Ex. 1 $\vec{V}(x,y) = y\,\vec{i} + (x+\cos y)\,\vec{j} + 0\cdot\vec{k}$

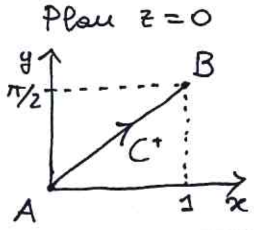
1. $\vec{\mathrm{rot}}\,\vec{V} = \vec{k}\cdot\left[\frac{\partial}{\partial x}(x+\cos y) - \frac{\partial}{\partial y}(y)\right] = \vec{k}\cdot(1-1) = \vec{0}$

Le domaine de définition de $\vec{V}$ est $\mathbb{R}^3$ qui est simplement connexe, donc $\vec{V} = \vec{\nabla} f \Leftrightarrow \vec{\mathrm{rot}}\,\vec{V} = 0$. Puisque $\vec{\mathrm{rot}}\,\vec{V} = 0$, $\vec{V}$ est bien le gradient d'une fonction $f : \mathbb{R}^3 \to \mathbb{R}$. Cherchon f t.q. $\vec{V} = \vec{\nabla} f$:

$$\begin{cases} \frac{\partial f}{\partial x} = y & \Longrightarrow f(x,y) = \int y\,dx + g(y) = xy + g(y) \\ \frac{\partial f}{\partial y} = x + \cos y & \hookrightarrow \frac{\partial f}{\partial y} = x + g'(y) = x + \cos y \Leftrightarrow g'(y) = \cos y \\ & \Leftrightarrow g(y) = \sin y + c. \\ \frac{\partial f}{\partial z} = 0 & \Rightarrow f \text{ ne dépend pas de } z \end{cases}$$

$$\Longrightarrow f(x,y,z) = xy + \sin y + c.$$

2. Plan $z=0$

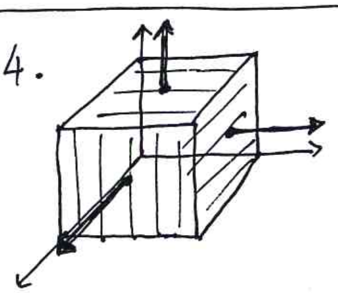


$$\int_{C^+} \vec{V}\cdot\vec{d\ell} = \int_{C^+} \vec{\nabla} f\cdot\vec{d\ell} = f(B) - f(A) = f(1,\pi/2,0) - f(0,0,0)$$
$$= 1\cdot\pi/2 + \sin\pi/2 + \cancel{c} - 0 - \sin 0 - \cancel{c} = \frac{\pi}{2} + 1.$$

3. $\mathrm{div}\,\vec{V} = \frac{\partial y}{\partial x} + \frac{\partial(x+\cos y)}{\partial y} = 0 - \sin y = -\sin y.$

Sur $\mathbb{R}^3$, $\vec{V} = \vec{\mathrm{rot}}\,\vec{U} \Leftrightarrow \mathrm{div}\,\vec{V} = 0$. Puisque $\mathrm{div}\,\vec{V} = -\sin y$ n'est pas toujours nulle, $\vec{V}$ n'est pas le rotationnel d'un champ $\vec{U}$.

4.



Le cube S est une surface fermée de $\mathbb{R}^3$, donc $S = \partial D$ où D est l'intérieur du cube :

$$D = \left\{ (x,y,z) \in \mathbb{R}^3,\ 0 \le x \le \pi/2,\ 0 \le y \le \pi/2,\ 0 \le z \le \pi/2 \right\}.$$

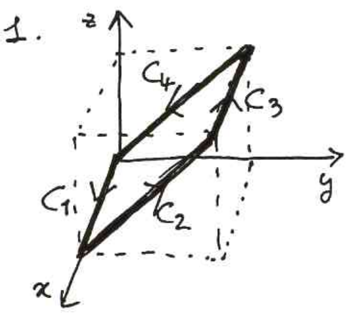
Par le théorème d'Ostrogradski, on a alors :

$$\oint\!\!\oint_S \vec{V}\cdot\vec{dS} = \iiint_D \mathrm{div}\,\vec{V}\cdot dx\,dy\,dz = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2} (-\sin y)\,dx\,dy\,dz$$

$$= \int_0^{\pi/2} dx \int_0^{\pi/2} (-\sin y)\,dy \int_0^{\pi/2} dz = \left[x\right]_0^{\pi/2} \left[\cos y\right]_0^{\pi/2} \left[z\right]_0^{\pi/2} = \left(\frac{\pi}{2}-0\right)(0-1)\left(\frac{\pi}{2}-0\right) = -\frac{\pi^2}{4}.$$

2/ <u>Ex. 2</u> $\vec{\mathcal{U}}(x,y,z) = x^2\,\vec{i} + xz\,\vec{j} + yz\,\vec{k}$

1.



