac{1}{3} - \frac{1}{2}$$

$$\int_{C_4} \vec{U}\cdot\vec{dl} = \int_{C_4} \Big(xz\,\underset{0}{\underset{\shortparallel}{dx}} + y^2 dy + yz\,\underset{0}{\underset{\shortparallel}{dz}}\Big) = \int_1^0 y^2 dy = \Big[\tfrac{1}{3}y^2\Big]_1^0 = -\frac{1}{3}.$$

$$\Rightarrow \oint_C \vec{U}\cdot\vec{dl} = \cancel{\tfrac{1}{3}} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{3}} - \frac{1}{2} - \cancel{\tfrac{1}{3}} = -\frac{1}{2}.$$

2.



S = rectangle t.q. $\partial S = C$.
Vu l'orientation de S donnée par $\vec{N}$, le bord orienté de S est exactement la courbe C avec l'orientation donnée.

Par le théorème de Stokes on a donc :

$$\iint_S \vec{\mathrm{rot}}\,\vec{U}\cdot\vec{dS} = \oint_{C=\partial S} \vec{U}\cdot\vec{dl} = -\frac{1}{2}.$$