

REPÈRES MOBILES ET CHAMPS DE VECTEURS

CONVENTION STANDARD ISO 80000-2

https://en.wikipedia.org/wiki/Vector_calculus_identities

	Coordonnées cartésiennes	Coordonnées cylindriques	Coordonnées sphériques
Point P	$(x, y, z) \quad x, y, z \in \mathbb{R}$	$(\rho, \varphi, z) \quad \begin{cases} \rho \geq 0 \\ \varphi \in [0, 2\pi[\\ z \in \mathbb{R} \end{cases}$	$(r, \theta, \varphi) \quad \begin{cases} r \geq 0 \\ \theta \in [0, \pi] \\ \varphi \in [0, 2\pi[\end{cases}$
Changement coordonnées	$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \varphi = \begin{cases} \arctan \frac{y}{x} & \text{si } x, y > 0 \\ \text{etc} & \text{sinon} \end{cases} \\ z = z \end{cases}$ $\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \varphi = \text{idem} \end{cases}$	$\begin{cases} x = \rho \cos \varphi \\ y = \rho \sin \varphi \\ z = z \end{cases}$ $\begin{cases} r = \sqrt{\rho^2 + z^2} \\ \theta = \arccos \frac{z}{\sqrt{\rho^2 + z^2}} \\ \varphi = \varphi \end{cases}$	$\begin{cases} x = r \cos \varphi \sin \theta \\ y = r \sin \varphi \sin \theta \\ z = r \cos \theta \end{cases}$ $\begin{cases} \rho = r \sin \theta \\ \varphi = \varphi \\ z = r \cos \theta \end{cases}$
Élément de volume	$dx dy dz$	$\rho d\rho d\varphi dz$	$r^2 \sin \theta dr d\theta d\varphi$
Repère en P (orthonormal direct)	$(P; \vec{i}, \vec{j}, \vec{k}) \quad \begin{cases} \vec{i} = \frac{\partial}{\partial x} \\ \vec{j} = \frac{\partial}{\partial y} \\ \vec{k} = \frac{\partial}{\partial z} \end{cases}$	$(P; \vec{e}_\rho, \vec{e}_\varphi, \vec{k}) \quad \begin{cases} \vec{e}_\rho = \frac{\partial}{\partial \rho} \\ \vec{e}_\varphi = \frac{1}{\rho} \frac{\partial}{\partial \varphi} \\ \vec{k} = \frac{\partial}{\partial z} \end{cases}$	$(P; \vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi) \quad \begin{cases} \vec{e}_r = \frac{\partial}{\partial r} \\ \vec{e}_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} \\ \vec{e}_\varphi = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \end{cases}$
Changement repère	$\begin{cases} \vec{e}_\rho = \cos \varphi \vec{i} + \sin \varphi \vec{j} \\ \vec{e}_\varphi = -\sin \varphi \vec{i} + \cos \varphi \vec{j} \\ \vec{k} = \vec{k} \end{cases}$ $\begin{cases} \vec{e}_r = \cos \varphi \sin \theta \vec{i} + \sin \varphi \sin \theta \vec{j} + \cos \theta \vec{k} \\ \vec{e}_\theta = \cos \varphi \cos \theta \vec{i} + \sin \varphi \cos \theta \vec{j} - \sin \theta \vec{k} \\ \vec{e}_\varphi = -\sin \varphi \vec{i} + \cos \varphi \vec{j} \end{cases}$	$\begin{cases} \vec{i} = \cos \varphi \vec{e}_\rho - \sin \varphi \vec{e}_\varphi \\ \vec{j} = \sin \varphi \vec{e}_\rho + \cos \varphi \vec{e}_\varphi \\ \vec{k} = \vec{k} \end{cases}$ $\begin{cases} \vec{e}_r = \sin \theta \vec{e}_\rho + \cos \theta \vec{k} \\ \vec{e}_\theta = \cos \theta \vec{e}_\rho - \sin \theta \vec{k} \\ \vec{e}_\varphi = \vec{e}_\varphi \end{cases}$	$\begin{cases} \vec{i} = \cos \varphi \sin \theta \vec{e}_r + \cos \varphi \cos \theta \vec{e}_\theta - \sin \varphi \vec{e}_\varphi \\ \vec{j} = \sin \varphi \sin \theta \vec{e}_r + \sin \varphi \cos \theta \vec{e}_\theta + \cos \varphi \vec{e}_\varphi \\ \vec{k} = \cos \theta \vec{e}_r - \sin \theta \vec{e}_\theta \end{cases}$ $\begin{cases} \vec{e}_\rho = \sin \theta \vec{e}_r + \cos \theta \vec{e}_\theta \\ \vec{e}_\varphi = \vec{e}_\varphi \\ \vec{k} = \cos \theta \vec{e}_r - \sin \theta \vec{e}_\theta \end{cases}$
Courbe paramétrée	$(x(t), y(t), z(t)) \quad t \in \mathbb{R}$	$(\rho(t), \varphi(t), z(t)) \quad t \in \mathbb{R}$	$(r(t), \theta(t), \varphi(t)) \quad t \in \mathbb{R}$
Vecteur position $\vec{x}(t)$	$x(t) \vec{i} + y(t) \vec{j} + z(t) \vec{k}$	$\rho(t) \vec{e}_\rho(t) + z(t) \vec{k}$	$r(t) \vec{e}_r(t)$
Vecteur vitesse $\dot{\vec{x}}(t)$	$\dot{x}(t) \vec{i} + \dot{y}(t) \vec{j} + \dot{z}(t) \vec{k}$	$\dot{\rho}(t) \vec{e}_\rho(t) + \rho(t) \dot{\varphi}(t) \vec{e}_\varphi(t) + \dot{z}(t) \vec{k}$	$\dot{r}(t) \vec{e}_r(t) + r(t) \dot{\theta}(t) \vec{e}_\theta(t) + r(t) \sin \theta(t) \dot{\varphi}(t) \vec{e}_\varphi(t)$
Vitesse du repère sur une courbe	$\begin{cases} \dot{x}(t) = 0 \\ \dot{y}(t) = 0 \\ \dot{z}(t) = 0 \end{cases}$	$\begin{cases} \dot{\vec{e}}_\rho(t) = \dot{\varphi}(t) \vec{e}_\varphi(t) \\ \dot{\vec{e}}_\varphi(t) = -\dot{\varphi}(t) \vec{e}_\rho(t) \\ \dot{\vec{k}}(t) = 0 \end{cases}$	$\begin{cases} \dot{\vec{e}}_r(t) = \dot{\varphi}(t) \sin \theta(t) \vec{e}_\varphi(t) + \dot{\theta}(t) \vec{e}_\theta(t) \\ \dot{\vec{e}}_\theta(t) = -\dot{\theta}(t) \vec{e}_r(t) + \dot{\varphi}(t) \cos \theta(t) \vec{e}_\varphi(t) \\ \dot{\vec{e}}_\varphi(t) = -\dot{\varphi}(t) \sin \theta(t) \vec{e}_r(t) - \dot{\varphi}(t) \cos \theta(t) \vec{e}_\theta(t) \end{cases}$