C_1 : $x : 0 \to 1$, $y = 0$, $dy = 0$, $z = 0$, $dz = 0$

C_2 : $x = 1$, $dx = 0$, $y : 0 \to 1$, $z = y$, $dz = dy$

C_3 : $x : 1 \to 0$, $y = 1$, $dy = 0$, $z = 1$, $dz = 0$

C_4 : $x = 0$, $dx = 0$, $y : 1 \to 0$, $z = y$, $dz = dy$

$$\oint_C \vec{U}\cdot\vec{d\ell} = \int_{C_1} \vec{U}\cdot\vec{d\ell} + \int_{C_2} \vec{U}\cdot\vec{d\ell} + \int_{C_3} \vec{U}\cdot\vec{d\ell} + \int_{C_4} \vec{U}\cdot\vec{d\ell} \quad , \text{ où}$$

$$\vec{d\ell} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \quad \text{et} \quad \vec{\mathcal{U}}\cdot\vec{d\ell} = \begin{pmatrix} x^2 \\ xz \\ yz \end{pmatrix}\cdot\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = x^2dx + xz\,dy + yz\,dz .$$

Alors :

$$\int_{C_1} \vec{U}\cdot d\ell = \int_{C_1} \Big(x^2dx + xz\underset{0}{\underset{\shortparallel}{dy}} + yz\underset{0}{\underset{\shortparallel}{dz}}\Big) = \int_0^1 x^2\,dx = \Big[\tfrac{1}{3}x^3\Big]_0^1 = \tfrac{1}{3}$$

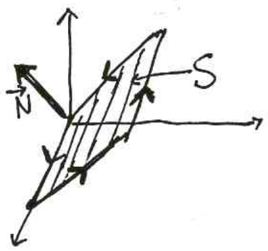
$$\int_{C_2} \vec{U}\cdot d\ell = \int_{C_2} \Big(x^2\underset{0}{\underset{\shortparallel}{dx}} + \underset{1}{\underset{\shortparallel}{x}}\,\underset{y}{\underset{\shortparallel}{z}}\,dy + y\underset{y}{\underset{\shortparallel}{z}}\,\underset{dy}{\underset{\shortparallel}{dz}}\Big) = \int_0^1 (y+y^2)\,dy = \Big[\tfrac{1}{2}y^2 + \tfrac{1}{3}y^3\Big]_0^1 = \tfrac{1}{2} + \tfrac{1}{3}$$

$$\int_{C_3} \vec{U}\cdot\vec{d\ell} = \int_{C_3} \Big(x^2dx + xz\underset{0}{\underset{\shortparallel}{dy}} + yz\underset{0}{\underset{\shortparallel}{dz}}\Big) = \int_1^0 x^2\,dx = \Big[\tfrac{1}{3}x^3\Big]_1^0 = -\tfrac{1}{3}$$

$$\int_{C_4} \vec{U}\cdot\vec{d\ell} = \int_{C_4} \Big(x^2\underset{0}{\underset{\shortparallel}{dx}} + \underset{0}{\underset{\shortparallel}{x}}\,\underset{y}{\underset{\shortparallel}{z}}\,dy + y\underset{y}{\underset{\shortparallel}{z}}\,\underset{dy}{\underset{\shortparallel}{dz}}\Big) = \int_1^0 y^2\,dy = \Big[\tfrac{1}{3}y^3\Big]_1^0 = -\tfrac{1}{3}$$

donc $\displaystyle\oint_C \vec{U}\cdot\vec{d\ell} = \cancel{\tfrac{1}{3}} + \tfrac{1}{2} + \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{3}} = \tfrac{1}{2}$.

2.



S = rectangle t.q. $\partial S = C$.

Vu l'orientation de S donnée par $\vec{N}$, le bord orienté de S est exactement la courbe C avec l'orientation donnée.

Par le théorème de Stokes, on a donc

$$\iint_S \vec{\text{rot}}\,\vec{U}\cdot\vec{dS} = \oint_{\partial S} \vec{\mathcal{U}}\cdot\vec{d\ell} = \frac{1}{2} .$$