Math 2 - Examen final 1ère session - 17/05/2018 - Corrigé

QCM - c a d b a

Exercice 1 : $u(x,t) = x\sin(xt) \Rightarrow \vec{\nabla}u(x,t) = \begin{pmatrix} \sin(xt) + xt\cos(xt) \\ x^2\cos(xt) \end{pmatrix}$

$$\Delta u(x,t) = t\cos(xt) + t\cos(xt) - xt^2\sin(xt) - x^3\sin(xt)$$
$$= 2t\cos(xt) - x(t^2+x^2)\sin(xt)$$

Exercice 2 : $\Omega = \{(x,y,z) \mid x^2+y^2 \leq 1,\ 0 \leq z \leq 3\}$, $\mu(x,y,z) = \dfrac{x^2+y^2}{z+1}$

a) $\tilde{\Omega} = \{(\rho,\varphi,z) \mid \rho \leq 1,\ 0 \leq \varphi \leq 2\pi,\ 0 \leq z \leq 3\}$

$\tilde{\mu}(\rho,\varphi,z) = \dfrac{\rho^2}{z+1}$

b) $$M = \iiint_\Omega \mu(x,y,z)\,dx\,dy\,dz = \iiint_{\tilde{\Omega}} \tilde{\mu}(\rho,\varphi,z)\,\rho\,d\rho\,d\varphi\,dz$$
$$= \iiint_{\tilde{\Omega}} \frac{\rho^3}{z+1}\,d\rho\,d\varphi\,dz = \int_0^1 \rho^3 d\rho \int_0^{2\pi} d\varphi \int_0^3 \frac{1}{z+1}\,dz = \left[\frac{1}{4}\rho^4\right]_0^1 \left[\varphi\right]_0^{2\pi} \left[\ln(z+1)\right]_0^3$$
$$= \frac{1}{4}\cdot 2\pi \cdot (\ln 4 - \underbrace{\ln 1}_{=0}) = \frac{2\pi \cdot 2\ln 2}{4} = \pi \ln 2$$

c) $G(x_G, y_G, z_G)$ centre de masse :

$x_G = 0$ et $y_G = 0$ car Ω est symétrique par rapport à l'axe $\vec{Oz}$ et $\mu(x,y,z)$ aussi. Vérifions :

$$x_G = \frac{1}{M}\iiint_\Omega x\,\mu(x,y,z)\,dx\,dy\,dz = \frac{1}{M}\iiint_{\tilde{\Omega}} \rho\cos\varphi\,\tilde{\mu}(\rho,\varphi,z)\,\rho\,d\rho\,d\varphi\,dz$$
$$= \frac{1}{\pi\ln 2}\int_0^1 \rho^4 d\rho \int_0^{2\pi}\cos\varphi\,d\varphi \int_0^3 \frac{1}{z+1}\,dz = \frac{1}{\pi\ln 2}\left[\frac{1}{5}\rho^5\right]_0^1 \underbrace{\left[\sin\varphi\right]_0^{2\pi}}_{=0} \left[\ln(z+1)\right]_0^3 = 0$$

idem pour $y_G = \dfrac{1}{\pi\ln 2}\left[\dfrac{1}{5}\rho^5\right]_0^1 \underbrace{\left[-\cos\varphi\right]_0^{2\pi}}_{=0} \left[\ln(z+1)\right]_0^3 = 0$

$$z_G = \frac{1}{M}\iiint_\Omega z\,\mu(x,y,z)\,dx\,dy\,dz = \frac{1}{M}\iiint_{\tilde{\Omega}} z\,\tilde{\mu}(\rho,\varphi,z)\,\rho\,d\rho\,d\varphi\,dz$$
$$= \frac{1}{\pi\ln 2}\int_0^1 \rho^3 d\rho \int_0^{2\pi} d\varphi \int_0^3 \frac{z}{z+1}\,dz$$

où $\displaystyle\int_0^3 \frac{z}{z+1}\,dz = \int_0^3 \frac{z+1-1}{z+1}\,dz = \int_0^3 \left(1 - \frac{1}{z+1}\right)dz = \left[z - \ln(z+1)\right]_0^3 = 3 - \ln 4 = 3 - 2\ln 2$

donc $z_G = \dfrac{1}{\cancel{\pi}\ln 2}\cdot\dfrac{1}{4}\cdot 2\cancel{\pi}\cdot(3-2\ln 2) = \dfrac{3-2\ln 2}{2\ln 2}$.

Conclusion: $G\left(0, 0, \dfrac{3-2\ln 2}{2\ln 2}\right)$.

