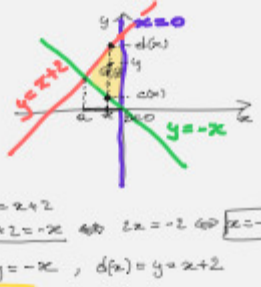
Exo 35 c) D = partie bornée du plan délimitée par les droites $x=0$, $y=x+2$, $y=-x$



• Représentation de D sous la forme "Fubini"

$D = \{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ c(x)\le y\le d(x)\}$

Pour trouver a et b : a = sol. du système $\begin{cases} y=x+2 \\ y=-x \end{cases}$ $\begin{cases} =x+2 \\ x+2=-x \end{cases}$ $\Leftrightarrow 2x=-2 \Leftrightarrow x=-1=a$

$a=-1$, $b=0$ on le voit sur le dessin.

Pour trouver $c(x)$ et $d(x)$ on regarde le dessin : $c(x)=y=-x$, $d(x)=y=x+2$

$\Rightarrow D = \{(x,y)\in\mathbb{R}^2 \mid -1\le x\le 0,\ -x\le y\le x+2\}$

• $\iint_D (1+x+x^3+y^2+y^4)\,dx\,dy \overset{\text{Fubini}}{=} \int_{-1}^{0} dx \int_{-x}^{x+2} (1+x+x^3+y^2+y^4)\,dy$

$= \int_{-1}^{0} dx \left[(1+x+x^3)y + \frac{1}{3}y^3 + \frac{1}{5}y^5\right]_{y=-x}^{y=x+2} = \int_{-1}^{0} \Big((1+x+x^3)(x+2) + \frac{1}{3}(x+2)^3 + \frac{1}{5}(x+2)^5 - (1+x+x^3)(-x) - \frac{1}{3}(-x)^3 - \frac{1}{5}(-x)^5\Big)\,dx$

$= \int_{-1}^{0} \Big(\underbrace{(1+x+x^3)}_{\text{multiplier}}\underbrace{(x+2+x)}_{2+2x} + \underbrace{\frac{1}{3}(x+2)^3}_{\text{dont var } x+2} + \underbrace{\frac{1}{5}(x+2)^5}_{\text{directe}} + \frac{1}{3}x^3 + \frac{1}{5}x^5\Big)\,dx$

$= \int_{-1}^{0} \Big(2+2x+2x+2x^2+2x^3+2x^4+\frac{1}{3}x^3+\frac{1}{5}x^5\Big)\,dx + \int_{-1}^{0} \Big(\frac{1}{3}\underbrace{(x+2)}_{u}^3 + \frac{1}{5}\underbrace{(x+2)}_{u}^5\Big)\,dx$

$2+4x+2x^2+\frac{7}{3}x^3+2x^4+\frac{1}{5}x^5$; $u=x+2 \Rightarrow x=u-2 \Rightarrow dx=du$; $x=-1 \Rightarrow u=-1+2=1$, $x=0 \Rightarrow u=2$

$= \left[2x+2x^2+\frac{7}{12}x^4+\frac{2}{5}x^5+\frac{1}{30}x^6\right]_{-1}^{0} + \int_1^2 \Big(\frac{1}{3}u^3+\frac{1}{5}u^5\Big)\,du$

$= \cdots\cdots + \left[\frac{1}{12}u^4+\frac{1}{30}u^6\right]_1^2$

$= 0 - \Big(-2+2-\frac{2}{3}+\frac{7}{12}-\frac{1}{5}+\frac{1}{30}\Big) + \Big(\frac{1}{12}\cdot 16 + \frac{1}{30}\cdot 64\Big) - \Big(\frac{1}{12}+\frac{1}{30}\Big)$

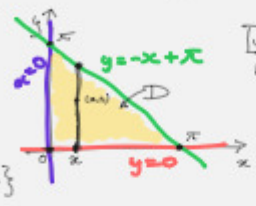
$= \frac{2}{3} - \frac{7}{12} + \frac{2}{5} - \frac{1}{30} + \frac{8}{5} + \frac{32}{15} - \frac{1}{12} - \frac{1}{30}$

