Fiche 7

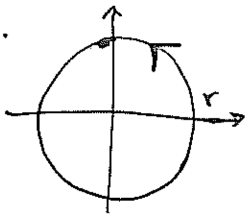
**Ex 1 :** $\vec{V} = \frac{x-y}{x^2+y^2}\,\vec{i} + \frac{x+y}{x^2+y^2}\,\vec{j}$

1. $D = \mathbb{R}^2 - \{(0,0)\}$ non simplement connexe.

2. $$\vec{\mathrm{rot}}\,\vec{V} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \frac{x-y}{x^2+y^2} & \frac{x+y}{x^2+y^2} & 0 \end{vmatrix} = \left[\frac{\partial}{\partial x}\left(\frac{x+y}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{x-y}{x^2+y^2}\right)\right]\vec{k}$$

$$= \frac{x^2+y^2-2x(x+y)}{(x^2+y^2)^2} - \frac{-x^2-y^2-2y(x-y)}{(x^2+y^2)^2}\,\vec{k} = \frac{\cancel{x^2+y^2}-\cancel{2x^2}-2xy+\cancel{x^2+y^2}-2xy+\cancel{2y^2}}{(x^2+y^2)^2}\,\vec{k}$$

$$= \frac{-4xy}{(x^2+y^2)^2}\,\vec{k}.$$ $\vec{V}$ n'est pas un champ de gradient sur des domaines simpl. connexes de $\mathbb{R}^2 \setminus O$.

3. 

$\gamma(t) = (r\cos t, r\sin t) \quad t \in [0,2\pi]$

$\gamma'(t) = (-r\sin t, r\cos t)$

$\vec{dl} = \gamma'(t)\,dt$

$x-y = r(\cos t - \sin t)$
$x+y = r(\cos t + \sin t)$
$x^2+y^2 = r^2$

$$\oint_{C+} \vec{V}\cdot\vec{dl} = \int_0^{2\pi}\left(-\frac{r(\cos t-\sin t)}{\cancel{r^2}}\cdot\cancel{r\sin t} + \frac{\cancel{r}(\cos t+\sin t)}{\cancel{r^2}}\cdot r\cos t\right)dt$$

$$= \int_0^{2\pi}\left(-\cancel{\cos t\sin t} + \underbrace{\sin^2 t + \cos^2 t}_{1} + \cancel{\sin t\cos t}\right)dt = \int_0^{2\pi} dt = 2\pi.$$

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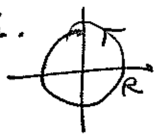
**Ex2.** $\vec{U} = (2x-y)\,\vec{i} + (x+y)\,\vec{j}$, $\quad \vec{V} = \vec{\mathrm{rot}}\,\vec{U} = \left[\overset{1}{\frac{\partial}{\partial x}(x+y)} - \overset{-1}{\frac{\partial}{\partial y}(2x-y)}\right]\vec{k} = 2\vec{k}$

1. 

$\gamma(t) = (R\cos t, R\sin t) \quad t \in [0,2\pi]$

$\gamma'(t) = (-R\sin t, R\cos t) \quad \vec{dl} = \gamma'(t)\,dt$

$2x-y = R(2\cos t - \sin t)$
$x+y = R(\cos t + \sin t)$

$u = 2t \quad t = 2\pi \Rightarrow u = 4\pi$
$du = 2\,dt$

$$\oint_{C+} \vec{U}\cdot\vec{dl} = \int_0^{2\pi}\left(-R^2\sin t(2\cos t - \sin t) + R^2\cos t(\cos t + \sin t)\right)dt =$$

$$= R^2\int_0^{2\pi}\left(-2\sin t\cos t + \underbrace{\sin^2 t + \cos^2 t}_{1} + \cos t\sin t\right)dt = R^2\int_0^{2\pi}(1 - \underset{\frac{1}{2}\sin(2t)}{\underset{\parallel}{\sin t\cos t}})\,dt = R^2\int_0^{2\pi} dt - \frac{1}{4}R^2\int_0^{4\pi}\sin u$$

$$= R^2\cdot 2\pi + \frac{1}{4}R^2\left[\cos u\right]_0^{4\pi} = 2\pi R^2 + \frac{R^2}{4}\underbrace{(1-1)}_{=0} = 2\pi R^2$$

2. $S = \{(\rho,\theta),\ 0 \le \rho \le R,\ 0 \le \theta \le 2\pi\}$ $\qquad dx\,dy = \rho\,d\rho\,d\theta$

$$\iint_S \vec{V}\cdot\vec{dS} = \iint_S 2\,dx\,dy = 2\int_0^R \rho\,d\rho\int_0^{2\pi} d\theta = 2\left[\frac{1}{2}\rho^2\right]_0^R\cdot\left[\theta\right]_0^{2\pi} = \cancel{2}\cdot\frac{R^2}{\cancel{2}}\cdot 2\pi = 2\pi R^2$$