2)

Exercice 3 : $\vec{E}(x,y,z) = 2xz\,\vec{\imath} + \ln z\,\vec{\jmath} + (x^2 + \frac{y}{z})\,\vec{k}$

$D = \{(x,y,z) \in \mathbb{R}^3 \mid z > 0\}$

a) $\vec{E}$ est conservatif, i.e. $\exists$ potentiel scalaire, car :

$$\vec{\mathrm{rot}}\,\vec{E} = \det\begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ 2xz & \ln z & x^2 + \frac{y}{z} \end{pmatrix} = \vec{\imath}\left(\tfrac{1}{z} - \tfrac{1}{z}\right) - \vec{\jmath}(2x - 2x) + \vec{k}(0-0) = \vec{0}$$

et puisque D est simplement convexe, par le lemme de Poincaré I

$\vec{E}$ ~~[illegible]~~ est un champ de gradient, $\vec{E} = \vec{\nabla} f$.

b) Potentiel scalaire $f : D \to \mathbb{R}$ tq. $\vec{E} = \vec{\nabla} f$:

$$\begin{cases} \frac{\partial f}{\partial x} = 2xz & (1) \\ \frac{\partial f}{\partial y} = \ln z & (2) \\ \frac{\partial f}{\partial q} = x^2 + \frac{y}{z} & (3) \end{cases}$$

$\Rightarrow f(x,y,z) = \int 2xz\,dx + g(y,z) = x^2 z + g(y,z)$

$\frac{\partial f}{\partial y} = 0 + \frac{\partial g}{\partial y} \overset{(2)}{=} \ln z \Rightarrow g(y,z) = \int \ln z\,dy + h(z) = y\ln z + h(z)$

$\Rightarrow f(x,y,z) = x^2 z + y\ln z + h(z)$

$\frac{\partial f}{\partial z} = x^2 + \frac{y}{z} + h'(z) \overset{(3)}{=} x^2 + \frac{y}{z} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C \in \mathbb{R}$

conclusion : $f(x,y,z) = x^2 z + y\ln z + C$.

c) Circulation de $\vec{E}$ le long d'un cercle (= courbe fermée) :

$$\oint_\gamma \vec{E} \cdot \vec{d\ell} = \oint_\gamma \vec{\mathrm{grad}}\, f \cdot \vec{d\ell} = 0.$$

Exercice 4 : $\vec{V}(x,y,z) = \vec{\imath} + xz\,\vec{\jmath} + x^2 y\,\vec{k}$

$S = \{ f(u,v) = (u, u^3, v) \mid u \in [-1,1],\ v \in [0,1] \}$

$\vec{V}(f(u,v)) = \vec{\imath} + uv\,\vec{\jmath} + u^2 u^3\,\vec{k} = \vec{\imath} + uv\,\vec{\jmath} + u^5\,\vec{k}$

$$\vec{n}_S = \frac{\partial f}{\partial u} \wedge \frac{\partial f}{\partial v} = \begin{pmatrix} 1 \\ 3u^2 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3u^2 \\ -1 \\ 0 \end{pmatrix} = 3u^2\,\vec{\imath} - \vec{\jmath}$$

$$\iint_S \vec{V} \cdot \vec{dS} = \int_{-1}^{1} du \int_0^1 dv\ \vec{V}(f(u,v)) \cdot \vec{n}_S = \int_{-1}^{1} du \int_0^1 dv \begin{pmatrix} 1 \\ uv \\ u^5 \end{pmatrix} \cdot \begin{pmatrix} 3u^2 \\ -1 \\ 0 \end{pmatrix}$$

$$= \int_{-1}^{1} du \int_0^1 dv\,(3u^2 - uv) = \int_{-1}^{1} du \left[3u^2 v - \tfrac{1}{2}uv^2\right]_{v=0}^{v=1} = \int_{-1}^{1} \left(3u^2 - \tfrac{1}{2}u\right) du$$

$$= \left[u^3 - \tfrac{1}{4}u^2\right]_{-1}^{1} = \left(1 - \tfrac{1}{4}\right) - \left((-1)^3 - \tfrac{1}{4}(-1)^2\right) = 1 \cancel{-\tfrac{1}{4}} + 1 \cancel{+\tfrac{1}{4}} = 2.$$

Exercice 5 : $\vec{B}(x,y,z) = y^2\,\vec{\imath} - x^3 z\,\vec{\jmath} + y\,\vec{k}$ $\quad S =$ sphère (surface fermée)

Gauss : $\mathrm{div}\,\vec{B} = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(-x^3 z) + \frac{\partial}{\partial z}(y) = 0$

$$\Rightarrow \oiint_S \vec{B} \cdot \vec{dS} = \iiint_\Omega \mathrm{div}\,\vec{B} \cdot dx\,dy\,dz = 0.$$

où $\partial\Omega = S$