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PCSI L1 UE Math2
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REPÈRES MOBILES ET CHAMPS DE VECTEURS

CONVENTION STANDARD ISO 80000-2

http://en.wikipedia.org/wiki/ISO 80000-2

	Coordonnées cartesiennes	**Coordonnées cylindriques**	**Coordonnées sphériques**
Point P	$(x,y,z) \quad x,y,z \in \mathbb{R}$	$(\rho,\varphi,z) \quad \left[\begin{array}{l}\rho \geq 0\\ \varphi \in [\,0,2\pi[\\ z \in \mathbb{R}\end{array}\right.$	$(r,\varphi,\theta) \quad \left[\begin{array}{l}r \geq 0\\ \varphi \in [\,0,2\pi[\\ \theta \in [0,\pi]\end{array}\right.$
Chang. coordonnées	$\left\{\begin{array}{l}\rho = \sqrt{x^2+y^2}\\ \varphi = \left\{\begin{array}{ll}\arctan\frac{y}{x} & \text{si } x,y>0\\ \text{etc} & \text{sinon}\end{array}\right.\\ z = z\end{array}\right.$ $\left\{\begin{array}{l}r = \sqrt{x^2+y^2+z^2}\\ \varphi = \text{idem}\\ \theta = \arccos\dfrac{z}{\sqrt{x^2+y^2+z^2}}\end{array}\right.$	$\left\{\begin{array}{l}x = \rho\ \cos\varphi\\ y = \rho\ \sin\varphi\\ z = z\end{array}\right.$ $\left\{\begin{array}{l}r = \sqrt{\rho^2+z^2}\\ \varphi = \varphi\\ \theta = \arccos\dfrac{z}{\sqrt{\rho^2+z^2}}\end{array}\right.$	$\left\{\begin{array}{l}x = r\ \cos\varphi\ \sin\theta\\ y = r\ \sin\varphi\ \sin\theta\\ z = r\ \cos\theta\end{array}\right.$ $\left\{\begin{array}{l}\rho = r\ \sin\theta\\ \varphi = \varphi\\ z = r\ \cos\theta\end{array}\right.$
Repère en P	$(P;\ \vec{i},\vec{j},\vec{k}) \quad \left[\begin{array}{l}\vec{i} = \dfrac{\partial}{\partial x}\\ \vec{j} = \dfrac{\partial}{\partial y}\\ \vec{k} = \dfrac{\partial}{\partial z}\end{array}\right.$	$(P;\ \vec{e_\rho},\vec{e_\varphi},\vec{k}) \quad \left[\begin{array}{l}\vec{e_\rho} = \dfrac{\partial}{\partial \rho}\\ \vec{e_\varphi} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \varphi}\\ \vec{k} = \dfrac{\partial}{\partial z}\end{array}\right.$	$(P;\ \vec{e_r},\vec{e_\varphi},\vec{e_\theta}) \quad \left[\begin{array}{l}\vec{e_r} = \dfrac{\partial}{\partial r}\\ \vec{e_\varphi} = \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \varphi}\\ \vec{e_\theta} = \dfrac{1}{r}\dfrac{\partial}{\partial \theta}\end{array}\right.$
Chang. repères	$\left\{\begin{array}{l}\vec{e_\rho} = \cos\varphi\ \vec{i} + \sin\varphi\ \vec{j}\\ \vec{e_\varphi} = -\sin\varphi\ \vec{i} + \cos\varphi\ \vec{j}\\ \vec{k} = \vec{k}\end{array}\right.$ $\left\{\begin{array}{l}\vec{e_r} = \cos\varphi\ \sin\theta\ \vec{i} + \sin\varphi\ \sin\theta\ \vec{j} + \cos\theta\ \vec{k}\\ \vec{e_\varphi} = -\sin\varphi\ \vec{i} + \cos\varphi\ \vec{j}\\ \vec{e_\theta} = \cos\varphi\ \cos\theta\ \vec{i} + \sin\varphi\ \cos\theta\ \vec{j} - \sin\theta\ \vec{k}\end{array}\right.$	$\left\{\begin{array}{l}\vec{i} = \cos\varphi\ \vec{e_\rho} - \sin\varphi\ \vec{e_\varphi}\\ \vec{j} = \sin\varphi\ \vec{e_\rho} + \cos\varphi\ \vec{e_\varphi}\\ \vec{k} = \vec{k}\end{array}\right.$ $\left\{\begin{array}{l}\vec{e_r} = \sin\theta\ \vec{e_\rho} + \cos\theta\ \vec{k}\\ \vec{e_\varphi} = \vec{e_\varphi}\\ \vec{e_\theta} = \cos\theta\ \vec{e_\rho} - \sin\theta\ \vec{k}\end{array}\right.$	$\left\{\begin{array}{l}\vec{i} = \cos\varphi\sin\theta\ \vec{e_r} - \sin\varphi\ \vec{e_\varphi} + \cos\varphi\cos\theta\ \vec{e_\theta}\\ \vec{j} = \sin\varphi\sin\theta\ \vec{e_r} + \cos\varphi\ \vec{e_\varphi} + \sin\varphi\cos\theta\ \vec{e_\theta}\\ \vec{k} = \cos\theta\ \vec{e_r} - \sin\theta\ \vec{e_\theta}\end{array}\right.$ $\left\{\begin{array}{l}\vec{e_\rho} = \sin\theta\ \vec{e_r} + \cos\theta\ \vec{e_\theta}\\ \vec{e_\varphi} = \vec{e_\varphi}\\ \vec{k} = \cos\theta\ \vec{e_\rho} - \sin\theta\ \vec{e_\theta}\end{array}\right.$

