

FORMULAIRE SUR LES CHAMPS VECTORIELS DE $\mathbb{R}^3$

	Coordonnées cartésiennes (x, y, z)	Coordonnées cylindriques (ρ, θ, z)	Coordonnées sphériques (r, θ, φ)
Champ de vecteurs $\vec{V}$	$V_x \vec{i} + V_y \vec{j} + V_z \vec{k}$	$V_\rho \vec{e}_\rho + V_\theta \vec{e}_\theta + V_z \vec{k}$	$V_r \vec{e}_r + V_\theta \vec{e}_\theta + V_\varphi \vec{e}_\varphi$
Gradient $\overrightarrow{\text{grad}} f = \vec{\nabla} f$	$\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$	$\frac{\partial f}{\partial \rho} \vec{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{\partial f}{\partial z} \vec{k}$	$\frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \varphi} \vec{e}_\varphi + \frac{1}{r \cos \varphi} \frac{\partial f}{\partial \theta} \vec{e}_\theta$
Divergence $\text{div } \vec{V} = \nabla \cdot \vec{V}$	$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$	$\frac{1}{\rho} \frac{\partial(\rho V_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z}$	$\frac{1}{r^2} \frac{\partial(r^2 V_r)}{\partial r} + \frac{1}{r \cos \varphi} \frac{\partial(\cos \varphi V_\varphi)}{\partial \varphi} + \frac{1}{r \cos \varphi} \frac{\partial V_\theta}{\partial \theta}$
Rotationnel $\overrightarrow{\text{rot}} \vec{V} = \vec{\nabla} \times \vec{V}$	$\begin{pmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{pmatrix} \begin{matrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{matrix}$	$\begin{pmatrix} \frac{1}{\rho} \frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z} \\ \frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho} \\ \frac{1}{\rho} \left(\frac{\partial(\rho V_\theta)}{\partial \rho} - \frac{\partial V_\rho}{\partial \theta} \right) \end{pmatrix} \begin{matrix} \vec{e}_\rho \\ \vec{e}_\theta \\ \vec{k} \end{matrix}$	$\begin{pmatrix} \frac{1}{r \cos \varphi} \left(\frac{\partial(\cos \varphi V_\theta)}{\partial \varphi} - \frac{\partial V_\varphi}{\partial \theta} \right) \\ \frac{1}{r} \left(\frac{1}{\cos \varphi} \frac{\partial V_r}{\partial \theta} - \frac{\partial(r V_\theta)}{\partial r} \right) \\ \frac{1}{r} \left(\frac{\partial(r V_\varphi)}{\partial r} - \frac{\partial V_r}{\partial \varphi} \right) \end{pmatrix} \begin{matrix} \vec{e}_r \\ \vec{e}_\varphi \\ \vec{e}_\theta \end{matrix}$
Laplacien $\Delta f = \vec{\nabla} \cdot \vec{\nabla} f$ $= \text{div}(\overrightarrow{\text{grad}} f)$	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial f}{\partial \varphi} \right) + \frac{1}{r^2 \cos^2 \varphi} \frac{\partial^2 f}{\partial \theta^2}$
Laplacien vectoriel $\Delta \vec{V}$	$\Delta V_x \vec{i} + \Delta V_y \vec{j} + \Delta V_z \vec{k}$	(affreux...)	(horrible!)

Notations : $\vec{\nabla} = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}$ $\Delta \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ $\vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$
 $\vec{V} \cdot \vec{\nabla} = V_x \frac{\partial}{\partial x} + V_y \frac{\partial}{\partial y} + V_z \frac{\partial}{\partial z}$ $(\vec{V} \cdot \vec{\nabla}) \vec{U} = V_x \frac{\partial U_x}{\partial x} \vec{i} + V_y \frac{\partial U_y}{\partial y} \vec{j} + V_z \frac{\partial U_z}{\partial z} \vec{k}$

Propriétés : $\overrightarrow{\text{grad}}(fg) = (\overrightarrow{\text{grad}} f)g + f(\overrightarrow{\text{grad}} g)$ $\overrightarrow{\text{grad}}(\vec{U} \cdot \vec{V}) = (\vec{U} \cdot \vec{\nabla}) \vec{V} + (\vec{V} \cdot \vec{\nabla}) \vec{U} + \vec{U} \wedge \overrightarrow{\text{rot}} \vec{V} + \vec{V} \wedge \overrightarrow{\text{rot}} \vec{U}$
 $\text{div}(f\vec{V}) = (\overrightarrow{\text{grad}} f) \cdot \vec{V} + f(\text{div } \vec{V})$ $\text{div}(\vec{U} \wedge \vec{V}) = (\overrightarrow{\text{rot}} \vec{U}) \cdot \vec{V} - \vec{U} \cdot (\overrightarrow{\text{rot}} \vec{V})$
 $\overrightarrow{\text{rot}}(f\vec{V}) = (\overrightarrow{\text{grad}} f) \wedge \vec{V} + f(\overrightarrow{\text{rot}} \vec{V})$ $\overrightarrow{\text{rot}}(\vec{U} \wedge \vec{V}) = (\text{div } \vec{V}) \vec{U} - \vec{V}(\text{div } \vec{U}) + (\vec{V} \cdot \vec{\nabla}) \vec{U} - (\vec{U} \cdot \vec{\nabla}) \vec{V}$
 $\Delta(fg) = f \Delta g + 2 \overrightarrow{\text{grad}} f \cdot \overrightarrow{\text{grad}} g + \Delta f g$

Identités : $\overrightarrow{\text{rot}}(\overrightarrow{\text{grad}} f) = \vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$ $\text{div}(\overrightarrow{\text{rot}} \vec{V}) = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}) = 0$
 $\overrightarrow{\text{rot}}(\overrightarrow{\text{rot}} \vec{V}) = \vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \Delta \vec{V} = \overrightarrow{\text{grad}}(\text{div } \vec{V}) - \Delta \vec{V}$

Théorème de Poincaré : Sur $D \subset \mathbb{R}^3$ simplement connexe : $\vec{V} = \overrightarrow{\text{grad}} f \iff \overrightarrow{\text{rot}} \vec{V} = \vec{0}$
Sur $D \subset \mathbb{R}^3$ contractile : $\vec{V} = \overrightarrow{\text{rot}} \vec{U} \iff \text{div } \vec{V} = 0$

Théorème de Stokes : Si $\vec{V} = \overrightarrow{\text{rot}} \vec{U}$: $\iint_{S^+} \vec{V} \cdot d\vec{S} = \oint_{\partial S^+} \vec{U} \cdot d\vec{\ell}$

Théorème de Ostrogradski : Si $S^+ = \partial\Omega$ est une surface fermée : $\oint\!\!\!\oint_{S^+} \vec{V} \cdot d\vec{S} = \iiint_{\Omega} \text{div } \vec{V} \, dx \, dy \, dz$

Corollaires : Si $\vec{V} = \overrightarrow{\text{grad}} f$ et C^+ est une courbe fermée : $\oint_{C^+} \vec{V} \cdot d\vec{\ell} = 0$
Si $\vec{V} = \overrightarrow{\text{grad}} f$ et C^+ est une courbe qui relie A à B : $\int_{C^+} \vec{V} \cdot d\vec{\ell} = f(B) - f(A)$