Exercice 1 (6) $f(x,y) = x^3 - y^2 - 3x$

$\vec{\nabla} f(x,y) = \begin{pmatrix} 3x^2-3 \\ -2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 3(x^2-1)=0 \\ y=0 \end{cases} \Leftrightarrow \begin{cases} x = \pm 1 \\ y = 0 \end{cases}$ il y a deux points critiques : $(-1,0)$ et $(1,0)$.

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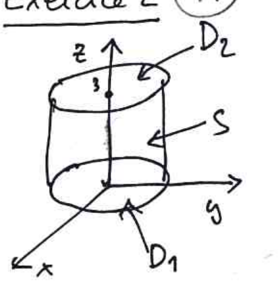
$\text{Hess} f(x,y) = \begin{pmatrix} 6x & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \det \text{Hess} f(x,y) = -12x$ 1

en $(-1,0)$: $\det \text{Hess} f(-1,0) = -12\cdot(-1) = 12 > 0$, puisque $6x = -6 < 0$, $(-1,0)$ est un max. local. 1

en $(1,0)$: $\det \text{Hess} f(1,0) = -12 < 0$, donc $(1,0)$ est un point col. 1

Exercice 2 (17)



1. $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 \le 1,\ 0 \le z \le 3\}$

(2) $\text{Vol}\,\Omega = \iiint_\Omega dx\,dy\,dz = \iint_{x^2+y^2\le 1} dx\,dy \cdot \int_0^3 dz = \int_0^1 \rho\,d\rho \int_0^{2\pi} d\theta \int_0^3 dz$

$\{(x,y) \mid x^2+y^2 \le 1\} = \{(\rho,\theta) \mid 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi\}$

$x = \rho\cos\theta$, $y = \rho\sin\theta$ $\Rightarrow dx\,dy = \rho\,d\rho\,d\theta$

$= \left[\frac{1}{2}\rho^2\right]_0^1 \left[\theta\right]_0^{2\pi} \left[z\right]_0^3 = \frac{1}{2}\cdot 2\pi \cdot 3 = 3\pi.$

2. $\vec{V}(x,y,z) = y\vec{i} + x\vec{j} - 2z\vec{k}$

(2) $\text{div}\,\vec{V} = \frac{\partial y}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial(-2z)}{\partial z} = 0 + 0 - 2 = -2$

$\vec{V}$ n'est pas le rotationnel d'un champ de vecteurs car $\text{div}\,\vec{V} \neq \vec{0}$.

3. $\Sigma = D_1 \cup D_2 \cup S$ est une surface fermée $= \partial\Omega$

(2) $\oiint_{\Sigma=\partial\Omega} \vec{V}\cdot d\vec{S} = \iiint_\Omega \text{div}\,\vec{V}\,dx\,dy\,dz = -2\iiint_\Omega dx\,dy\,dz = -2\,\text{vol}\,\Omega = -6\pi.$

4. ~~[illegible]~~

(2) $D_1 = \{(x,y,z) \mid x^2+y^2 \le 1,\ z = 0\}$ param. $\sigma(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, 0)$ $\rho \in [0,1]$, $\theta \in [0,2\pi]$.

$\begin{cases} \sigma_\rho = (\cos\theta, \sin\theta, 0) \\ \sigma_\theta = (-\rho\sin\theta, \rho\cos\theta, 0) \end{cases} \Rightarrow \sigma_\rho \wedge \sigma_\theta = (0, 0, \rho\cos^2\theta + \rho\sin^2\theta) = (0,0,\rho)$

cette param. donne l'orientation en haut, donc il faudra changer le signe au flux :

$\iint_{D_1^+} \vec{V}\cdot d\vec{S} = -\int_0^1 d\rho \int_0^{2\pi} d\theta\, \vec{V}\cdot(\sigma_\rho\wedge\sigma_\theta) = -\int_0^1\int_0^{2\pi} \underbrace{\begin{pmatrix} \rho\sin\theta \\ \rho\cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix}}_{=0} d\rho\,d\theta = 0.$

5. $D_2 = \{(x,y,z) \mid x^2+y^2 \le 1,\ z = 3\}$ param. $\sigma(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, 3)$ $\rho \in [0,1]$, $\theta \in [0,2\pi]$

$\Rightarrow \sigma_\rho \wedge \sigma_\theta = (0,0,\rho)$

(2) $\iint_{D_2^+} \vec{V}\cdot d\vec{S} = \int_0^1 d\rho \int_0^{2\pi} d\theta \cdot \begin{pmatrix} \rho\sin\theta \\ \rho\cos\theta \\ -6 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix} = \int_0^{2\pi} d\theta \cdot \int_0^1 (-6\rho)\,d\rho = 2\pi\cdot(-3)\left[\rho^2\right]_0^1 = -6\pi,$