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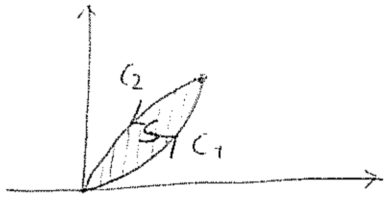
**Green-Riemann :**
$$\iint_S\left[\frac{\partial}{\partial x}P(x,y) - \frac{\partial}{\partial y}Q(x,y)\right]dx\,dy = \oint_{\partial S = C+}\left[P(x,y)\,dx + Q(x,y)\,dy\right].$$

Ex. 3 : $\vec{u} = (2xy - x^2)\,\vec{i} + (x+y^2)\,\vec{j}$ , $\vec{V} = \vec{rot}\,\vec{u}$

1)




$C^+ = C_1^+ \cup C_2^+$

$x : 0 \to 1$

$C_1^+ = \{(x,y),\ 0 \le x \le 1,\ y = x^2\}$

$C_2^+ = \{(x,y),\ y : 1 \to 0,\ x = y^2\}$

$$\oint_{C^+} \vec{u}\cdot\vec{dl} = \oint_{C^+}\left[(2xy-x^2)\,dx + (x^2+y^2)\,dy\right]$$

$$= \int_{C_1^+}\left[(2xy-x^2)\,dx + (x+y^2)\,dy\right] + \int_{C_2^+}\left[(2xy-x^2)\,dx + (x+y^2)\,dy\right]$$

$C_1^+$ : $x : 0 \to 1$ , $y : 0 \to 1$ ; $y = x^2$ , $x = \sqrt{y}$ ; $dy = 2x\,dx$

$C_2^+$ : $x : 1 \to 0$ , $y : 1 \to 0$ ; $x = y^2$ , $dx = 2y\,dy$ ; $y = \sqrt{x}$

$$= \int_0^1 (2x^3 - x^2)\,dx + \int_0^1 (x+x^4)\,2x\,dx - \int_0^1 (2y^3 - y^4)\,2y\,dy + \underbrace{(y^2+y^2)}_{2y^2}\,dy$$

$$= \int_0^1 (\underbrace{\cancel{2x^2} - x^2}_{x^2} + 2x^3 + 2x^5) - \int_0^1 (2y^2 + 4y^4 - 2y^5)\,dy$$

$$= \left[\frac{1}{3}x^3 + \frac{2}{4}x^4 + \frac{2}{6}x^6\right]_0^1 - \left[\frac{2}{3}y^3 + \frac{4}{5}y^5 - \frac{2}{6}y^6\right]_0^1$$

$$= \left(\frac{1}{3} + \frac{1}{2} + \frac{1}{3}\right) - \left(\frac{2}{3} + \frac{4}{5} - \frac{1}{3}\right) = \left(\frac{2}{3} + \frac{1}{2}\right) - \left(\frac{1}{3} + \frac{4}{5}\right)$$

$$= \frac{4+3}{6} - \frac{5+12}{15} = \frac{7}{6} - \frac{17}{15} = \frac{35-34}{30} = \frac{1}{30}$$

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2) $\vec{V} = \text{rot}\,\vec{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xy - x^2 & x+y^2 & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1-2x \end{pmatrix} = (1-2x)\,\vec{k}$

$$\iint_S \vec{V}\cdot\vec{dS} = \iint_S (1-2x)\,dx\,dy = \int_0^1 (1-2x)\,dx \int_{x^2}^{\sqrt{x}} dy = \int_0^1 (1-2x)(\sqrt{x} - x^2)\,dx$$

où $\partial S = C$ ; $S = \{(x,y),\ 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}$

$$= \int_0^1 \left(x^{\frac{1}{2}} - 2x^{\frac{1}{2}} - x^2 + 2x^3\right)dx = \left[\frac{1}{\frac{1}{2}+1}x^{\frac{3}{2}} - \frac{2}{\frac{3}{2}+1}x^{\frac{5}{2}} - \frac{1}{3}x^3 + \frac{2}{4}x^4\right]_0^1 = \frac{2}{3} - \frac{4}{5} - \frac{1}{3} + \frac{1}{2}$$

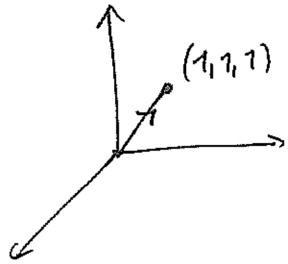
$= \frac{1}{3} + \frac{1}{2} - \frac{4}{5}$

$= \frac{5}{6} - \frac{4}{5} = \frac{25-24}{30}$

$= \frac{1}{30}$

Ex 4 $\quad \vec{E} = (y^2, z, 0)$

1.

