Non-commutative Hopf algebra of formal diffeomorphisms

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Abstract

The subject of this paper are two Hopf algebras which are the non-commutative analogues of two different groups of formal power series. The first group is the set of invertible series with the group law being multiplication of series, while the second group is the set of formal diffeomorphisms with the group law being composition of series. The motivation to introduce these Hopf algebras comes from the study of formal series with non-commutative coefficients. Invertible series with non-commutative coefficients still form a group, and we interpret the corresponding new non-commutative Hopf algebra as an alternative to the natural Hopf algebra given by the co-ordinate ring of the group, which has the advantage of being functorial in the algebra of coefficients. For the formal diffeomorphisms with non-commutative coefficients, this interpretation fails, because in this case the composition is not associative anymore. However, we show that for the dual non-commutative algebra there exists a natural co-associative co-product defining a non-commutative Hopf algebra. Moreover, we give an explicit formula for the antipode, which represents a non-commutative version of the Lagrange inversion formula, and we show that its coefficients are related to planar binary trees. Then we extend these results to the semi-direct co-product of the previous Hopf algebras, and to series in several variables. Finally, we show how the non-commutative Hopf algebras of formal series are related to some renormalization Hopf algebras, which are combinatorial Hopf algebras motivated by the renormalization procedure in quantum field theory, and to the renormalization functor given by the double tensor algebra on a bi-algebra.

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Contents

Introduction

In the well-known paper [12], A. Connes and D. Kreimer introduced a Hopf algebra structure on the set of Feynman graphs which allows one to describe the combinatorial part of the renormalization of quantum fields in a very elegant and simple way. The renormalization procedure affects the coefficients of the perturbative series which describe the propagators and the coupling constants in quantum field theory, transforming them from infinite to well-defined finite quantities. The perturbative series involved are traditionally expanded over the set of Feynman graphs, but recent works by two of the authors [6, 7], showed that the amplitudes of Feynman graphs can be regrouped to form amplitudes associated to other combinatorial objects, such as rooted planar binary trees, or, at the coarsest level, positive integers. These new amplitudes correspond to new expansions of the perturbative series, and they turned out to be compatible with the renormalization. The renormalization is then encoded in the co-product of some Hopf algebras constructed on the set of rooted planar binary trees [9] or on the set of positive integers [8]. The refinement of precision in the computation of the coefficients of the perturbative series, which is typical in quantum field theory, corresponds to a sequence of inclusions of Hopf algebras, the smallest one having generators labelled by the integers, the intermediate one with generators labelled by the trees, and the largest with generators labelled by the Feynman graphs.

In this context, the use of Hopf algebras can be explained as a "local co-ordinate" approach to the study of the renormalization groups, which are given on the sets of perturbative series relevant to each specific field theory. Accordingly, the Hopf algebras are never co-commutative, and they happen to be commutative if the amplitudes of the Feynman graphs are complex numbers, that is, if the quantum field considered is scalar. In this case, the renormalization Hopf algebras are exactly the co-ordinate rings of the renormalization groups. If the perturbative series are expanded over the integers, the co-products are constructed from the operations which are exactly the duals of the multiplication and the composition of usual formal series. On the contrary, when the series are expanded over trees, or over Feynman graphs, the dual operations of multiplication and composition of series have to be defined "ad hoc" in a way which generalises the usual operations. These new groups of series will be described in the upcoming paper [16] by one of the authors.

However, there are cases in which it might be useful to consider series with non-scalar coefficients. This is the case, for instance, in quantum electro-dynamics [7], where the quantum fields are 4-vectors or spinors, and the propagators are 4×4 matrices. This is also the case for certain types of infrared renormalization [2, 3], for the mass renormalization of a fermion family [5], or for quantum field theory over noncommutative geometries [40]. For these latter cases, we are led to study the multiplication and the composition of formal series with non-scalar coefficients.

In this paper, we consider two sets of formal power series with non-commutative coefficients. The first one is the set of invertible series with the multiplication law. Even with non-commutative coefficients, these series still form a group, and we show that its usual commutative co-ordinate ring can be replaced by a non-commutative Hopf algebra which is functorial in the algebra of coefficents. In fact, what we obtain is an example of a co-group element in the category of associative algebras, studied by B. Fresse in [17], if we read it in the appropriate way, i.e., if we replace tensor products by free products among the algebras in the image of the co-product.

The second set of formal series is that of formal diffeomorphisms on a line, with group law given by the composition of series. While this set forms a group if the coefficients of the series are scalar numbers, the composition fails to be associative when the coefficients are taken in an arbitrary non-commutative algebra. However, we show that on the dual algebra of local co-ordinates there is a natural co-product which is co-associative, and gives rise to a Hopf algebra which is neither commutative nor co-commutative. This Hopf algebra is related to the renormalization of quantum electrodynamics. More precisely, in Section 4 we show that the non-commutative Hopf algebra of formal diffeomorphisms is a Hopf sub-algebra of the non-commutative Hopf algebra on planar binary trees, introduced in [9], which represents at the same time the charge and the photon renormalization Hopf algebras.

Unlike the non-commutative Hopf algebra of invertible series, the non-commutative Hopf algebra of formal diffeomorphisms is not an example of a co-group element in associative algebras, because the co-product with image in the free product of algebras is not co-associative. However it has some remarkable properties, among which self-duality, which make it an oustanding example for many theories developed recently.

For instance, in [18], F. Gavarini applies the Quantum Duality Principle developed in [19] to the non-commutative Hopf algebra of formal diffeomorphisms, via four one-parameter deformations, to get four quantum groups with semiclassical limits given by some Poisson geometrical symmetries.

Similarly, P. van der Laan describes in [49] a general procedure to obtain canonically a Hopf algebra from an operad, and he shows that one obtains the non-commutative Hopf algebra of formal diffeomorphisms in the case of the operad of associative algebras.

Finally, the non-commutative Hopf algebra of formal diffeomorphisms is the simplest example of a Hopf algebra obtained via the renormalization functor constructed on the double tensor algebra of a bi-algebra by W. Schmitt and one of the authors in [11]. This example will be treated in detail in Section 5. Because of this, the dual Hopf algebra of the Hopf algebra of formal diffeomorphisms is also the simplest non-trivial example of a Hopf algebra coming from a dendriform algebra, as introduced by J.–L. Loday in [34], and further studied with M. Ronco in [35, 36, 44, 45]. In particular, the primitive elements of the dual Hopf algebra of formal diffeomorphisms are endowed with a very simple structure of a brace algebra (cf. [44, 45]), which is under investigation by some of the authors.

Our paper is organized as follows. In Section 1, we recall the definition of the group of invertible series with the multiplication law, and of its co-ordinate ring. We first treat the case of series with scalar coefficients, and then consider series with non-commutative coefficients. We show that the classical co-ordinate ring can be replaced by a non-commutative Hopf algebra, and that the group can still be reconstructed as a group of characters with non-commutative values.

In Section 2, we recall the definition of the group of formal diffeomorphisms with the composition law, and of its co-ordinate ring. Next, we consider series with non-commutative coefficients, and we show that, even if they do not form a group, on the algebra of local co-ordinates there is a natural structure of a Hopf algebra, which is neither commutative nor co-commutative. For this Hopf algebra, we present an explicit non-recursive formula for the antipode, which generalises the Lagrange Inversion Formula of formal series to the non-commutative context. We remark, that there appears already a non-commutative version of the Lagrange Inversion Formula in the literature, which is due to I. M. Gessel [20]. However, the inversion problem which is solved in [20] is inequivalent to ours. Finally, we explain in detail the labelling of the antipode coefficients in terms of planar binary trees.

In Section 3, we show that the classical action of the group of formal diffeomorphisms on the group of invertible series can be generalized to the dual non-commutative context by means of a suitable co-action among Hopf algebras. This co-action allows one to construct the semi-direct (or smash) Hopf algebra of the previous Hopf algebras, studied by R. K. Molnar in [39] and S. Majid in [38].

The operations introduced in Section 3 are the ingredients which we need for explaining the relationship between the non-commutative Hopf algebras of formal series and the renormalization Hopf algebras on planar binary trees used in [7, 9]. The latter Hopf algebras are related to the renormalization of quantum electrodynamics. In Section 4, we prove that the algebras of series are Hopf sub-algebras of the corresponding algebras on trees.

In Section 5, we show that the non-commutative Hopf algebra of formal diffeomorphisms can be obtained also via the renormalization functor described in [11], applied to the simplest possible bi-algebra, the trivial one. On the one hand, this result places the non-commutative Hopf algebra of formal diffeomorphisms in the context of a different approach of the renormalization procedure of quantum fields, namely the Epstein–Glaser renormalization on the configuration space, cf. [14], and to its interesting further developement by G. Pinter, cf. [41, 42]. On the other hand, it relates the non-commutative Hopf algebra of formal diffeomorphisms to a large class of special algebras, such as dipterous, dendriform, brace and B_∞ algebras, which were recently discovered by J.–L. Loday and collaborators, cf. [34, 35, 36, 44, 45].

