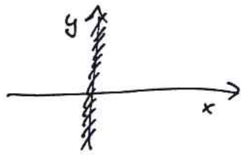
Ex. 1 $\quad f(x,y) = \frac{y}{x}$

1. $D_f = \{ (x,y) \in \mathbb{R}^2 \mid x \neq 0 \} = \mathbb{R}^2 \setminus$ droite $x=0$



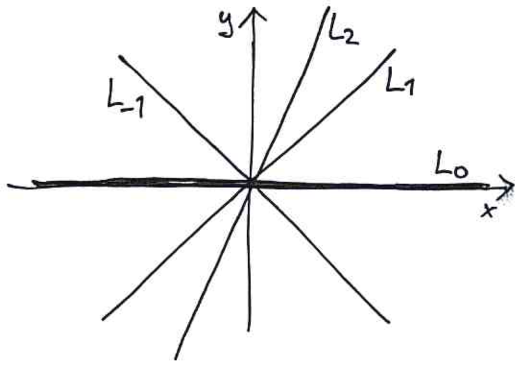
2. $\forall k \in \mathbb{R}$: $L_k(f) = \{ (x,y) \in D_f \mid f(x,y) = k \} = \{ (x,y) \in \mathbb{R}^2 \mid x \neq 0$ et $y = kx \}$

$k=0 \Rightarrow y=0$ axe $\vec{Ox}$ moins $(0,0)$

$k=1 \Rightarrow y=x$ droite bissectrice 1er quadr. moins pt. $(0,0)$

$k=2 \Rightarrow y=2x$

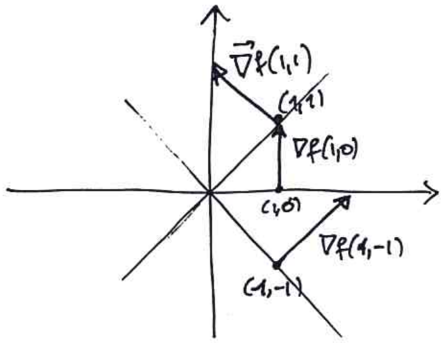
$k=-1 \Rightarrow y=-x$ bissectrice 2ème quadrant moins pt. $(0,0)$



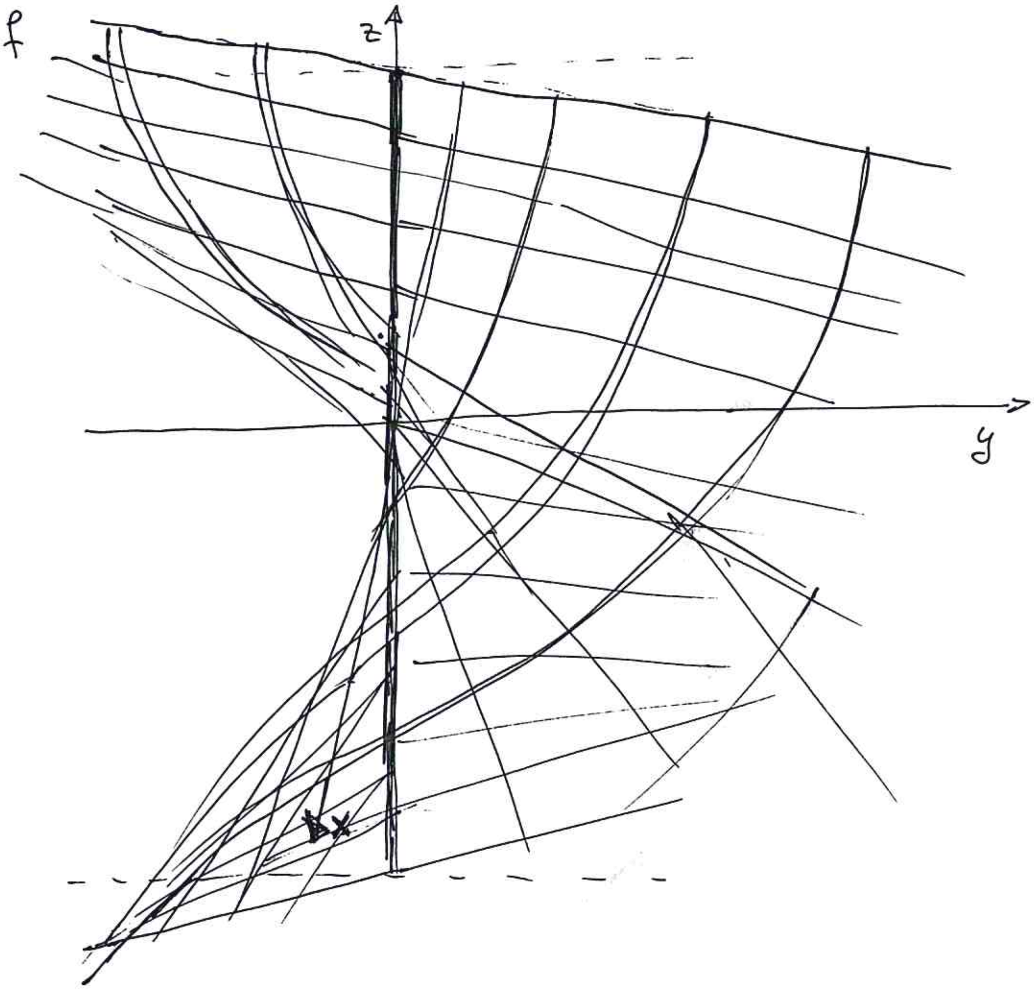
3. $\vec{\nabla} f(x,y) = \begin{pmatrix} -y/x^2 \\ 1/x \end{pmatrix}$ $\quad \vec{\nabla} f(1,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{\nabla} f(1,1) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

