TMB — Contrôle terminal automne 2016 2ème session — Corrigé — 21 juin 2017

Exo 1 a) $\vec{u} = (3,1,-1)$ $\vec{v} = (2,0,1)$

$$\vec{u}\cdot\vec{v} = \begin{pmatrix}3\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix} = 6+0-1 = 5 \qquad \vec{u}\wedge\vec{v} = \begin{pmatrix}3\\1\\-1\end{pmatrix}\wedge\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}1-0\\-(3+2)\\0-2\end{pmatrix} = \begin{pmatrix}1\\-5\\-2\end{pmatrix}$$

b) $f(x,y,z) = (-y,\ x+3z,\ 2y-5x)$ est linéaire car ses composantes sont des polynômes de degré 1 sans termes constants.

Sa matrice associée est $A = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 3\\ -5 & 2 & 0\end{pmatrix}$ et $\det A = -(-1)\begin{vmatrix}1 & 3\\ -5 & 0\end{vmatrix} = 0-(-15) = 15.$

$g(x,y,z) = (x^2y,\ y^2-z^2,\ z+1)$ n'est pas linéaire car il y a des carrés et des termes constants.

Exo 2 $f(x) = \dfrac{x-3}{x+1}$ a) $D_f = \{x\in\mathbb{R} \mid x\neq -1\}$

b) $f'(x) = \dfrac{(x+1)-(x-3)}{(x+1)^2} = \dfrac{x+1-x+3}{(x+1)^2} = \dfrac{4}{(x+1)^2}$

c) $f''(x) = -\dfrac{4}{(x+1)^4}\cdot 2(x+1) = -\dfrac{8}{(x+1)^3}$

d) $f(0) = -3$, $f'(0) = 4$, $f''(0) = -8$ $\Longrightarrow$

$$f(x) \underset{x\to 0}{\sim} f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 + o(x^2) = -3+4x-4x^2+o(x^2).$$

Exo 3 $x = \sin\theta \implies dx = \cos\theta\, d\theta$, $\sin 0 = 0$, $\sin \pi/2 = 1$

$$\int_0^{\pi/2} 3\cos\theta\sin\theta\, e^{2\sin\theta}\,d\theta = \int_{\sin 0}^{\sin(\pi/2)} 3x\,e^{2x}\,dx$$

par partie : $u(x) = 3x$ $v'(x) = e^{2x}$; $u'(x) = 3$ $v(x) = \frac{1}{2}e^{2x}$

$$= \left[\tfrac{3}{2}x e^{2x}\right]_0^1 - \int_0^1 \tfrac{3}{2}e^{2x}\,dx = \left[\tfrac{3}{2}xe^{2x} - \tfrac{3}{4}e^{2x}\right]_0^1 = \left[\tfrac{3}{4}e^{2x}(2x-1)\right]_0^1$$

$$= \tfrac{3}{4}e^2(2-1) - \tfrac{3}{4}e^0(0-1) = \tfrac{3}{4}e^2 + \tfrac{3}{4} = \tfrac{3}{4}(e^2+1).$$

Exo 4 a) (E) $\dot{u}(t) = \dfrac{2t}{t^2+1}u(t) + 3$

(E_0) $\dot{u}(t) = \dfrac{2t}{t^2+1}u(t) \implies$ sol. $u_0(t) = \lambda e^{\int \frac{2t}{t^2+1}dt} = \lambda e^{\ln(t^2+1)} = \lambda(t^2+1)$, $\lambda\in\mathbb{R}$.

Sol. particulière de (E) : $u_p(t) = \lambda(t)(t^2+1)$ avec $\lambda(t)$ t.q.

$$\lambda'(t)(t^2+1) + \cancel{\lambda(t)\cdot 2t} = \cancel{\frac{2t}{t^2+1}\cdot\lambda(t)\cdot(t^2+1)} + 3 \iff \lambda'(t) = \frac{3}{t^2+1}$$

$$\lambda(t) = \int\frac{3}{t^2+1}\,dt = 3\arctan t \implies u_p(t) = 3\arctan(t)\cdot(t^2+1)$$

$$\implies u(t) = u_0(t) + u_p(t) = (\lambda + 3\arctan t)(t^2+1)$$

b) $u(0) = (\lambda + 3\arctan 0)(0+1) = \lambda = 7 \implies$

$u(t) = (7+\arctan t)(t^2+1).$