Finally, in Section 6, we briefly sketch how to generalise the non-commutative Hopf algebra of formal diffeomorphisms to series with several variables. The practical applications of such formulae can be found in the renormalization of massive quantum field theory, cf. [10].

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1 Non-commutative Hopf algebra of invertible series

In this section we introduce the Abelian group of invertible formal power series with scalar coefficients and its co-ordinate ring, which carries the structure of a commutative and co-commutative Hopf algebra. We use this simple example to describe the duality between a group and its co-ordinate ring.

Then we introduce the group of invertible formal power series with coefficients from an arbitrary associative and unital algebra $\mathcal{A}$. Aside from the usual commutative co-ordinate ring, which depends on the chosen algebra $\mathcal{A}$, we present another Hopf algebra related to this group, which is no longer commutative and no longer dual to the group, but turns out to be functorial in $\mathcal{A}$.

1.1 Group of invertible series

Consider the set

$$G^{\mathrm{inv}} = \left\{ f(x) = 1 + \sum_{n=1}^{\infty} f_n x^n,\ f_n \in \mathbb{C} \right\} \tag{1.1}$$

of invertible formal power series in a variable x with complex coefficients, where, for simplicity, we fix the invertible constant term f_0 to be 1. This set forms an Abelian group, with the multiplication

$$(fg)(x) := f(x)g(x) = 1 + \sum_{n=1}^{\infty} \sum_{k=0}^{n} f_k g_{n-k}\ x^n, \tag{1.2}$$

the unit being given by the constant series $1(x) = 1$, and where the inverse f^{-1} of a series f can be found recursively, for instance by using the Wronski formula (cf. [23, p. 17]). The first few coefficients of f^{-1} are $(f^{-1})_0 = 1$, $(f^{-1})_1 = -f_1$, $(f^{-1})_2 = -f_2 + f_1^2$.

The group G^{inv} is a projective limit of affine groups, so we can consider its co-ordinate ring $\mathbb{C}(G^{\mathrm{inv}})$ (defined below), a commutative algebra related to the scalar functions on G^{inv}. We remark that, since G^{inv} is not compact, and also not locally compact, its co-ordinate ring cannot be defined in the classical way as the algebra of representative functions on the group (i.e., the polynomials in the matrix elements of the finite dimensional representations of the group, cf. [1, Sec. 2.2]). However, we can define $\mathbb{C}(G^{\mathrm{inv}})$ to be the set of functions $G^{\mathrm{inv}} \to \mathbb{C}$ which are polynomial with respect to an appropriate basis. As basis, we choose the functions b_n, $n = 1, 2, \ldots$, where b_n associates to each element of G^{inv} its n-th coefficient. That is, we may interpret b_n as the normalized n^{th}-derivative evaluated at $x = 0$,

$$b_n(f) = \frac{1}{n!} \frac{\mathrm{d}^n f(0)}{\mathrm{d}x^n} = f_n.$$

Thus, $\mathbb{C}(G^{\mathrm{inv}})$ is isomorphic to the polynomial ring $\mathbb{C}[b_1, b_2, \ldots]$.

The action of the functions on the elements of the group gives a duality pairing $\langle b_n, f\rangle := b_n(f) = f_n$ between $\mathbb{C}(G^{\mathrm{inv}})$ and G^{inv}. Through this pairing, the group structure on G^{inv} induces the structure of a commutative Hopf algebra on $\mathbb{C}(G^{\mathrm{inv}})$, as it happens for affine or classical compact groups, cf. [1, 25, 24]. In particular, the group law on G^{inv} induces a dual co-product Δ^{inv} on $\mathbb{C}(G^{\mathrm{inv}})$, that is, a map $\Delta^{\mathrm{inv}} : \mathbb{C}(G^{\mathrm{inv}}) \otimes \mathbb{C}(G^{\mathrm{inv}}) \longrightarrow \mathbb{C}(G^{\mathrm{inv}})$ such that

$$\langle \Delta^{\mathrm{inv}} b_n, f \otimes g\rangle = \langle b_n, fg\rangle,$$

where, as usual, $\langle a \otimes b, f \otimes g\rangle = \langle a, f\rangle\langle b, g\rangle$. In this case, it is easy to verify that the induced co-product on $\mathbb{C}(G^{\mathrm{inv}})$ has the form

$$\Delta^{\mathrm{inv}} b_n = \sum_{k=0}^{n} b_k \otimes b_{n-k}, \quad (b_0 := 1), \tag{1.3}$$

on the generators, and therefore it is co-commutative. Still by duality, the unit 1 of the group induces a co-unit ε on $\mathbb{C}(G^{\mathrm{inv}})$, that is, a map $\varepsilon : \mathbb{C}(G^{\mathrm{inv}}) \longrightarrow \mathbb{C}$ such that

$$\varepsilon(b_n) = \langle b_n, 1\rangle.$$

Again, it is easy to verify that the co-unit has values $\varepsilon(1) = 1$ and $\varepsilon(b_n) = 0$ for $n \geq 1$. Finally, by duality, the operation of inversion in G^{inv} gives rise to an antipode on $\mathbb{C}(G^{\mathrm{inv}})$, that is, a map $S : \mathbb{C}(G^{\mathrm{inv}}) \longrightarrow \mathbb{C}(G^{\mathrm{inv}})$ such that

$$\langle S(b_n), f\rangle = \langle b_n, f^{-1}\rangle.$$

In fact, the defining relation for the antipode yields a recursive formula for the action of the antipode on the generators. Thus, all these data together define the structure of a commutative and co-commutative graded connected Hopf algebra on $\mathbb{C}(G^{\mathrm{inv}})$.

Finally, as is also the case for affine or classical Lie groups, cf. [29, 48] (see e.g. [24, Ch. VII, § 30] or [25, Theorem 3.5]), the group G^{inv} can be reconstructed completely from its co-ordinate ring $\mathbb{C}(G^{\mathrm{inv}})$, as the group of algebra homomorphisms (characters) $\mathrm{Hom}_{Alg}(\mathbb{C}(G^{\mathrm{inv}}), \mathbb{C})$ with the convolution product defined on the generators by

$$(\alpha\beta)(b_n) := m \circ (\alpha \otimes \beta) \circ \Delta^{\mathrm{inv}} b_n, \tag{1.4}$$

for any algebra homomorphisms α, β on $\mathbb{C}(G^{\mathrm{inv}})$. Here, m denotes the multiplication in $\mathbb{C}$. In other words, we have an isomorphism of groups $G^{\mathrm{inv}} \cong \mathrm{Hom}_{Alg}(\mathbb{C}(G^{\mathrm{inv}}), \mathbb{C})$ which associates to a series $f \in G^{\mathrm{inv}}$ the algebra homomorphism α_f on $\mathbb{C}(G^{\mathrm{inv}})$ given on the generators by $\alpha_f(b_n) = \langle b_n, f\rangle$.

1.2 Invertible series with non-commutative coefficients

Let $\mathcal{A}$ be an associative unital algebra, and consider the set

$$G^{\mathrm{inv}}(\mathcal{A}) = \left\{ f(x) = 1 + \sum_{n=1}^{\infty} f_n x^n, \ f_n \in \mathcal{A} \right\} \tag{1.5}$$

of invertible formal power series with coefficients in $\mathcal{A}$. The product (1.2) still makes $G^{\mathrm{inv}}(\mathcal{A})$ into a group, which is Abelian only if $\mathcal{A}$ is commutative.

As before, $G^{\mathrm{inv}}(\mathcal{A})$ can be recovered from its co-ordinate ring $\mathbb{C}(G^{\mathrm{inv}}(\mathcal{A}))$, which is still a polynomial ring in infinitely many variables, now depending on the chosen algebra $\mathcal{A}$. For instance, if $\mathcal{A} = M_2(\mathbb{C})$ is the algebra of 2×2 matrices with complex entries, then $f_n = \big(f_n^{ij}\big)_{i,j=1,2}$ with $f_n^{ij} \in \mathbb{C}$. Thus, we can choose the matrix elements as generators for the co-ordinate ring, and

$$\mathbb{C}(G^{\mathrm{inv}}(M_2(\mathbb{C}))) \cong \mathbb{C}[b_1^{ij}, b_2^{ij}, \ldots \mid i,j = 1,2]$$

is a polynomial algebra on 4 times infinitely many variables. The group law on $G^{\mathrm{inv}}(\mathcal{A})$ induces again a dual co-product on $\mathbb{C}(G^{\mathrm{inv}}(\mathcal{A}))$. For instance, if $\mathcal{A} = M_2(\mathbb{C})$ and $\mathbb{C}(G^{\mathrm{inv}}(\mathcal{A})) = \mathbb{C}[b_n^{ij} | n \in \mathbb{N},\ i,j = 1,2]$, then

$$\Delta b_n^{ij} = \sum_{k=0}^{n} \sum_{l=1,2} b_k^{il} \otimes b_{n-k}^{lj}.$$

As a result, $\mathbb{C}(G^{\mathrm{inv}}(\mathcal{A}))$ is still a commutative Hopf algebra, which is co-commutative only if $\mathcal{A}$ is commutative.