$= \frac{26}{12} - \frac{8}{12} + \frac{2}{6} - \frac{1}{30} + \frac{32}{15} = 2 - \frac{2}{3} + \frac{2}{5} + \frac{31}{15} = \frac{30-10+6+31}{15}$

$= \frac{67-10}{15} = \frac{57}{15} = \frac{19}{5}$ ok!

d) On le saute car les calculs sont très long et ça apporte rien de nouveau sur les intégrales doubles.

e) $D = \{(x,y)\in\mathbb{R}^2 \mid x\ge 0,\ y\ge 0,\ x+y\le\pi\}$



$y \le \pi - x$
bord : $y=\pi-x$
$y=-x+\pi$

⚠ ce n'est pas la "forme de Fubini"

? ≤ x ≤ ? ? ≤ y ≤ ?

$D = \{(x,y)\in\mathbb{R}^2 \mid 0\le x\le\pi,\ 0\le y\le\pi-x\}$

$\iint_D \sin(x+y)\,dx\,dy \overset{\text{Fubini}}{=} \int_0^{\pi} dx \int_0^{\pi-x} \sin(\underbrace{x+y}_{u})\,\underbrace{dy}_{du}$

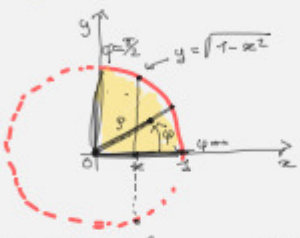
$u=x+y$; $y=u-x \Rightarrow dy=du$; $y=0 \Rightarrow u=x$, $y=\pi-x \Rightarrow u=\pi$

$= \int_0^{\pi} dx \int_x^{\pi} \sin(u)\,du = \int_0^{\pi} dx \left[-\cos u\right]_{u=x}^{u=\pi} = \int_0^{\pi} \Big(\underbrace{-\cos(\pi)}_{-1} + \cos(x)\Big)\,dx$

$= \int_0^{\pi} (1+\cos x)\,dx = \left[x+\sin x\right]_0^{\pi} = \pi + \underbrace{\sin\pi}_{0} - 0 - \underbrace{\sin 0}_{0} = \pi$.

f) $D = \{(x,y) \mid x\ge 0,\ y\ge 0,\ x^2+y^2\le 1\}$

$D = \{(x,y) \mid 0\le x\le 1,\ 0\le y\le \underbrace{\sqrt{1-x^2}}_{\text{fonction !!}}\}$



⚠ Changt de variables : en coord. polaires !

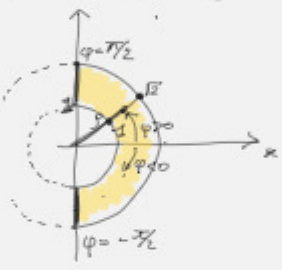
$\tilde{D} = \{(\rho,\varphi) \mid 0\le\varphi\le 1,\ 0\le\varphi\le\frac{\pi}{2}\}$

$\iint_D \underbrace{(1-(x^2+y^2))}_{1-\rho^2}\,\underbrace{dx\,dy}_{\rho\,d\rho\,d\varphi} \overset{\text{Changt var}}{=} \iint_{\tilde{D}} (1-\rho^2)\,\rho\,d\rho\,d\varphi \overset{\text{Fubini}}{=} \int_0^1 \rho(1-\rho^2)\,d\rho \cdot \int_0^{\pi/2} d\varphi$

$= \left[2\rho^2-\frac{1}{4}\rho^4\right]_0^1 \cdot \left[\varphi\right]_0^{\pi/2} = \frac{\pi}{2}\Big(2-\frac{1}{4}\Big) = \frac{\pi}{2}\cdot\frac{8-1}{4} = \frac{7\pi}{8}$.

g) $D = \{(x,y)\in\mathbb{R}^2 \mid x\ge 0,\ 1\le \underbrace{x^2+y^2}_{\rho^2} \le 2\}$