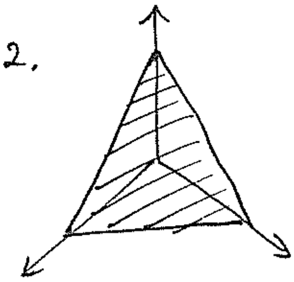


$$C^+ = \{ \gamma(t) = O + t(1,1,1) = (t,t,t) \ , \ 0 \leq t \leq 1 \}$$

$$\gamma'(t) = (1,1,1) \qquad \vec{d\ell} = \begin{pmatrix}1\\1\\1\end{pmatrix} dt$$

$$\int_{C^+} \vec{E}\cdot\vec{d\ell} = \int_0^1 \begin{pmatrix}t^2\\t\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix} dt = \int_0^1 (t^2+t)\,dt = \left[\tfrac{1}{3}t^3 + \tfrac{1}{2}t^2\right]_0^1 = \tfrac{1}{3} + \tfrac{1}{2} = \tfrac{5}{6} .$$

2.



$$S = \{ (x,y,z) \in \mathbb{R}^3, \ x+y+z=1, \ x,y,z \geq 0 \}$$

param. : $\quad S^+ = \{ \sigma(u,v) = (1-u-v, u, v) \ , \ 0 \leq u \leq 1 \ , \ 0 \leq v \leq 1-u \}$

$$\begin{cases} \dfrac{\partial\sigma}{\partial u} = (-1,1,0) \\ \dfrac{\partial\sigma}{\partial v} = (-1,0,1) \end{cases} \qquad \frac{\partial\sigma}{\partial u}\wedge\frac{\partial\sigma}{\partial v} = \begin{pmatrix}1\\1\\1\end{pmatrix} \qquad \vec{dS} = \begin{pmatrix}1\\1\\1\end{pmatrix} du\,dv$$

$$E|_S = (u^2, v, 0)$$

$$\iint_{S^+} \vec{E}\cdot\vec{d\ell} = \iint_S (u^2+v)\,du\,dv = \int_0^1 du \int_0^{1-u} (u^2+v)\,dv = \int_0^1 \left[u^2v + \tfrac{1}{2}v^2\right]_0^{1-u} du$$

$$= \int_0^1 \left(u^2(1-u) + \tfrac{1}{2}(1-u)^2\right) du = \int_0^1 \left(u^2 - u^3 + \tfrac{1}{2} - u + \tfrac{1}{2}u^2\right) du = \int_0^1 \left(\tfrac{1}{2} - u + \tfrac{3}{2}u^2 - u^3\right) du$$

$$= \left[\tfrac{1}{2}u - \tfrac{1}{2}u^2 + \tfrac{1}{2}u^3 - \tfrac{1}{4}u^4\right]_0^1 = \cancel{\tfrac{1}{2}} - \cancel{\tfrac{1}{2}} + \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4} .$$

cartesien : $\quad \vec{dS} = \begin{pmatrix} dy\,dz \\ -dx\,dz \\ dx\,dy \end{pmatrix} \qquad x+y+z=1 \Rightarrow x = 1-y-z \ , \ dx = -dy - dz$

$$dx\,dz = -dy\,dz$$

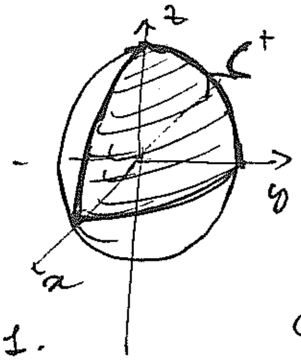
$$\iint_S \vec{E}\cdot\vec{dS} = \iint_S \left(y^2 dy\,dz - z\,dx\,dz\right) = \iint_S (y^2+z)\,dy\,dz \qquad \begin{aligned} y &: 0 \to 1 \\ z &: 0 \to 1-y \end{aligned}$$

$$= \int_0^1 dy \int_0^{1-y} (y^2+z)\,dz = \int_0^1 dy \left[y^2z + \tfrac{1}{2}z^2\right]_0^{1-y} = \int_0^1 \left(y^2(1-y) + \tfrac{1}{2}(1-y)^2\right) dy$$

$$= \int_0^1 \left(y^2 - y^3 + \tfrac{1}{2} - y + \tfrac{1}{2}y^2\right) dy = \int_0^1 \left(\tfrac{1}{2} - y + \tfrac{3}{2}y^2 - y^3\right) dy$$

$$= \left[\tfrac{1}{2}y - \tfrac{1}{2}y^2 + \tfrac{1}{2}y^3 - \tfrac{1}{4}y^4\right]_0^1 = \cancel{\tfrac{1}{2} - \tfrac{1}{2}} + \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4} .$$