Alternatively, we can associate to $G^{\mathrm{inv}}(\mathcal{A})$ a non-commutative Hopf algebra $\mathcal{H}^{\mathrm{inv}}$, which has the advantage of being functorial in $\mathcal{A}$, that is, it does not depend on the chosen algebra $\mathcal{A}$. To do this, we consider the set $\mathcal{H}^{\mathrm{inv}}$ of $\mathcal{A}$-valued polynomial functions on $G^{\mathrm{inv}}(\mathcal{A})$. Then, as an algebra, $\mathcal{H}^{\mathrm{inv}}$ is isomorphic to the free unital associative algebra (tensor algebra) on infinitely many variables b_n,

$$\mathcal{H}^{\mathrm{inv}} \cong \mathbb{C}\langle b_1, b_2, \ldots \rangle, \tag{1.6}$$

and the formula (1.3) still defines a co-associative co-product which makes $\mathcal{H}^{\mathrm{inv}}$ into a non-commutative co-commutative Hopf algebra. Note that we can recover $\mathbb{C}(G^{\mathrm{inv}})$ from $\mathcal{H}^{\mathrm{inv}}$ by simple Abelianisation, and, thus, $\mathcal{H}^{\mathrm{inv}}$ can be considered as a non-commutative analogue of the group G^{inv}.

Of course, in this case, the group $G^{\mathrm{inv}}(\mathcal{A})$ cannot anymore be reconstructed from $\mathcal{H}^{\mathrm{inv}}$, because the multiplication $m : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ is not anymore an algebra homomorphism, and therefore the convolution product $\alpha\beta$ of two algebra homomorphisms $\alpha, \beta \in \mathrm{Hom}_{Alg}(\mathcal{H}^{\mathrm{inv}}, \mathcal{A})$ defined by formula (1.4) is not anymore an element of $\mathrm{Hom}_{Alg}(\mathcal{H}^{\mathrm{inv}}, \mathcal{A})$. However, the group $G^{\mathrm{inv}}(\mathcal{A})$ can be easily reconstructed as follows.

Let $*$ denote the free product of associative algebras (in the terminology of [32] or [50]), which is the co-product or sum in the category of associative algebras (in the terminology of [31, Section I.7]). We recall a few basic facts about the free product $*$.

Given two unital associative algebras $\mathcal{A}$ and $\mathcal{B}$, the free product $\mathcal{A} * \mathcal{B}$ can be defined as the universal unital associative algebra which, for any given unital algebra C and any algebra homomorphisms $\alpha : \mathcal{A} \to \mathcal{C}$ and $\beta : \mathcal{B} \to \mathcal{C}$, makes the following diagram commutative:

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[dr, "\alpha"'] & \mathcal{A} * \mathcal{B} \arrow[d, "\exists!"'] & \mathcal{B} \arrow[l] \arrow[dl, "\beta"] \\ & \mathcal{C} & \end{tikzcd}$$

As a vector space, $\mathcal{A} * \mathcal{B}$ is generated by the tensor products in which elements of $\mathcal{A}$ and $\mathcal{B}$ alternate, that is

$$\mathcal{A} * \mathcal{B} = \Big(\bigoplus_{k=0}^{\infty} (\mathcal{A} \otimes \mathcal{B})^{\otimes k}\Big) \oplus \Big(\mathcal{B} \otimes \bigoplus_{k=0}^{\infty} (\mathcal{A} \otimes \mathcal{B})^{\otimes k}\Big) \oplus \Big(\bigoplus_{k=1}^{\infty} (\mathcal{B} \otimes \mathcal{A})^{\otimes k}\Big) \oplus \Big(\bigoplus_{k=0}^{\infty} \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{A})^{\otimes k}\Big).$$

In particular, $\mathcal{A} \otimes \mathcal{B}$ is a sub-space of $\mathcal{A} * \mathcal{B}$. We endow $\mathcal{A} * \mathcal{B}$ with a product $*$, which is, essentially, the concatenation product, except that any occurrence of $\cdots \otimes a \otimes a' \otimes \cdots$ is replaced by $\cdots \otimes aa' \otimes \cdots$ for any $a, a' \in \mathcal{A}$, and any occurrence of $\cdots \otimes b \otimes b' \otimes \cdots$ is replaced by $\cdots \otimes bb' \otimes \cdots$ for any $b, b' \in \mathcal{B}$. That is, for any $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$, we have, for example, $(a \otimes b) * (a' \otimes b') = a \otimes b \otimes a' \otimes b'$ and

$(a \otimes b) * (b' \otimes a') = a \otimes (bb') \otimes a'$. Then, there is an obvious projection from $\mathcal{A} * \mathcal{B}$ to $\mathcal{A} \otimes \mathcal{B}$, which maps $a^1 \otimes b^1 \otimes a^2 \otimes b^2 \otimes \cdots a^k \otimes b^k$ to $a^1 a^2 \cdots a^k \otimes b^1 b^2 \cdots b^k$, and similarly for the other basis elements of $\mathcal{A} * \mathcal{B}$. This map is a homomorphism of algebras.

Proposition 1.1. *Let $\Delta^{\mathrm{inv}}_* : \mathcal{H}^{\mathrm{inv}} \longrightarrow \mathcal{H}^{\mathrm{inv}} * \mathcal{H}^{\mathrm{inv}}$ be the operator defined on the generators by formula* (1.3)*, and extended as an algebra homomorphism. Then Δ^{inv}_* is co-associative.*

Moreover, if we denote by $\mathcal{H}^{\mathrm{inv}}_*$ the algebra $\mathcal{H}^{\mathrm{inv}}$ endowed with Δ^{inv}_*, then $\mathrm{Hom}_{Alg}(\mathcal{H}^{\mathrm{inv}}_*, \mathcal{A})$ is a group with group law given by the convolution. In addition, the groups $G^{\mathrm{inv}}(\mathcal{A})$ and $\mathrm{Hom}_{Alg}(\mathcal{H}^{\mathrm{inv}}_*, \mathcal{A})$ are isomorphic to each other.

Proof. Since $\mathcal{H}^{\mathrm{inv}} \otimes \mathcal{H}^{\mathrm{inv}}$ is a sub-space of $\mathcal{H}^{\mathrm{inv}} * \mathcal{H}^{\mathrm{inv}}$, formula (1.3) yields a well-defined operator on $\mathcal{H}^{\mathrm{inv}}$. The fact that Δ^{inv} is co-associative is easily checked, so it only remains to prove that $\mathrm{Hom}_{Alg}(\mathcal{H}^{\mathrm{inv}}_*, \mathcal{A})$ is a group, and that it is isomorphic to $G^{\mathrm{inv}}(\mathcal{A})$.

The multiplication m on $\mathcal{A}$, that is, the map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, can be extended to a map $m_* : \mathcal{A} * \mathcal{A} \to \mathcal{A}$. Unlike m, the extension m_* is an algebra homomorphism. Therefore, given $\alpha, \beta \in \mathrm{Hom}_{Alg}(\mathcal{H}^{\mathrm{inv}}_*, \mathcal{A})$, the convolution defined by

$$\alpha\beta := m_* \circ (\alpha * \beta) \circ \Delta^{\mathrm{inv}} \tag{1.7}$$

is an element of $\mathrm{Hom}_{Alg}(\mathcal{H}^{\mathrm{inv}}_*, \mathcal{A})$. The rest of the proof that $\mathrm{Hom}_{Alg}(\mathcal{H}^{\mathrm{inv}}_*, \mathcal{A})$ is a group, is completely analogous to the proof in the commutative case, as, for example, given in [24] or [25].

That $G^{\mathrm{inv}}(\mathcal{A})$ and $\mathrm{Hom}_{Alg}(\mathcal{H}^{\mathrm{inv}}_*, \mathcal{A})$ are isomorphic to each other is evident from the construction. □

Note that the new type of Hopf algebra $\mathcal{H}^{\mathrm{inv}}_*$ is an example of a co-group in the category of associative algebras, as considered by B. Fresse [17] and by G. M. Bergman and A. O. Hausknecht [4, Sec. 60–62]. In particular, there the reader may find more details on the group structure of $\mathrm{Hom}_{Alg}(\mathcal{H}^{\mathrm{inv}}_*, \mathcal{A})$ and on the generalization of this construction to algebras over any operad.

Finally, note also that the co-product Δ^{inv} is just the composition of Δ^{inv}_* by the natural projection $\mathcal{H}^{\mathrm{inv}} * \mathcal{H}^{\mathrm{inv}} \to \mathcal{H}^{\mathrm{inv}} \otimes \mathcal{H}^{\mathrm{inv}}$. Therefore, the co-associativity of Δ^{inv} follows from the co-associativity of Δ^{inv}_*, but the converse is not true.