	Coordonnées cartesiennes (x, y, z)	**Coordonnées cylindriques** (ρ, φ, z)	**Coordonnées sphériques** (r, φ, θ)
Champ de vecteurs $\vec{A}$	$A_x\ \vec{i} + A_y\ \vec{j} + A_z\ \vec{k}$	$A_\rho\ \vec{e_\rho} + A_\varphi\ \vec{e_\varphi} + A_z\ \vec{k}$	$A_r\ \vec{e_r} + A_\varphi\ \vec{e_\varphi} + A_\theta\ \vec{e_\theta}$
Gradient $\overrightarrow{\text{grad}}\, f = \vec{\nabla} f$	$\dfrac{\partial f}{\partial x}\ \vec{i} + \dfrac{\partial f}{\partial y}\ \vec{j} + \dfrac{\partial f}{\partial z}\ \vec{k}$	$\dfrac{\partial f}{\partial \rho}\ \vec{e_\rho} + \dfrac{1}{\rho}\dfrac{\partial f}{\partial \varphi}\ \vec{e_\varphi} + \dfrac{\partial f}{\partial z}\ \vec{k}$	$\dfrac{\partial f}{\partial r}\ \vec{e_r} + \dfrac{1}{r\sin\theta}\dfrac{\partial f}{\partial \varphi}\ \vec{e_\varphi} + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\ \vec{e_\theta}$
Divergence $\text{div}\, \vec{A} = \nabla \cdot \vec{A}$	$\dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$	$\dfrac{1}{\rho}\dfrac{\partial(\rho A_\rho)}{\partial \rho} + \dfrac{1}{\rho}\dfrac{\partial A_\varphi}{\partial \varphi} + \dfrac{\partial A_z}{\partial z}$	$\dfrac{1}{r^2}\dfrac{\partial(r^2 A_r)}{\partial r} + \dfrac{1}{r\sin\theta}\dfrac{\partial(\sin\theta\ A_\theta)}{\partial \theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial A_\varphi}{\partial \varphi}$
Rotationnel $\overrightarrow{\text{rot}}\, \vec{A} = \vec{\nabla} \times \vec{A}$	$\left(\dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z}\right)\ \vec{i}$ $+\left(\dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x}\right)\ \vec{j}$ $+\left(\dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y}\right)\ \vec{k}$	$\left(\dfrac{1}{\rho}\dfrac{\partial A_z}{\partial \varphi} - \dfrac{\partial A_\varphi}{\partial z}\right)\ \vec{e_\rho}$ $+\left(\dfrac{\partial A_\rho}{\partial z} - \dfrac{\partial A_z}{\partial \rho}\right)\ \vec{e_\varphi}$ $+\dfrac{1}{\rho}\left(\dfrac{\partial(\rho A_\varphi)}{\partial \rho} - \dfrac{\partial A_\rho}{\partial \varphi}\right)\ \vec{k}$	$\dfrac{1}{r\ \sin\theta}\left(\dfrac{\partial(\sin\theta\ A_\varphi)}{\partial \theta} - \dfrac{\partial A_\theta}{\partial \varphi}\right)\ \vec{e_r}$ $+\dfrac{1}{r}\left(\dfrac{\partial(r A_\theta)}{\partial r} - \dfrac{\partial A_r}{\partial \theta}\right)\ \vec{e_\varphi}$ $+\dfrac{1}{r}\left(\dfrac{1}{\sin\theta}\dfrac{\partial A_r}{\partial \varphi} - \dfrac{\partial(r A_\varphi)}{\partial r}\right)\ \vec{e_\theta}$
Laplacien $\Delta f = \vec{\nabla} \cdot \vec{\nabla} f$ $= \text{div}\,(\overrightarrow{\text{grad}}\, f)$	$\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2}$	$\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho\dfrac{\partial f}{\partial \rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 f}{\partial \varphi^2} + \dfrac{\partial^2 f}{\partial z^2}$	$\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial f}{\partial r}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 f}{\partial \varphi^2}$ $+\dfrac{1}{r^2\cos\theta}\dfrac{\partial}{\partial \theta}\left(\cos\theta\ \dfrac{\partial f}{\partial \theta}\right)$
Laplacien vectoriel $\Delta\vec{A}$	$\Delta A_x\ \vec{i} + \Delta A_y\ \vec{j} + \Delta A_z\ \vec{k}$	(affreux...)	(horrible !)