Ex.5 $\Sigma = \{ (x,y,z) \in \mathbb{R}^3,\ x^2+y^2+z^2 = R^2,\ x \geq 0,\ y \geq 0,\ z \geq 0 \}$




$$= \left\{ \sigma(\theta,\varphi) = R(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta) \quad \begin{matrix} 0 \leq \varphi \leq \frac{\pi}{2} \\ \frac{\pi}{2} \leq \theta \leq \pi \end{matrix} \right\}$$

$$\vec{E}(x,y,z) = (z,0,0).$$

1. $$C^+ = \{(0,y,z),\ y^2+z^2 = R^2,\ y,z \geq 0\}$$
$$= \{ \gamma(t) = R(0,\cos t, \sin t),\ 0 \leq t \leq \tfrac{\pi}{2} \} \qquad \gamma'(t) = R(0,-\sin t, \cos t)$$

$$\int_{C^+} \vec{E}\cdot\vec{dl} = \int_0^{\frac{\pi}{2}} 0 = 0.$$

2. param:
$$\frac{\partial\sigma}{\partial\theta} = R(\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta)$$
$$\frac{\partial\sigma}{\partial\varphi} = R(-\sin\theta\sin\varphi,\ \sin\theta\cos\varphi,\ 0) \qquad \vec{E}\big|_S = (R\cos\theta, 0, 0)$$

$$\frac{\partial\sigma}{\partial\theta}\wedge\frac{\partial\sigma}{\partial\varphi} = R^2(\sin^2\theta\cos\varphi,\ \sin^2\theta\sin\varphi,\ \cos\theta\sin\theta)$$

$$\iint_S \vec{E}\cdot\vec{dS} = \int_{\pi/2}^{\pi} d\theta \int_0^{\pi/2} d\varphi\ R^3\sin^2\theta\cos\theta\cos\varphi = R^3\int_{\pi/2}^{\pi}\sin^2\theta\cos\theta\, d\theta \int_0^{\pi/2}\cos\varphi\, d\varphi$$

$u = \sin\theta$, $du = \cos\theta\, d\theta$ $\quad \theta \in [\pi/2, \pi] \Rightarrow u \in [1, 0]$

$$= R^3 \int_0^1 u^2 du\ [\sin\varphi]_0^{\pi/2} = R^3 \frac{1}{3}[u^3]_0^1 \cdot 1 = \frac{R^3}{3}.$$

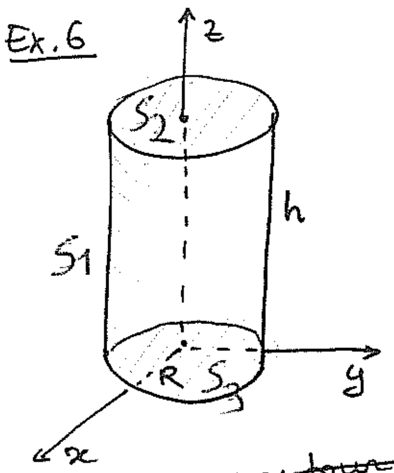
def implicit: $\vec{dS} = \begin{pmatrix} dy\,dz \\ -dx\,dz \\ dx\,dy \end{pmatrix}$

$$\iint_S \vec{E}\cdot\vec{dS} = \iint_S z\,dy\,dz = \int_0^R z\,dz\int_0^{\sqrt{R^2-z^2}} dy = \frac{1}{2}\int_0^R 2z\sqrt{R^2-z^2}\,dz = -\frac{1}{2}\int_{R^2}^0 u^{\frac{1}{2}}du$$

$u = R^2 - z^2$
$du = -2z\,dz$
$z = 0 \Rightarrow u = R^2$
$z = R \Rightarrow u = 0$

$$= -\frac{1}{2}\left[\frac{1}{\frac{1}{2}+1} u^{\frac{3}{2}}\right]_{R^2}^0 = \frac{1}{2}\cdot\frac{2}{3}\cdot(R^2)^{\frac{3}{2}} = \frac{R^3}{3}.$$

Ex. 6




$$S = \{(x,y,z)\in\mathbb{R}^3,\ x^2+y^2=R^2,\ 0\le z\le h\} \leftarrow S_1$$
$$\cup \{(x,y,0),\ x^2+y^2\le R^2\} \leftarrow S_2$$
$$\cup \{(x,y,h),\ x^2+y^2\le R^2\} \leftarrow S_3$$

$\vec{V}(x,y,z) = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ déf sur $\mathbb{R}^3$.