2 Non-commutative Hopf algebra of series with composition

In this section we introduce the group of formal power series with the composition law, which we call *formal diffeomorphisms*, and its co-ordinate ring.

Proceeding as in Section 1, we consider subsequently series with non-scalar coefficients and show that, even if these series do not anymore form a group, dually there exists a Hopf algebra which is neither commutative nor co-commutative, and which reproduces the co-ordinate ring of the group by Abelianisation.

For this new non-commutative Hopf algebra, we give an explicit formula for the co-product and an explicit non-recursive formula for the antipode.

2.1 Group of formal diffeomorphisms and the Faà di Bruno bi-algebra

We consider now the set

$$G^{\mathrm{dif}} = \left\{ \varphi(x) = x + \sum_{n=1}^{\infty} \varphi_n x^{n+1},\ \varphi_n \in \mathbb{C} \right\} \tag{2.1}$$

of formal power series in a variable x with complex coefficients, zero constant term, and invertible linear term φ_0, which we set equal to 1 for simplicity. This set forms a (non-Abelian) group with composition law

$$(\varphi \circ \psi)(x) := \varphi\big(\psi(x)\big) = \psi(x) + \sum_{n=1}^{\infty} \varphi_n \psi(x)^{n+1}. \tag{2.2}$$

The unit is given by the series $\mathrm{id}(x) = x$, and the (compositional) inverse $\varphi^{[-1]}(x)$ of a series $\varphi(x)$ can be found by use of the Lagrange inversion formula [30] (see e.g. [23] or [47, Theorem 5.4.2]). Such series are called *formal diffeomorphisms* (tangent to the identity).

An explicit expression for the composition can be easily derived directly from the definition (2.2). However, we shall not need it here. Instead, for later use, we propose an alternative expression of the composition of two series, in form of the (*formal*) residue (see [13] for an exposition of formal residue calculus, in the commutative setting, however). Given a Laurent series $F(z)$ in z, we write $\langle z^{-1}\rangle F(z)$ for the formal residue of $F(z)$, that is, its coefficient of z^{-1}. Using this notation, the composition of φ and ψ can be written as

$$(\varphi \circ \psi)(x) = \langle z^{-1}\rangle \frac{\varphi(z)}{z-\psi(x)}. \tag{2.3}$$

This is justified, if we interpret $(z-\psi(x))^{-1}$ as a formal power series in z^{-1}, that is, using the expansion of the geometric series,

$$\frac{1}{z-\psi(x)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{\psi(x)}{z}\right)^n = \sum_{n=0}^{\infty}\psi(x)^n z^{-n-1},$$

because then

$$\frac{\varphi(z)}{z-\psi(x)} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\varphi_m\psi(x)^n z^{m-n},$$

from which (2.3) follows immediately. In the sequel, we shall *always* adopt this convention.

As before, this group is the projective limit of affine groups, and it can be reconstructed from its co-ordinate ring $\mathbb{C}(G^{\mathrm{dif}})$. The latter can be defined as the polynomial ring $\mathbb{C}[a_1, a_2, \ldots]$ in infinitely many variables a_n, with $n \in \mathbb{N}$, where a_n is the function on G^{dif} acting as the normalized $(n+1)^{st}$-derivative evaluated at $x = 0$, that is

$$a_n(\varphi) = \frac{1}{(n+1)!}\frac{\mathrm{d}^{n+1}\varphi(0)}{\mathrm{d}x^{n+1}} = \varphi_n.$$

The group structure of G^{dif} induces a Hopf algebra structure on $\mathbb{C}(G^{\mathrm{dif}})$. The co-product for the generators of $\mathbb{C}(G^{\mathrm{dif}})$ can be extracted from the standard duality condition

$$\langle \Delta^{\mathrm{dif}} a_n, \varphi \otimes \psi\rangle = a_n(\varphi \circ \psi),$$

where $\langle a_n, \varphi\rangle = a_n(\varphi)$ and $\langle a_n \otimes a_m, \varphi \otimes \psi\rangle = a_n(\varphi)a_m(\psi)$.

Remark 2.1. A slight variation of this Hopf algebra is known as the *Faà di Bruno bi-algebra*, an algebra based on the original computations made by Faà di Bruno in [15] on the derivatives of the composition of two functions. The Faà di Bruno bi-algebra is in fact the coordinate ring of the semigroup of formal series of the form $\varphi(x) = \sum_{n=1}^{\infty}\varphi_n\frac{x^n}{n!}$, with φ_1 not necessarily equal to 1. Repeating the duality procedure described above, we can identify the Faà di Bruno bi-algebra in its standard form (cf. [27] or [38, Section 5.1]) with the graded polynomial ring $\mathcal{B}^{\mathrm{FdB}} = \mathbb{C}[u_1, u_2, \ldots]$ in infinitely many variables, with the degree of u_n being defined by $n-1$. The co-product in $\mathcal{B}^{\mathrm{FdB}}$, dual to the composition, takes the form

$$\Delta u_n = \sum_{k=1}^{n} u_k \otimes \sum_{\substack{\alpha_1+2\alpha_2+\cdots+n\alpha_n=n\\ \alpha_1+\alpha_2+\cdots+\alpha_n=k}} \frac{n!}{\alpha_1!\,\alpha_2!\cdots\alpha_n!}\frac{u_1^{\alpha_1}u_2^{\alpha_2}\cdots u_n^{\alpha_n}}{1!^{\alpha_1}2!^{\alpha_2}\cdots n!^{\alpha_n}} \tag{2.4}$$

on the generators u_n, and the co-unit is defined by $\varepsilon(u_n) = \delta_{n,0}$. For instance,

$$\begin{aligned}
\Delta u_1 &= u_1 \otimes u_1,\\
\Delta u_2 &= u_1 \otimes u_2 + u_2 \otimes u_1^2,\\
\Delta u_3 &= u_1 \otimes u_3 + u_2 \otimes 3u_1u_2 + u_3 \otimes u_1^3,\\
\Delta u_4 &= u_1 \otimes u_4 + u_2 \otimes 4u_1u_3 + u_2 \otimes 3u_2^2 + u_3 \otimes 6u_1^2u_2 + u_4 \otimes u_1^4.
\end{aligned}$$

An explicit expression of the co-product Δ^{dif} (in $\mathcal{H}^{\mathrm{dif}}$) on the generators a_n can be obtained by replacing Δ by Δ^{dif} and u_n by $n!\,a_{n-1}$ in (2.4), and by setting $u_1 = a_0 = 1$.

Remark 2.2. The co-ordinate ring $\mathbb{C}(G^{\mathrm{dif}})$ with its induced Hopf algebra structure appeared also as a particular example of an incidence Hopf algebra in the article [46, Ex. 14.2] by W. R. Schmitt.

Remark 2.3. A convenient way to present the co-product Δ^{dif} for all generators a_n in compact form is by means of the generating series

$$A(x) = x + \sum_{n=1}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} a_n x^{n+1}, \quad (a_0 := 1), \tag{2.5}$$

since it allows to reconstruct each series $\varphi \in G^{\mathrm{dif}}$ by duality:

$$\langle A(x), \varphi \rangle = \sum_{n=0}^{\infty} \langle a_n, \varphi \rangle x^{n+1} = \sum_{n=0}^{\infty} \varphi_n x^{n+1} = \varphi(x). \tag{2.6}$$

In fact, more generally, we have

$$\begin{aligned} \langle A(x)^m, \varphi \rangle &= \left\langle \sum_{n_1,\ldots,n_m \geq 0} a_{n_1} a_{n_2} \cdots a_{n_m} x^{(n_1+1)+\cdots+(n_m+1)}, \varphi \right\rangle \\ &= \sum_{n_1,\ldots,n_m \geq 0} \langle a_{n_1} a_{n_2} \cdots a_{n_m}, \varphi \rangle x^{(n_1+1)+\cdots+(n_m+1)} \\ &= \sum_{n_1,\ldots,n_m \geq 0} \varphi_{n_1} \varphi_{n_2} \cdots \varphi_{n_m} x^{(n_1+1)+\cdots+(n_m+1)} = \varphi(x)^m. \end{aligned} \tag{2.7}$$

