Ex. 1 $f(x,y) = e^{x^3+y-xy}$, $(x,y)\in\mathbb{R}^2$

1. $$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} (3x^2-y)\, e^{x^3+y-xy} \\ (1-x)\, e^{x^3+y-xy} \end{pmatrix}.$$

2. $$df_{(-1,1)} = \frac{\partial f}{\partial x}(-1,1)\,dx + \frac{\partial f}{\partial y}(-1,1)\,dy = 2e\,dx + 2e\,dy .$$

3. $$\text{Hess } f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} \left[6x+(3x^2-y)^2\right] e^{x^3+y-xy} & \left[-1+(1-x)(3x^2-y)\right] e^{x^3+y-xy} \\ & (1-x)^2\, e^{x^3+y-xy} \end{pmatrix}$$

4. (x,y) est point critique $\Leftrightarrow \nabla f(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} (3x^2-y)\, e^{x^3+y-xy} = 0 \\ (1-x)\, e^{x^3+y-xy} = 0 \end{cases} \Leftrightarrow \begin{cases} y = 3x^2 = 3 \\ x = 1 \end{cases}$

Donc il y a un seul point critique $(1,3)$.

5. Nature du point critique $(1,3)$:

$\text{Hess } f(1,3) = \begin{pmatrix} 6e & -e \\ -e & 0 \end{pmatrix} \Rightarrow \det \text{Hess } f(1,3) = 0 - (-e)^2 = -e^2 < 0$

donc $(1,3)$ est un point col.

6. Taylor en (a,b) à l'ordre 2 :

$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + \frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b) + {}$$
$$+ \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2 + o\left((x-a)^2+(y-b)^2\right).$$

$\Rightarrow$ Taylor en $(0,0)$:

$$e^{x^3+y-xy} = e^0 + \cancel{(3\cdot 0-0)}e^0\cdot x + (1-0)e^0\cdot y + \cancel{\frac{(6\cdot 0+0)}{2}}e^0\cdot x^2 + \left[-1+\cancel{(1-0)\cdot 0}\right]e^0\, xy + \frac{1}{2}(1-0)e^0\, y^2 + o(x^2+y^2)$$
$$= 1 + y - xy + \frac{1}{2}y^2 + o(x^2+y^2).$$

7. Taylor en $(1,3)$: puisque $\nabla f(1,3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ et $\text{Hess } f(1,3) = \begin{pmatrix} 6e & -e \\ -e & 0 \end{pmatrix}$, on a :

$$e^{x^3+y-xy} = e^{1+\cancel{3-3}} + \frac{6e}{2}(x-1)^2 - e(x-1)(y-3) + 0\cdot(y-3)^2 + o\left((x-1)^2+(y-3)^2\right)$$
$$= e + 3e(x-1)^2 - e(x-1)(y-3) + o\left((x-1)^2+(y-3)^2\right).$$

2)

Ex. 2 $\quad f(x,y) = xy \quad , \quad (x,y) \in \mathbb{R}^2$

1. $\dfrac{\partial f}{\partial x}(x,y) = y \qquad\qquad \dfrac{\partial f}{\partial y}(x,y) = x$

2. $F(u,v) = f(x(u,v), y(u,v)) \implies$

$$\frac{\partial F}{\partial u}(u,v) = \frac{\partial f}{\partial x}(x(u,v), y(u,v)) \cdot \frac{\partial x}{\partial u}(u,v) + \frac{\partial f}{\partial y}(x(u,v), y(u,v)) \cdot \frac{\partial y}{\partial u}(u,v)$$

$$= y(u,v) \cdot \frac{\partial x}{\partial u}(u,v) + x(u,v) \cdot \frac{\partial y}{\partial u}(u,v)$$

$$\frac{\partial F}{\partial v}(u,v) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} = y(u,v) \cdot \frac{\partial x}{\partial v}(u,v) + x(u,v) \cdot \frac{\partial y}{\partial v}(u,v) .$$

3. $F(t) = f(x(t), y(t)) \implies$

$$F'(t) = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot x'(t) + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot y'(t)$$

$$= y(t) \cdot x'(t) + x(t) \cdot y'(t) .$$