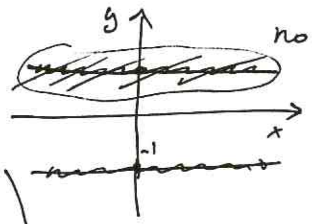
$\vec{\nabla} f(1,-1) = \begin{pmatrix} +1 \\ 1 \end{pmatrix}$



4. graphe de f



Ex. 2 $\qquad f(x,y) = \dfrac{x^3 - xy}{y+1}$



1. $D_f = \{(x,y) \in \mathbb{R}^2 \mid y \neq -1\} = \mathbb{R}^2 \setminus$ droite $y = -1$

2. $$\vec{\nabla} f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} \frac{3x^2 - y}{y+1} \\ \frac{-x(y+1) - (3x^2 - y)}{(y+1)^2} \end{pmatrix} = \begin{pmatrix} \frac{3x^2 - y}{y+1} \\ -\frac{x + x^3}{(y+1)^2} \end{pmatrix}$$

3. $$df_{(x,y)} = \frac{\partial f}{\partial x}(x,y)\, dx + \frac{\partial f}{\partial y}(x,y)\, dy = \frac{3x^2 - y}{y+1}\, dx - \frac{x(1+x^2)}{(y+1)^2}\, dy$$

4. $$df_{(1,0)} = \frac{3-0}{0+1}\, dx - \frac{1+1}{(0+1)^2}\, dy = 3\,dx - 2\,dy$$

5. $$df_{(1,0)}(2,4) = 3 \cdot 2 - 2 \cdot 4 = 6 - 8 = -2$$

6. $$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} \frac{6x}{y+1} & -\frac{1+3x^2}{(y+1)^2} \\ -\frac{1+3x^2}{(y+1)^2} & \frac{2x(1+x^2)}{(y+1)^3} \end{pmatrix}$$

Ex. 3 $\qquad f : \mathbb{R}^2 \to \mathbb{R}$ diff. sur $D = \{(x,y) \in \mathbb{R}^2 \mid x > 0,\ y > 0\}$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{x} \quad \text{et} \quad \frac{\partial f}{\partial y}(x,y) = \frac{1}{y}.$$

1. $F(u,v) = f(3v+1, u^2 - v)$ $\quad$ i.e. $\begin{cases} x = 3v + 1 \\ y = u^2 - v \end{cases}$

$$\frac{\partial F}{\partial u}(u,v) = \frac{\partial f}{\partial x}(3v+1, u^2 - v) \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}(3v+1, u^2 - v) \cdot \frac{\partial y}{\partial u}$$
$$= \frac{1}{3v+1} \cdot 0 + \frac{1}{u^2 - v} \cdot 2u = \frac{2u}{u^2 - v}$$

$$\frac{\partial F}{\partial v}(u,v) = \frac{\partial f}{\partial x}(3v+1, u^2 - v) \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}(3v+1, u^2 - v) \frac{\partial y}{\partial u}$$
$$= \frac{1}{3v+1} \cdot 3 + \frac{1}{u^2 - v} \cdot (-1) = \frac{3}{3v+1} - \frac{1}{u^2 - v}.$$

2. $G(t) = f\left(t, \frac{1}{3t}\right)$ $\quad$ i.e. $\begin{cases} x = t \\ y = \frac{1}{3t} \end{cases}$

$$G'(t) = \frac{\partial f}{\partial x}\left(t, \frac{1}{3t}\right) \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}\left(t, \frac{1}{3t}\right) \cdot \frac{dy}{dt} =$$
$$= \frac{1}{t} \cdot 1 + \frac{1}{\frac{1}{3t}} \cdot \left(-\frac{1}{3t^2}\right) = \frac{1}{t} - \frac{\cancel{3t}}{\cancel{3t^2}} = \frac{1}{t} - \frac{1}{t} = 0.$$