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6. $\displaystyle \oiint_{\Sigma} \vec{V}\cdot\vec{dS} = \iint_{D_1} \vec{V}\cdot\vec{dS} + \iint_{D_2} \vec{V}\cdot\vec{dS} + \iint_{S} \vec{V}\cdot\vec{dS}$

2 $\displaystyle \Longrightarrow \iint_{S} \vec{V}\cdot\vec{dS} = \oiint_{\Sigma} \vec{V}\cdot\vec{dS} - \iint_{D_1} \vec{V}\cdot\vec{dS} - \iint_{D_2} \vec{V}\cdot\vec{dS} = -6\pi - 0 - (-6\pi) = 0.$

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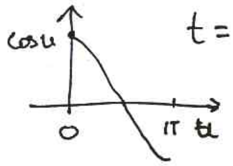
7. $S = \{ \sigma(u,v) = (\cos u, \sin u, v) \mid 0 \le u \le 2\pi ,\ 0 \le v \le 3 \}$

$\begin{cases} \sigma_u = (-\sin u, \cos u, 0) \\ \sigma_v = (0,0,1) \end{cases} \Longrightarrow \sigma_u \wedge \sigma_v = (\cos u, \sin u, 0)$

2 $\displaystyle \iint_S \vec{V}\cdot\vec{dS} = \int_0^{2\pi} du \int_0^3 dv \cdot \underbrace{\begin{pmatrix} \sin u \\ \cos u \\ -2v \end{pmatrix}\cdot\begin{pmatrix} \cos u \\ \sin u \\ 0 \end{pmatrix}}_{2\sin u\cos u} = \int_0^{2\pi} 2\sin u \cos u \, du \cdot \int_0^3 dv$

$\displaystyle = 2 \cdot \int_0^{\pi} 2\sin u \cos u \, du \cdot [v]_0^3 = 6 \cdot \int_{\cos 0}^{\cos \pi} -2t \, dt = -6 \int_1^{-1} 2t \, dt = -6 \left[t^2\right]_1^{-1}$



$t = \cos u$ bijective entre 0 et π $\Longrightarrow dt = -\sin u \, du$

$= -6 \cdot 1 - (-6)\cdot 1$

$= 0.$

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8. $\vec{W}$ t.q. $\vec{\mathrm{rot}}\, \vec{W} = -\frac{1}{3} \vec{V}$

3 $\displaystyle \oint_{\gamma = \partial D_2} \vec{W}\cdot\vec{dl} = \iint_{D_2} \vec{\mathrm{rot}}\, \vec{W}\cdot\vec{dS} = -\frac{1}{3} \iint_{D_2} \vec{V}\cdot\vec{dS} = -\frac{1}{3}\cdot(-6\pi) = 2\pi.$

CORRIGÉ DU CC4 :

QCM : b d c c d c a e

Q9 Si $V(x,y,z) = f(x,y,z)\,\vec{i} + g(x,y,z)\,\vec{j} + h(x,y,z)\,\vec{k}$, alors

$$\int_{C^+} \vec{V}\cdot\vec{dl} = \int_{C^+} \begin{pmatrix} f \\ g \\ h \end{pmatrix}\cdot\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \int_{C^+} f\,dx + g\,dy + h\,dz .$$

Si C^+ est une courbe param. par $\gamma(t) = (x(t), y(t), z(t))$, $t \in [a,b]$, alors

$$\int_{C^+} \vec{V}\cdot\vec{dl} = \int_a^b \begin{pmatrix} f \\ g \\ h \end{pmatrix}\cdot \gamma'(t)\, dt = \int_a^b \Big(f(x(t), y(t), z(t))\, x'(t) + g \cdot y'(t) + h \cdot z'(t) \Big) dt.$$

Q10 $\displaystyle \iint_{S^+} \vec{V}\cdot\vec{dS} = \iint_{S^+} \begin{pmatrix} f \\ g \\ h \end{pmatrix}\cdot\begin{pmatrix} dy\,dz \\ -dx\,dz \\ dx\,dy \end{pmatrix} = \iint_{S^+} \left[f\,dy\,dz - g\,dx\,dz + h\,dx\,dy \right]$

Si S^+ est param. par $\sigma(u,v) = (x(u,v), y(u,v), z(u,v))$ avec $u \in I$, $v \in J$, alors

$$\iint_{S^+} \vec{V}\cdot\vec{dS} = \int_I du \int_J dv \begin{pmatrix} f \\ g \\ h \end{pmatrix}\cdot(\sigma_u \wedge \sigma_v) .$$