If we now set $\Delta^{\mathrm{dif}} A(x) := \sum \Delta^{\mathrm{dif}} a_n x^n$, then we obtain

$$\begin{aligned} \langle \Delta^{\mathrm{dif}} A(x), \varphi \otimes \psi \rangle &= \sum_{n=0}^{\infty} \langle \Delta^{\mathrm{dif}} a_n, \varphi \otimes \psi \rangle x^{n+1} = \sum_{n=0}^{\infty} a_n (\varphi \circ \psi) x^{n+1} = (\varphi \circ \psi)(x) \\ &= \langle z^{-1} \rangle \frac{\varphi(z)}{z - \psi(x)} \\ &= \langle z^{-1} \rangle \left(\langle A(z), \varphi \rangle \, \langle \frac{1}{z - A(x)}, \psi \rangle \right) = \langle z^{-1} \rangle \langle A(z) \otimes \frac{1}{z - A(x)}, \varphi \otimes \psi \rangle, \end{aligned}$$

where we have used (2.6) and (2.7) to go from the second to the third line. Therefore, the co-product of the generating series $A(x)$ is given by

$$\Delta^{\mathrm{dif}} A(x) = \langle z^{-1} \rangle A(z) \otimes \frac{1}{z - A(x)}. \tag{2.8}$$

To find $\Delta^{\mathrm{dif}} a_n$ for each $n \geq 1$, it suffices to evaluate this residue, where, again, the inverse of $z - A(x)$ has to be interpreted as

$$\frac{1}{z - A(x)} = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{A(x)}{z} \right)^n = \sum_{n=0}^{\infty} A(x)^n z^{-n-1}.$$

2.2 Formal diffeomorphisms with non-commutative coefficients

Let $\mathcal{A}$ be an associative unital algebra, and consider the set

$$G^{\mathrm{dif}}(\mathcal{A}) = \left\{ \varphi(x) = \sum_{n=0}^{\infty} \varphi_n x^{n+1}, \varphi_n \in \mathcal{A}, \varphi_0 = 1 \right\}$$

of formal power series with coefficients in $\mathcal{A}$. Proceeding in analogy to Section 1.2, where we adopted formula (1.2) for invertible series for the case of not necessarily commutative coefficients, it seems natural to adopt formula (2.2) as the definition of the composition $\varphi \circ \psi$ for series φ and ψ in not necessarily commutative coefficients. However, such a composition is not associative unless $\mathcal{A}$ is commutative, since the associator

$$\big(\varphi \circ (\psi \circ \eta)\big)(x) - \big((\varphi \circ \psi) \circ \eta\big)(x) = x^4 \big(\varphi_1 \eta_1 \psi_1 - \varphi_1 \psi_1 \eta_1\big) + O(x^5)$$

is non-zero if the coefficients do not commute.

Thus, the set of series with non-commutative coefficients, with composition defined by (2.2), does not define a group. However, we shall see in this section that, dually, there exists a co-associative co-product which gives rise to a non-commutative Hopf algebra. In other words, if we consider the free non-commutative algebra generated by the elementary functions $a_n(\varphi) = \varphi_n \in \mathcal{A}$, for $\varphi \in G^{\mathrm{dif}}(\mathcal{A})$, then the dual co-product of the composition (2.2) is co-associative.

Definition 2.4. Let $\mathcal{H}^{\mathrm{dif}} = \mathbb{C}\langle a_1, a_2, \ldots \rangle$ denote the free associative algebra in infinitely many variables a_n, for $n \in \mathbb{N}$. As for the commutative case (2.1), we consider the generating series $A(x) = x + \sum_{n=1}^{\infty} a_n x^{n+1}$. Then we define a co-product on the generators of $\mathcal{H}^{\mathrm{dif}}$ by the global formula (2.8),

$$\Delta^{\mathrm{dif}} A(x) = \langle z^{-1} \rangle A(z) \otimes \frac{1}{z - A(x)}, \tag{2.9}$$

and we extend it multiplicatively to products of elements of $\mathcal{H}^{\mathrm{dif}}$. As a co-unit on $\mathcal{H}^{\mathrm{dif}}$, we take the standard graded co-unit $\varepsilon(1) = 1$ and $\varepsilon(a_n) = 0$, in other words, $\varepsilon(A(x)) = x$. We show in Theorem 2.11 that $\mathcal{H}^{\mathrm{dif}}$ is indeed a Hopf algebra. Of course, we can obtain $\mathbb{C}(G^{\mathrm{dif}})$ from $\mathcal{H}^{\mathrm{dif}}$ by taking the Abelianisation.

For the structural analysis of $\mathcal{H}^{\mathrm{dif}}$ and its co-product Δ^{dif}, we shall make frequent use of certain non-commutative polynomials, which we define next.

Definition 2.5. For $m, n \geq 0$, we define the polynomials $Q_m^{(n)}(a)$ in m variables $a_1, a_2, \ldots, a_m$ by

$$Q_m^{(n)}(a) = \sum_{\substack{j_0 + \cdots + j_n = m \\ j_0, \ldots, j_n \geq 0}} a_{j_0} \cdots a_{j_n},$$

where, as before, a_0 is interpreted as 1. For convenience, we set $Q_m^{(-1)}(a) = 0$ if $m > 0$ and $Q_0^{(-1)}(a) = 1$.

According to this definition, we have $Q_0^{(n)}(a) = 1$ for all n, $Q_m^{(0)}(a) = a_m$ for all m, and

$$\begin{aligned}
Q_1^{(n)}(a) &= (n+1)a_1, \\
Q_2^{(n)}(a) &= (n+1)a_2 + \frac{n(n+1)}{2} a_1^2, \\
Q_3^{(n)}(a) &= (n+1)a_3 + \frac{n(n+1)}{2}(a_1 a_2 + a_2 a_1) + \frac{(n+1)n(n-1)}{6} a_1^3,
\end{aligned}$$

for any $n \geq 0$. More generally, for $n \geq 0$, we may write $Q_m^{(n)}(a)$ in the form

$$Q_m^{(n)}(a) = \sum_{l=0}^{\infty} \binom{n+1}{l} \sum_{\substack{h_1 + \cdots + h_l = m \\ h_1, \ldots, h_l \geq 1}} a_{h_1} \cdots a_{h_l}. \tag{2.10}$$

It follows directly from the definition that

$$A(x)^{n+1} = x^{n+1} \left(1 + \sum_{p=1}^{\infty} a_p x^p \right)^{n+1} = \sum_{m=0}^{\infty} Q_m^{(n)}(a)\; x^{m+n+1}, \tag{2.11}$$

in other words, $Q_m^{(n)}(a)$ is the coefficient of x^{m+n+1} in $A(x)^{n+1}$. From this generating function, we can easily derive two equations satisfied by the $Q_m^{(n)}(a)$ which we shall frequently use later on.

Lemma 2.6. *For $n, m \geq 0$, the polynomials $Q_m^{(n)}(a)$ satisfy the recurrence*

$$Q_m^{(n)}(a) = \sum_{l=0}^{m} a_l Q_{m-l}^{(n-1)}(a) = \sum_{l=0}^{m} Q_{m-l}^{(n-1)}(a) a_l.$$

Proof. The equation follows directly by comparing coefficients of x^{n+m+1} in the generating function identity $A(x)^{n+1} = A(x)A(x)^n = A(x)^n A(x)$, using (2.11). $\square$

Lemma 2.7. *For $l, m, n \geq 0$, the polynomials $Q_m^{(n)}(a)$ satisfy the quadratic relation*

$$Q_m^{(l+n+1)}(a) = \sum_{k=0}^{m} Q_k^{(l)}(a) Q_{m-k}^{(n)}(a). \tag{2.12}$$

This relation holds as well if one of l and n is equal to -1.

Proof. The relation follows directly by comparing coefficients of $x^{m+l+n+2}$ in the generating function equation $A(x)^{l+n+2} = A(x)^{l+1} A(x)^{n+1}$, using (2.11). $\square$

Another simple corollary of (2.11) is an explicit expression for the double generating function for the $Q_m^{(n)}(a)$.

Corollary 2.8. *The generating function*

$$Q(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} x^n y^m Q_m^{(n)}(a) \tag{2.13}$$

is the Green function[1] *at x of the Hamiltonian $H(y) = \left(\frac{A(y)}{y}\right)^{-1}$, where, as before, $A(y) = \sum_{n=0}^{\infty} a_n y^{n+1}$. That is, the generating function is*

$$Q(x, y) = \left(\left(\frac{A(y)}{y} \right)^{-1} - x \right)^{-1} = \left(1 - x \frac{A(y)}{y} \right)^{-1} \frac{A(y)}{y}. \tag{2.14}$$

Moreover, it satisfies the resolvent equation

$$Q(x, y) = Q(z, y) + (x - z) Q(x, y) Q(z, y). \tag{2.15}$$

Proof. For proving the first equation, we use (2.11) to rewrite $Q(x, y)$ in the form

$$Q(x, y) = \sum_{n=0}^{\infty} x^n \sum_{m=0}^{\infty} y^m Q_m^{(n)}(a) = \sum_{n=0}^{\infty} x^n \left(\frac{A(y)}{y} \right)^{n+1}.$$

The sum over n is a geometric series and can therefore be evaluated. This yields (2.14). Eq. (2.15) can now easily be verified by substituting (2.14) in (2.15). $\square$

We are now in the position to explicitly describe the action of Δ^{dif} on the generators a_n.