	Coordonnées cartésiennes (x, y, z)	Coordonnées cylindriques (ρ, φ, z)	Coordonnées sphériques (r, θ, φ)
Champ de vecteurs $\vec{A}$	$A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$	$A_\rho \vec{e}_\rho + A_\varphi \vec{e}_\varphi + A_z \vec{k}$	$A_r \vec{e}_r + A_\theta \vec{e}_\theta + A_\varphi \vec{e}_\varphi$
Gradient $\overrightarrow{\text{grad}} f = \vec{\nabla} f$	$\frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$	$\frac{\partial f}{\partial \rho} \vec{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \varphi} \vec{e}_\varphi + \frac{\partial f}{\partial z} \vec{k}$	$\frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \vec{e}_\varphi$
Divergence $\text{div } \vec{A} = \nabla \cdot \vec{A}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{\rho} \frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r^2} \frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\sin \theta A_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\varphi}{\partial \varphi}$
Rotationnel $\overrightarrow{\text{rot}} \vec{A} = \vec{\nabla} \times \vec{A}$	$\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \vec{i}$ $+ \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \vec{j}$ $+ \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \vec{k}$	$\left(\frac{1}{\rho} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) \vec{e}_\rho$ $+ \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \vec{e}_\varphi$ $+ \frac{1}{\rho} \left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi} \right) \vec{k}$	$\frac{1}{r \sin \theta} \left(\frac{\partial(\sin \theta A_\varphi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi} \right) \vec{e}_r$ $+ \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \varphi} - \frac{\partial(r A_\varphi)}{\partial r} \right) \vec{e}_\theta$ $+ \frac{1}{r} \left(\frac{\partial(r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \vec{e}_\varphi$
Laplacien $\Delta f = \vec{\nabla} \cdot \vec{\nabla} f$ $= \text{div}(\overrightarrow{\text{grad}} f)$	$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$	$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \cos \theta} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial f}{\partial \theta} \right)$ $+ \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2}$
Laplacien vectoriel $\Delta \vec{A}$	$\Delta A_x \vec{i} + \Delta A_y \vec{j} + \Delta A_z \vec{k}$	(affreux...)	(horrible!)