$\tilde{D} = \{(\rho,\varphi) \mid 1\le\rho\le\sqrt{2},\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\}$



$\iint_D \underbrace{x^2}_{\rho^2\cos^2\varphi}\,\underbrace{dx\,dy}_{\rho\,d\rho\,d\varphi} \overset{\text{Changt var}}{=} \iint_{\tilde{D}} \rho^3\cos^2\varphi\,d\rho\,d\varphi \overset{\text{Fubini}}{=} \int_1^{\sqrt{2}} \rho^3\,d\rho \cdot \underbrace{\int_{-\pi/2}^{\pi/2} \cos^2\varphi\,d\varphi}_{\cos^2\varphi+1}$

$= \left[\frac{1}{4}\rho^4\right]_1^{\sqrt{2}} \cdot \frac{\pi}{2}$

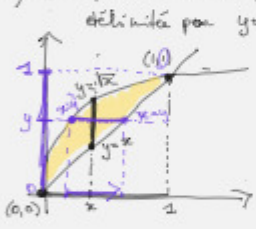
← primitive : p. 88
→ intégrale (connue !) : p. 95 $= \pi/2$

$= \left(\frac{1}{4}(\sqrt{2})^4 - \frac{1}{4}\right)\frac{\pi}{2} = \left(\frac{4}{4}-\frac{1}{4}\right)\frac{\pi}{2} = \frac{3\pi}{8}$.

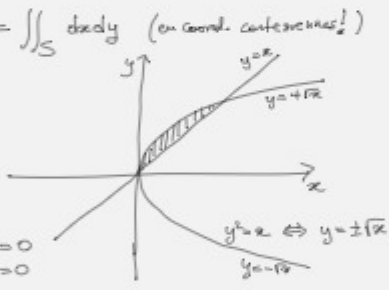
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Exercice 36 : $S\subseteq\mathbb{R}^2$ $\quad \text{Aire}(S) := \iint_S dx\,dy$ (en coord. cartésiennes !)

a) S = partie bornée du plan délimitée par $y=x$, $y^2=x$



$\begin{cases} y=x \\ y^2=x \end{cases} \Leftrightarrow \begin{cases} y=x \\ x^2-x=0 \\ x(x-1)=0 \end{cases}$ $\begin{cases} x=0 \\ y=0 \end{cases}$ $\begin{cases} x=1 \\ y=1 \end{cases}$



$\int x^n\,dx = \frac{1}{n+1}x^{n+1}$

$S = \{(x,y) \mid 0\le x\le 1,\ x\le y\le\sqrt{x}\}$

$S = \{(x,y) \mid 0\le y\le 1,\ y^2\le x\le y\}$

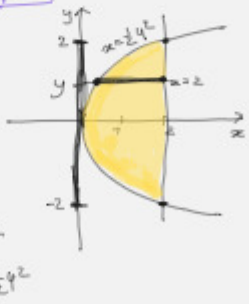
$\text{Aire}(S) \overset{\text{déf}}{=} \iint_S dx\,dy \overset{\text{Fubini}}{=} \int_0^1 dx \int_x^{\sqrt{x}} dy = \int_0^1 dx\,[y]_x^{\sqrt{x}} = \int_0^1 (\sqrt{x}-x)\,dx$

$= \int_0^1 (x^{\frac{1}{2}}-x)\,dx = \left[\underbrace{\frac{1}{\frac{1}{2}+1}}_{=\frac{2}{3}}x^{\frac{1}{2}+1} - \frac{1}{2}x^2\right]_0^1 = \frac{2}{3}-\frac{1}{2}-(0) = \frac{4-3}{6} = \boxed{\frac{1}{6}}$

$\text{Aire}(S) \overset{\text{déf}}{=} \iint_S dx\,dy \overset{\text{Fubini}}{=} \int_0^1 dy \int_{y^2}^{y} dx = \int_0^1 dy\,[x]_{y^2}^{y} = \int_0^1 (y-y^2)\,dy$