Lemma 2.9. *On the generators a_n of $\mathcal{H}^{\mathrm{dif}}$, the co-product is given by*

$$\Delta^{\mathrm{dif}} a_n = \sum_{k=0}^{n} a_k \otimes Q_{n-k}^{(k)}(a).$$

Proof. According to Definition 2.4, we find $\Delta^{\mathrm{dif}} A_n$ by extracting the coefficient of x^{n+1} from $\Delta^{\mathrm{dif}} A(x)$, as given by (2.9). Consequently, we expand the right-hand side of (2.9), and we obtain

$$\begin{aligned}
\Delta^{\mathrm{dif}} A(x) &= \langle z^{-1} \rangle A(z) \otimes \frac{1}{z - A(x)} \\
&= \sum_{n=0}^{\infty} \langle z^{-1} \rangle z^{-n-1} A(z) \otimes A(x)^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k \otimes A(x)^n \langle z^{-1} \rangle z^{k-n} \\
&= \sum_{k=0}^{\infty} a_k \otimes A(x)^{k+1}.
\end{aligned} \tag{2.16}$$

[1]If H is an operator on a Hilbert space, the *resolvent* or *Green function* of H is the operator $R(z) = (H - z)^{-1}$, and it satisfies the *resolvent identity* $R(z) = R(z') + (z - z') R(z) R(z')$, cf. [28].

Use of (2.11) thus yields our claim. ☐

For instance, we have

$$\begin{aligned}
\Delta^{\text{dif}} a_1 &= a_1 \otimes 1 + 1 \otimes a_1, \\
\Delta^{\text{dif}} a_2 &= a_2 \otimes 1 + 1 \otimes a_2 + 2a_1 \otimes a_1, \\
\Delta^{\text{dif}} a_3 &= a_3 \otimes 1 + 1 \otimes a_3 + 3a_2 \otimes a_1 + 2a_1 \otimes a_2 + a_1 \otimes a_1^2.
\end{aligned}$$

From the value of $\Delta^{\text{dif}} a_3$, we see that the co-product is not co-commutative.

There is, as well, an elegant description of the co-product on the polynomials $Q_m^{(n)}(a)$, which we give in the corollary below. In particular, it shows that the sub-algebra of $\mathcal{H}^{\text{dif}}$ generated by the $Q_m^{(n)}(a)$'s is in fact a *Hopf* sub-algebra of $\mathcal{H}^{\text{dif}}$.

Corollary 2.10. *For $m, n \geq 0$, the co-product of $Q_m^{(n)}(a)$ is given by*

$$\Delta^{\text{dif}} Q_m^{(n)}(a) = \sum_{k=0}^{m} Q_{m-k}^{(n)}(a) \otimes Q_k^{(n+m-k)}(a). \tag{2.17}$$

Proof. Since, by (2.11), the generating function of the $Q_m^{(n)}(a)$ is $A(x)^{n+1}$, we compute the image of a power of $A(x)$ under Δ^{dif}. Using (2.9), we have

$$\begin{aligned}
\Delta^{\text{dif}}\big(A(x)^2\big) = \Delta^{\text{dif}} A(x)\ \Delta^{\text{dif}} A(x) &= \sum_{m,n=0}^{\infty} a_m a_n \otimes A(x)^{m+n+2} \\
&= \langle z^{-1} \rangle \sum_{m,n,k=0}^{\infty} a_m a_n z^{m+n+2} \otimes A(x)^k z^{-k-1} \\
&= \langle z^{-1} \rangle A(z)^2 \otimes \frac{1}{z - A(x)}.
\end{aligned}$$

By induction on n, a similar reasoning for $n \geq 2$ shows that

$$\Delta^{\text{dif}}\big(A(x)^n\big) = \langle z^{-1} \rangle A(z)^n \otimes \frac{1}{z - A(x)}. \tag{2.18}$$

Hence, comparison of coefficients of x^{m+n+1} in (2.18) (with n replaced by $n+1$) yields (2.17). ☐

The results obtained so far, allow us not to prove that $\mathcal{H}^{\text{dif}}$ is a graded and connected Hopf algebra.

Theorem 2.11. *The algebra $\mathcal{H}^{\text{dif}}$ is a graded and connected Hopf algebra, which is neither commutative nor co-commutative.*

Proof. The algebra $\mathcal{H}^{\text{dif}}$ becomes a graded algebra and a graded co-algebra by defining the degree of a monomial $a_{j_1} a_{j_2} \cdots a_{j_m}$ to be $j_1 + j_2 + \cdots + j_m$. Moreover, $\mathcal{H}^{\text{dif}}$ is connected, that is, the zero degree part consists only of the scalars. Since, for a graded connected Hopf algebra, the antipode is given by the standard recursive formula

$$S a_n = -a_n - \sum_{p=1}^{n-1} a_p S\big(Q_{n-p}^{(p)}(a)\big) = -a_n - \sum_{p=1}^{n-1} (S a_p) Q_{n-p}^{(p)}(a), \tag{2.19}$$

the only statement which needs to be proved is the co-associativity of the co-product.

A recursive proof of the latter is given in [8]. It involves a number of technical computations in order to deduce some recurrence relations for the polynomials $Q_m^{(l)}(a)$. Here we present an alternative simple proof which justifies the introduction of residues and generating series.

If we consider a series $f(x) = \sum_{n=0}^{\infty} f_n x^n$ with complex coefficents $f_n \in \mathbb{C}$, then, because of (2.18), the image of the the composition $(f \circ A)(x) = \sum_{n=0}^{\infty} f_n A(x)^n$ under Δ^{dif} is given by

$$\Delta^{\text{dif}}(f \circ A)(x) = \langle z^{-1} \rangle (f \circ A)(z) \otimes \frac{1}{z - A(x)}.$$

In particular, for $f(x) = 1/(y-x)$ (regarded as a formal power series in x!) we obtain

$$\Delta^{\mathrm{dif}} \frac{1}{y - A(x)} = \langle z^{-1} \rangle \frac{1}{y - A(z)} \otimes \frac{1}{z - A(x)}. \tag{2.20}$$

With the help of this result, it is now very easy to prove the co-associativity of the co-product: on the one hand, we have

$$\begin{aligned}(\Delta^{\mathrm{dif}} \otimes 1) \circ \Delta^{\mathrm{dif}} A(x) &= \langle {z_1}^{-1} \rangle \Delta^{\mathrm{dif}} A(z_1) \otimes \frac{1}{z_1 - A(x)} \\ &= \langle {z_1}^{-1} \rangle \langle {z_2}^{-1} \rangle A(z_2) \otimes \frac{1}{z_2 - A(z_1)} \otimes \frac{1}{z_1 - A(x)},\end{aligned}$$

and, on the other hand, we have

$$\begin{aligned}(1 \otimes \Delta^{\mathrm{dif}}) \circ \Delta^{\mathrm{dif}} A(x) &= \langle {z_2}^{-1} \rangle A(z_2) \otimes \Delta^{\mathrm{dif}} \frac{1}{z_2 - A(x)} \\ &= \langle {z_2}^{-1} \rangle \langle {z_1}^{-1} \rangle A(z_2) \otimes \frac{1}{z_2 - A(z_1)} \otimes \frac{1}{z_1 - A(x)},\end{aligned}$$

yielding the same expression on the right-hand side. □

Remark 2.12. The Abelianisation of $\mathcal{H}^{\mathrm{dif}}$ gives the co-ordinate ring $\mathbb{C}(G^{\mathrm{dif}})$. In fact, if we suppose that the variables a_n commute, then, rewriting (2.10), the polynomial $Q_m^{(k)}(a)$ becomes

$$Q_m^{(k)}(a) = \sum_{l=0}^{\infty} \binom{k+1}{l} \sum_{\substack{p_1+2p_2+\cdots+mp_m=m \\ p_1+p_2+\cdots+p_m=l}} \frac{l!}{p_1! \cdots p_m!}\ a_1^{p_1} \cdots a_m^{p_m},$$

where the sum runs over $p_1, \ldots, p_m \geq 0$.[2] Consequently, the co-product Δ^{dif}, as given Lemma 2.9, becomes

$$\Delta^{\mathrm{dif}} a_n = \sum_{k=0}^{n} a_k \otimes \sum_{l=1}^{n-k} \frac{(k+1)!}{(k+1-l)!} \sum_{\substack{p_1+\cdots+(n-k)p_{n-k}=n-k \\ p_1+p_2+\cdots+p_{n-k}=l}} \frac{1}{p_1! \cdots p_{n-k}!}\ a_1^{p_1} \cdots a_{n-k}^{p_{n-k}} \tag{2.21}$$

in the commutative case. This agrees with the Faà di Bruno co-product on the variables $a_n = u_{n+1}/(n+1)!$, with $a_0 = u_1 = 1$. In fact, from (2.4), by setting $\beta_i = \alpha_{i+1}$, and then summing up over $l = \beta_0$, we obtain:

$$\begin{aligned}\Delta a_n &= \sum_{k=0}^{n} a_k \otimes \sum_{\substack{\beta_0+2\beta_1+\cdots+(n+1)\beta_n=n+1 \\ \beta_0+\beta_1+\cdots+\beta_n=k+1}} \frac{(k+1)!}{\beta_0!\beta_1! \cdots \beta_n!} a_1^{\beta_1} a_2^{\beta_2} \cdots a_n^{\beta_n} \\ &= \sum_{k=0}^{n} a_k \otimes \sum_{l=0}^{k} \frac{(k+1)!}{l!} \sum_{\substack{2\beta_1+\cdots+(n+1)\beta_n=n+1-l \\ \beta_1+\cdots+\beta_n=k+1-l}} \frac{1}{\beta_1! \cdots \beta_n!} a_1^{\beta_1} a_2^{\beta_2} \cdots a_n^{\beta_n} \\ &= \sum_{k=0}^{n} a_k \otimes \sum_{l=1}^{k+1} \frac{(k+1)!}{(k+1-l)!} \sum_{\substack{\beta_1+\cdots+n\beta_n=n-k \\ \beta_1+\cdots+\beta_n=l}} \frac{1}{\beta_1! \cdots \beta_n!} a_1^{\beta_1} a_2^{\beta_2} \cdots a_n^{\beta_n}.\end{aligned}$$