Notations : $\vec{\nabla} = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}$ $\Delta \equiv \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ $\vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$

$$\vec{A} \cdot \vec{\nabla} = A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z}$$

Propriétés : $\overrightarrow{\text{grad}}(fg) = (\overrightarrow{\text{grad}} f)g + f(\overrightarrow{\text{grad}} g)$ $\overrightarrow{\text{grad}}(\vec{B} \cdot \vec{A}) = (\vec{B} \cdot \vec{\nabla})\vec{A} + (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{B} \wedge \overrightarrow{\text{rot}} \vec{A} + \vec{A} \wedge \overrightarrow{\text{rot}} \vec{B}$
 $\text{div}(f\vec{A}) = (\overrightarrow{\text{grad}} f) \cdot \vec{A} + f(\text{div } \vec{A})$ $\text{div}(\vec{B} \wedge \vec{A}) = (\overrightarrow{\text{rot}} \vec{B}) \cdot \vec{A} - \vec{B} \cdot (\overrightarrow{\text{rot}} \vec{A})$
 $\overrightarrow{\text{rot}}(f\vec{A}) = (\overrightarrow{\text{grad}} f) \wedge \vec{A} + f(\overrightarrow{\text{rot}} \vec{A})$ $\overrightarrow{\text{rot}}(\vec{B} \wedge \vec{A}) = (\text{div } \vec{A}) \vec{B} - \vec{A}(\text{div } \vec{B}) + (\vec{A} \cdot \vec{\nabla}) \vec{B} - (\vec{B} \cdot \vec{\nabla}) \vec{A}$
 $\Delta(fg) = f \Delta g + 2 \overrightarrow{\text{grad}} f \cdot \overrightarrow{\text{grad}} g + \Delta f g$

Identités : $\overrightarrow{\text{rot}}(\overrightarrow{\text{grad}} f) = \vec{\nabla} \times (\vec{\nabla} f) = \vec{0}$ $\text{div}(\overrightarrow{\text{rot}} \vec{A}) = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$

$$\overrightarrow{\text{rot}}(\overrightarrow{\text{rot}} \vec{A}) = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \Delta \vec{A} = \overrightarrow{\text{grad}}(\text{div } \vec{A}) - \Delta \vec{A}$$

Lemme de Poincaré : Sur $D \subset \mathbb{R}^3$ simplement connexe : $\vec{A} = \overrightarrow{\text{grad}} f \iff \overrightarrow{\text{rot}} \vec{A} = \vec{0}$

Sur $D \subset \mathbb{R}^3$ contractile : $\vec{A} = \overrightarrow{\text{rot}} \vec{B} \iff \text{div } \vec{A} = 0$

Théorème de Stokes : Si $\vec{A} = \overrightarrow{\text{rot}} \vec{B}$: $\iint_{S^+} \vec{A} \cdot d\vec{S} = \oint_{\partial S^+} \vec{B} \cdot d\vec{\ell}$

Théorème de Gauss : Si $S^+ = \partial\Omega$ est une surface fermée : $\oint\!\!\!\oint_{S^+} \vec{A} \cdot d\vec{S} = \iiint_{\Omega} \text{div } \vec{A} \, dx \, dy \, dz$

Corollaires : Si $\vec{A} = \overrightarrow{\text{grad}} f$ et C^+ est une courbe fermée : $\oint_{C^+} \vec{A} \cdot d\vec{\ell} = 0$

Si $\vec{A} = \overrightarrow{\text{grad}} f$ et C^+ est une courbe qui relie P à Q : $\int_{C^+} \vec{A} \cdot d\vec{\ell} = f(Q) - f(P)$