1. Puisque $\mathbb{R}^3$ ~~autour de~~ S est simplement connexe, $\vec{V} = \vec{\nabla} f \Leftrightarrow \vec{\text{rot}}\,\vec{V} = 0$.

$$\vec{\text{rot}}\,\vec{V} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & y^2 & z^2 \end{vmatrix} = \vec{i}\left(\frac{\partial z^2}{\partial y} - \frac{\partial y^2}{\partial z}\right) + \vec{j}\left(\frac{\partial z^2}{\partial x} - \frac{\partial x^2}{\partial z}\right) + k \text{ idem} = 0.$$

$\Rightarrow \vec{V} = \vec{\nabla} f$.

Donc $\displaystyle\int_{\partial S=\emptyset} \vec{V}\cdot\vec{d\ell} = 0$ car $\partial S = \emptyset$ et donc $\displaystyle\iint_S \vec{\text{rot}}\,\vec{V}\cdot\vec{dS} = 0$ !

2. $\vec{V} = \vec{\text{rot}}\,\vec{U} \Leftrightarrow \text{div}\,\vec{V} = 0$, car $\vec{V}$ est déf sur un ens "étoilé" $\mathbb{R}^3$.

$$\text{div}\,\vec{V} = \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z} = 2x + 2y + 2z \neq 0. \quad \underline{\text{non}}!$$

3. $$\oiint_S \vec{V}\cdot\vec{dS} = \oiint_S \left(x^2 dy\,dz + y^2 dx\,dz + z^2 dx\,dy\right) \qquad S = S_1\cup S_2\cup S_3$$

$$\iint_{S_1} \vec{V}\cdot\vec{dS} = \iint_{\substack{x^2+y^2=R^2\\0\le z\le h}} \left(x^2 dy\,dz + y^2 dx\,dz + \underbrace{z^2 dx\,dy}_{=0}\right) = \int_{-R}^{R}(R^2-y^2)dy\int_0^h dz + \int_{-R}^{R}(R^2-x^2)dx\int_0^h dz + 0$$

$$= 2h\left[R^2 y - \frac{y^3}{3}\right]_{-R}^{R} = 2h\left(\cancel{R^3} - \frac{R^3}{3} - \cancel{(-R^3)} + \frac{(-R)^3}{3}\right) = 0.$$

$$\iint_{S_2} \vec{V}\cdot\vec{dS} = \iint_{\substack{x^2+y^2\le R^2\\z=h}} \left(\underbrace{x^2 dy\,dz}_{=0} + \underbrace{y^2 dx\,dz}_{=0} + \underbrace{z^2 dx\,dy}_{=h^2}\right) = h^2\iint_{x^2+y^2\le R^2} dx\,dy = h^2\int_0^{2\pi} d\theta\int_0^R \rho\,d\rho = h^2\, 2\pi\frac{R^2}{2}$$
$$= \pi h^2 R^2.$$

$$\iint_{S_3} \vec{V}\cdot\vec{dS} = \iint_{\substack{x^2+y^2\le 0\\z=0}} \left(x^2 dy\,dy + y^2 dx\,dz + z^2 dx\,dy\right) = 0.$$

4. $$\oiint_{S=\partial D} \vec{V}\cdot\vec{dS} = \iiint_D \text{div}(\vec{V})\,dx\,dy\,dz = \iiint_{\substack{x^2+y^2\le R^2\\0\le z\le h}} 2\overbrace{(x+y+z)}^{\rho(\cos\theta+\sin\theta)}\overbrace{dx\,dy\,dz}^{\rho\,d\rho\,d\theta}$$

$$= 2\int_0^h dz\int_0^R d\rho\int_0^{2\pi} d\theta\left[\rho^2(\cos\theta+\sin\theta) + \rho z\right] = 2\int_0^h dz\int_0^R d\rho\left[\rho^2\sin\theta - \rho^2\cos\theta + \rho z\theta\right]_0^{2\pi}$$

$$= 2\int dz\int d\rho\left[\cancel{-\rho^2} + \cancel{\rho^2} + 2\pi\rho z\right] = 4\pi\int_0^h z\,dz\int_0^R \rho\,d\rho = 4\pi\cdot\frac{1}{2}h^2\cdot\frac{1}{2}R^2 = \pi h^2 R^2.$$