Since the β_i are non-negative integers, the condition $\beta_1 + \cdots + n\beta_n = n-k$ implies that $\beta_{n-k+1} = \cdots = \beta_n = 0$. As a consequence, the condition $\beta_1 + \cdots + \beta_{n-k} = l$ implies that k runs from 1 to at most $n-k$. Therefore Δa_n gives exactly (2.21).

[2] It should be noted that these combinatorial factors are well known from Planck's quantum theory of blackbody radiation, see for instance [43].

Remark 2.13. In Proposition 1.1, we showed that the co-product Δ^{inv} on the Hopf algebra $\mathcal{H}^{\mathrm{inv}}$ can be lifted up to a new kind of co-product Δ_*^{inv}, with values in the free product $\mathcal{H}^{\mathrm{inv}} * \mathcal{H}^{\mathrm{inv}}$, which remains co-associative and gives $\mathcal{H}_*^{\mathrm{inv}} = (\mathcal{H}^{\mathrm{inv}}, \Delta_*^{\mathrm{inv}})$ the structure of a co-group in associative algebras, cf. [17]. By way of contrast, a lifting of the co-product Δ^{dif} with values in $\mathcal{H}^{\mathrm{dif}} * \mathcal{H}^{\mathrm{dif}}$ is *not* co-associative, and the new Hopf algebra $\mathcal{H}_*^{\mathrm{dif}} = (\mathcal{H}^{\mathrm{dif}}, \Delta_*^{\mathrm{dif}})$ is *not* a co-group in associative algebras. This reflects the fact that the set of formal diffeomorphisms with non-commutative coefficients fails to be a group because the composition fails to be associative. In fact, given a non-commutative algebra $\mathcal{A}$, the set $G^{\mathrm{dif}}(\mathcal{A})$ can still be reconstructed from $\mathcal{H}_*^{\mathrm{dif}}$ as the set $\mathrm{Hom}_{Alg}(\mathcal{H}_*^{\mathrm{dif}}, \mathcal{A})$ of characters with non-commutative values, via the adapted convolution defined by formula (1.7). In this duality, the non-co-associativity of Δ_*^{dif} corresponds exactly to the non-associativity of the composition of series. This makes it even more remarkable that the co-product Δ^{dif}, which is the composition of Δ_*^{dif} by the natural projection $\mathcal{H}^{\mathrm{dif}} * \mathcal{H}^{\mathrm{dif}} \to \mathcal{H}^{\mathrm{dif}} \otimes \mathcal{H}^{\mathrm{dif}}$, turns out to be co-associative.

Finally note that the problem of whether the Hopf algebra on trees introduced by J.–L. Loday and M. Ronco in [37] is a co-group was already investigated by R. Holtkamp in [26]. This Hopf algebra turns out to be isomorphic to the linear dual of the Hopf algebra $\widetilde{\mathcal{H}^{\alpha}}$ that we introduce in Section 4, which we show being an extension of $\mathcal{H}^{\mathrm{dif}}$. In Sections 3.5 and 3.6 of [26], Holtkamp shows that the Loday–Ronco Hopf algebra cannot be a co-group, and therefore our result on $\mathcal{H}^{\mathrm{dif}}$ agrees with his.

2.3 Explicit non-commutative Lagrange formula

In the commutative setting, the antipode of the co-ordinate ring $\mathbb{C}(G^{\mathrm{dif}})$ is of course the operation which is dual to the inversion $\phi \mapsto \phi^{[-1]}$ of a formal power series. The coefficients of the inverse series $\phi^{[-1]}$ are usually computed by using the Lagrange inversion formula.

As we outlined in Section 2.2, in the non-commutative setting, the analogue for the co-ordinate $\mathbb{C}(G^{\mathrm{dif}})$ is the (non-commutative) Hopf algebra $\mathcal{H}^{\mathrm{dif}}$. Its antipode can be computed using the recursive formula (2.19). In particular, in $\mathcal{H}^{\mathrm{dif}}$, the first few values of the antipode on the generators a_i are

$$\begin{aligned}
Sa_1 &= -a_1, \\
Sa_2 &= -a_2 + 2a_1^2, \\
Sa_3 &= -a_3 + (2a_1a_2 + 3a_2a_1) - 5a_1^3, \\
Sa_4 &= -a_4 + (2a_1a_3 + 3a_2^2 + 4a_3a_1) - (5a_1^2a_2 + 7a_1a_2a_1 + 9a_2a_1^2) + 14a_1^4.
\end{aligned}$$

It should be observed that the square of the antipode is equal to the identity only for a_1 and a_2.

In contrast to the commutative setting, where there exist both the recursive and the Lagrange inversion formula, for $\mathcal{H}^{\mathrm{dif}}$ there is no known analogue of the Lagrange inversion formula. In the theorem below, we present an explicit non-recursive expression for the antipode for $\mathcal{H}^{\mathrm{dif}}$. Clearly, the Abelianisation of this expression gives an explicit formula for the inversion of formal diffeomorphisms (which can be seen as an alternative version of the Lagrange inversion formula).

Theorem 2.14. *The action of the antipode of the Hopf algebra $\mathcal{H}^{\mathrm{dif}}$ on the generators has the following closed form:*

$$Sa_n = -a_n - \sum_{k=1}^{n-1} (-1)^k \sum_{\substack{n_1+\cdots+n_k+n_{k+1}=n \\ n_1,\ldots,n_{k+1}>0}} \lambda(n_1,\ldots,n_k)\ a_{n_1}\cdots a_{n_k} a_{n_{k+1}}, \tag{2.22}$$

with coefficients

$$\lambda(n_1,\ldots,n_k) = \sum_{\substack{m_1+\cdots+m_k=k \\ m_1+\cdots+m_h \geq h \\ h=1,\ldots,k-1}} \binom{n_1+1}{m_1} \cdots \binom{n_k+1}{m_k}. \tag{2.23}$$

It should be noted that the coefficient $\lambda(n_1,\ldots,n_k)$ of the monomial $a_{n_1}\cdots a_{n_k}a_{n_{k+1}}$ does not depend on the last factor $a_{n_{k+1}}$.

Proof. We prove formula (2.22) by induction on n, starting from the recursive definition (2.19) of the antipode S. Formula (2.22) is obviously correct for $n = 1$, and it is also correct for $n = 2$, because in the latter case the sum over k on the right-hand side gives only one term for $k = 1$, namely

$$(-1) \sum_{\substack{n_1+n_2=2 \\ n_1,n_2>0}} \lambda(n_1)\ a_{n_1} a_{n_2} = -\lambda(1)\ a_1^2 = -\binom{2}{1}\ a_1^2 = -2a_1^2.$$

For $n \geq 3$, suppose that formula (2.22) holds for Sa_p, $p = 1, \ldots, n-1$. We shall show that

$$\sum_{p=1}^{n-1} (Sa_p) Q^{(p)}_{n-p}(a) = \sum_{p=1}^{n-1} (-1)^p \sum_{\substack{n_1+\cdots+n_p+n_{p+1}=n \\ n_1,\ldots,n_{p+1}>0}} \lambda(n_1, \ldots, n_p)\ a_{n_1} \cdots a_{n_p} a_{n_{p+1}}. \tag{2.24}$$