Notations : $\vec{\nabla} = \dfrac{\partial}{\partial x}\ \vec{i} + \dfrac{\partial}{\partial y}\ \vec{j} + \dfrac{\partial}{\partial z}\ \vec{k}$ $\qquad \Delta \equiv \nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ $\qquad \vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

$\vec{A} \cdot \vec{\nabla} = A_x \dfrac{\partial}{\partial x} + A_y \dfrac{\partial}{\partial y} + A_z \dfrac{\partial}{\partial z}$ $\qquad (\vec{A} \cdot \vec{\nabla})\vec{B} = A_x \frac{\partial B_x}{\partial x}\ \vec{i} + A_y \frac{\partial B_y}{\partial y}\ \vec{j} + A_z \frac{\partial B_z}{\partial z}\ \vec{k}$

Proprietés :

$\overrightarrow{\text{grad}}\,(fg) = (\overrightarrow{\text{grad}}\, f)g + f(\overrightarrow{\text{grad}}\, g)$ $\qquad \overrightarrow{\text{grad}}\,(\vec{B} \cdot \vec{A}) = (\vec{B} \cdot \vec{\nabla})\vec{A} + (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{B} \wedge \overrightarrow{\text{rot}}\, \vec{A} + \vec{A} \wedge \overrightarrow{\text{rot}}\, \vec{B}$

$\text{div}\,(f\vec{A}) = (\overrightarrow{\text{grad}}\, f)\vec{A} + f(\text{div}\, \vec{A})$ $\qquad \text{div}\,(\vec{B} \wedge \vec{A}) = (\overrightarrow{\text{rot}}\, \vec{B}) \cdot \vec{A} - \vec{B} \cdot (\overrightarrow{\text{rot}}\, \vec{A})$

$\overrightarrow{\text{rot}}\,(f\vec{A}) = (\overrightarrow{\text{grad}}\, f) \wedge \vec{A} + f\ (\overrightarrow{\text{rot}}\, \vec{A})$ $\qquad \overrightarrow{\text{rot}}\,(\vec{B} \wedge \vec{A}) = (\text{div}\, \vec{A})\ \vec{B} - \vec{A}\ (\text{div}\, \vec{B}) + (\vec{A} \cdot \vec{\nabla})\ \vec{B} - (\vec{B} \cdot \vec{\nabla})\ \vec{A}$

$\Delta(fg) = f\,\Delta g + 2\,\overrightarrow{\text{grad}}\, f \cdot \overrightarrow{\text{grad}}\, g + \Delta f\ g$

Identités : $\overrightarrow{\text{rot}}\,(\overrightarrow{\text{grad}}\, f) = \vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$ $\qquad \text{div}\,(\overrightarrow{\text{rot}}\, \vec{A}) = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

$\overrightarrow{\text{rot}}\,(\overrightarrow{\text{rot}}\, \vec{A}) = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \Delta\vec{A} = \overrightarrow{\text{grad}}\,(\text{div}\, \vec{A}) - \Delta\vec{A}$

Théorème de Poincaré : Sur $D \subset \mathbb{R}^3$ *simplement connexe* : $\quad \vec{A} = \overrightarrow{\text{grad}}\, f \iff \overrightarrow{\text{rot}}\, \vec{A} = 0$

Sur $D \subset \mathbb{R}^3$ *contractile* : $\quad \vec{A} = \overrightarrow{\text{rot}}\, \vec{B} \iff \text{div}\, \vec{A} = 0$

Théorème de Stokes : Si $\vec{A} = \overrightarrow{\text{rot}}\, \vec{B}$: $\quad \iint_{S^+} \vec{A} \cdot d\vec{S} = \oint_{\partial S^+} \vec{B} \cdot d\vec{\ell}$

Théorème de Gauss : Si $S^+ = \partial\Omega$ est une surface fermée : $\quad \oiint_{S^+} \vec{A} \cdot d\vec{S} = \iiint_{\Omega} \text{div}\, \vec{A}\ dx\ dy\ dz$

Corollaires : Si $\vec{A} = \overrightarrow{\text{grad}}\, f$ et C^+ est une courbe fermée : $\quad \oint_{C^+} \vec{A} \cdot d\vec{\ell} = 0$

Si $\vec{A} = \overrightarrow{\text{grad}}\, f$ et C^+ est une courbe qui relie P à Q : $\quad \int_{C^+} \vec{A} \cdot d\vec{\ell} = f(Q) - f(P)$