$= \left[\frac{1}{2}y^2-\frac{1}{3}y^3\right]_0^1 = \frac{1}{2}-\frac{1}{3} = \frac{3-2}{6} = \boxed{\frac{1}{6}}$

b) $S = \{(x,y)\in\mathbb{R}^2 \mid \underbrace{\frac{y^2}{2}}_{x=\frac{1}{2}y^2} \le x \le \underbrace{2}_{x=2}\}$



$\begin{cases} x=2 \\ x=\frac{1}{2}y^2 \end{cases}$ $\begin{cases} x=2 \\ y^2=4 \quad y=\pm 2 \end{cases}$

$S = \{(x,y)\in\mathbb{R}^2 \mid -2\le y\le 2,\ \frac{1}{2}y^2\le x\le 2\}$

$\text{Aire}(S) = \iint_S dx\,dy \overset{\text{Fubini}}{=} \int_{-2}^{2} dy \int_{\frac{1}{2}y^2}^{2} dx = \int_{-2}^{2} dy\,[x]_{\frac{1}{2}y^2}^{2}$

$= \int_{-2}^{2} \underbrace{\Big(2-\frac{1}{2}y^2\Big)}_{\text{fct paire}}\,dy = 2\int_0^2 (2-\frac{1}{2}y^2)\,dy = 2\left[2y-\frac{1}{6}y^3\right]_0^2$ (*)

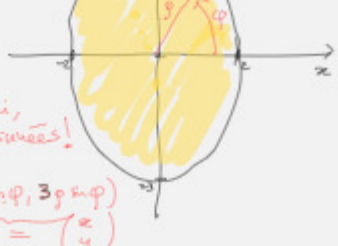
Remarque Si f est impaire, $f(-x)=-f(x)$, alors $\int_{-a}^{a} f(x)\,dx = 0$

Si f est paire, $f(-x)=f(x)$, alors $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

$(*) = 2\Big(4-\frac{8}{6}\Big) = 2\Big(4-\frac{4}{3}\Big) = 2\,\frac{12-4}{3} = \frac{16}{3}$.

c) S = partie du plan délimitée par l'ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$

$\left(\frac{x}{2}\right)^2+\left(\frac{y}{3}\right)^2=1$



$\text{Aire}(S) = \iint_S dx\,dx$ ⚠ avant Fubini, changt. coordonnées !

$h\begin{cases} x = 2\rho\cos\varphi \\ y = 3\rho\sin\varphi \end{cases}$ $\quad h(\rho,\varphi) = (x,y) = \underbrace{(2\rho\cos\varphi,\ 3\rho\sin\varphi)}_{=\begin{pmatrix} x \\ y \end{pmatrix}}$ $\quad h:\mathbb{R}^2\longrightarrow\mathbb{R}^2$

$Jh(\rho,\varphi) = \begin{pmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\varphi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\varphi} \end{pmatrix} = \begin{pmatrix} 2\cos\varphi & -2\rho\sin\varphi \\ 3\sin\varphi & 3\rho\cos\varphi \end{pmatrix}$

$\det Jh(\rho,\varphi) = 2\cos\varphi\cdot 3\rho\cos\varphi + 3\sin\varphi\cdot 2\rho\sin\varphi$

$= 6\rho\cos^2\varphi + 6\rho\sin^2\varphi = 6\rho\underbrace{(\cos^2\varphi+\sin^2\varphi)}_{1} = 6\rho$

$\Rightarrow \boxed{dx\,dy = 6\rho\,d\rho\,d\varphi}$

$\tilde{S} = \{(\rho,\varphi) \mid 0\le\rho\le 1,\ 0\le\varphi\le 2\pi\}$

$\text{Aire}(S) = \iint_S dx\,dy \overset{\text{changt var}}{=} \iint_{\tilde{S}} 6\rho\,d\rho\,d\varphi \overset{\text{Fubini}}{=} \int_0^1 6\rho\,d\rho \int_0^{2\pi} d\varphi$

$= \left[\frac{6}{2}\rho^2\right]_0^1 \cdot \left[\varphi\right]_0^{2\pi} = 3\cdot 2\pi = 6\pi$.

miro