In fact,

$$\begin{aligned}
\sum_{p=1}^{n-1} (Sa_p) Q^{(p)}_{n-p}(a) &= \sum_{p=1}^{n-1} \left[-a_p - \sum_{k=1}^{p-1} \sum_{\substack{p_1+\cdots+p_{k+1}=p \\ p_1,\ldots,p_{k+1}>0}} (-1)^k \lambda(p_1, \ldots, p_k)\ a_{p_1} \cdots a_{p_{k+1}} \right] \\
&\qquad \times \left[\sum_{l=1}^{n-p} \sum_{\substack{j_1+\cdots+j_l=n-p \\ j_1,\ldots,j_l>0}} \binom{p+1}{l}\ a_{j_1} \cdots a_{j_l} \right] \\
&= \sum_{p=1}^{n-1} \sum_{l=1}^{n-p} \sum_{\substack{j_1+\cdots+j_l=n-p \\ j_1,\ldots,j_l>0}} (-1) \binom{p+1}{l}\ a_p a_{j_1} \cdots a_{j_l} \\
&\quad + \sum_{p=1}^{n-1} \sum_{k=1}^{p-1} \sum_{l=1}^{n-p} \sum_{\substack{p_1+\cdots+p_{k+1}=p \\ j_1+\cdots+j_l=n-p}} (-1)^{k+1} \lambda(p_1, \ldots, p_k) \binom{p+1}{l}\ a_{p_1} \cdots a_{p_{k+1}} a_{j_1} \cdots a_{j_l} \\
&= \sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{\substack{p+j_1+\cdots+j_q=n \\ j_1,\ldots,j_q>0}} (-1) \binom{p+1}{q}\ a_p a_{j_1} \cdots a_{j_q} \\
&\quad + \sum_{q=1}^{n-1} \sum_{k=1}^{q-1} \sum_{\substack{p_1+\cdots+p_{k+1} \\ j_1+\cdots+j_{q-k}=n}} (-1)^{k+1} \lambda(p_1, \ldots, p_k) \binom{p_1+\cdots+p_{k+1}+1}{q-k}\ a_{p_1} \cdots a_{p_{k+1}} a_{j_1} \cdots a_{j_{q-k}} \\
&= \sum_{q=1}^{n-1} \sum_{\substack{n_1+\cdots+n_{q+1}=n \\ n_1,\ldots,n_{q+1}>0}} a_{n_1} \cdots a_{n_{q+1}} \\
&\qquad \times \left[-\binom{n_1+1}{q} + \sum_{k=1}^{q-1} (-1)^{k+1} \lambda(n_1, \ldots, n_k) \binom{n_1+\cdots+n_{k+1}+1}{q-k} \right].
\end{aligned}$$

Therefore, the identity (2.24) is verified if and only if for any $q = 1, \ldots, n-1$, and for any positive integers $n_1, \ldots, n_{q+1}$ with constant sum $n_1 + \cdots + n_{q+1} = n$, we have

$$(-1)^q \lambda(n_1, \ldots, n_q) = -\binom{n_1+1}{q} + \sum_{k=1}^{q-1} (-1)^{k+1} \lambda(n_1, \ldots, n_k) \binom{n_1+\cdots+n_{k+1}+1}{q-k}.$$

This identity is proved in the next lemma. (Recall the definition (2.23) of the coefficients $\lambda(n_1, \ldots, n_q)$.) $\square$

Lemma 2.15. *Let $n \geq 2$, then for any $q = 1, \dots, n-1$, and for any positive integers $n_1, \dots, n_{q+1}$, we have*

$$-\binom{n_1+1}{q} + \sum_{k=1}^{q} (-1)^{k+1} \sum_{\substack{m_1+\cdots+m_k=k \\ m_1+\cdots+m_h \geq h \\ h=1,\dots,k-1}} \binom{n_1+1}{m_1} \cdots \binom{n_k+1}{m_k} \binom{n_1+\cdots+n_{k+1}+1}{q-k} = 0. \tag{2.25}$$

Proof. Let us introduce some short notations: let $M_j := m_1 + \cdots + m_j$, $N_j := n_1 + \cdots + n_j$, denote by $\sum'_{m_1,m_2,\dots}$ the sum over non-negative integers $m_1, m_2, \dots$ such that $m_1 + \cdots + m_h \geq h$ for all h, $1 \leq h \leq k-1$, and finally set $\Pi_0 := 1$ and

$$\Pi_j := \prod_{i=1}^{j} \binom{n_i+1}{m_i}, \quad j > 0.$$

Using these notations, Equation (2.25) may be rewritten in the form

$$\sum_{k=0}^{q} (-1)^{k+1} \sum_{M_k=k}{}' \Pi_k \binom{N_{k+1}+1}{q-k} = 0.$$

Let $S(q)$ be the sum on the left-hand side. We claim that for $0 \leq m \leq q$ we have

$$S(q) = \sum_{k=0}^{q-m} (-1)^{k+1} \sum_{M_k=k}{}' \Pi_k \binom{N_{k+1}+1}{q-k} + \sum_{\ell=1}^{m} (-1)^{q+\ell} \sum_{M_{q+1-m}=q+1-\ell}{}' \Pi_{q+1-m} \binom{N_{q+1-m}+\ell-m}{\ell-1}. \tag{2.26}$$

We prove this claim by induction on m. Clearly, formula (2.26) holds for $m = 0$. So, let us suppose that it holds for m, and let us, under this hypothesis, do the following computation:

$$\begin{aligned} S(q) = & \sum_{k=0}^{q-m-1} (-1)^{k+1} \sum_{M_k=k}{}' \Pi_k \binom{N_{k+1}+1}{q-k} \\ & + (-1)^{q-m+1} \sum_{M_{q-m}=q-m}{}' \Pi_{q-m} \binom{N_{q-m+1}+1}{m} \\ & + \sum_{\ell=1}^{m} (-1)^{q+\ell} \sum_{s=0}^{m-\ell+1} \sum_{M_{q+1-m}=q+1-\ell-s}{}' \Pi_{q-m} \binom{n_{q-m+1}+1}{s} \binom{N_{q+1-m}+\ell-m}{\ell-1} \\ = & \sum_{k=0}^{q-m-1} (-1)^{k+1} \sum_{M_k=k}{}' \Pi_k \binom{N_{k+1}+1}{q-k} \\ & + \sum_{\ell=1}^{m+1} (-1)^{q+\ell} \sum_{s=0}^{m-\ell+1} \sum_{M_{q-m}=q+1-\ell-s}{}' \Pi_{q-m} \binom{n_{q-m+1}+1}{s} \binom{N_{q+1-m}+\ell-m}{\ell-1}. \end{aligned}$$

We now replace ℓ by $r - s$ and we interchange the inner sums:

$$\begin{aligned} S(q) = & \sum_{k=0}^{q-m-1} (-1)^{k+1} \sum_{M_k=k}{}' \Pi_k \binom{N_{k+1}+1}{q-k} \\ & + \sum_{r=1}^{m+1} \sum_{M_{q-m}=q+1-r}{}' \Pi_{q-m} \sum_{s=0}^{r-1} (-1)^{q+r-s} \binom{n_{q-m+1}+1}{s} \binom{N_{q+1-m}+r-s-m}{r-s-1}. \end{aligned} \tag{2.27}$$

In the second line, the inner sum over s can be evaluated using the Chu–Vandermonde formula (see e.g.

[22, Sec. 5.1, (5.27)]). Thus, we obtain:

$$\begin{aligned}
&\sum_{s=0}^{r-1}(-1)^{q+r-s}\binom{n_{q-m+1}+1}{s}\binom{N_{q+1-m}+r-s-m}{r-s-1}\\
&\qquad\qquad=\sum_{s=0}^{r-1}(-1)^{q+1}\binom{n_{q-m+1}+1}{s}\binom{-N_{q+1-m}+m-2}{r-s-1}\\
&\qquad\qquad=(-1)^{q+1}\binom{n_{q-m+1}-N_{q+1-m}+m-1}{r-1}\\
&\qquad\qquad=(-1)^{q+1}\binom{-N_{q-m}+m-1}{r-1}=(-1)^{q+r}\binom{N_{q-m}+r-m-1}{r-1}.
\end{aligned}$$

If we substitute this in (2.27), we obtain exactly formula (2.26) with m replaced by $m+1$.

To prove the lemma, we set $m=q$ in (2.26). This gives

$$\begin{aligned}
S(q)&=-\binom{n_1+1}{q}+\sum_{\ell=1}^{q}(-1)^{q+\ell}\binom{n_1+1}{q+1-\ell}\binom{n_1+\ell-q}{\ell-1}\\
&=\sum_{\ell=1}^{q+1}(-1)^{q+\ell}\binom{n_1+1}{q+1-\ell}\binom{n_1+\ell-q}{\ell-1}.
\end{aligned}$$

Again, the sum can be evaluated using the Chu–Vandermonde formula. As a result, we obtain

$$S(q)=(-1)^{q+1}\binom{q-1}{q}=0.$$

□

2.4 A tree labelling for the antipode coefficients

The coefficients $\lambda(n_1,n_2,\ldots,n_k)$ in formula (2.22) for the antipode are given, by means of (2.23), as a sum over k-tuples $(m_1,m_2,\ldots,m_k)$ of non-negative integers satisfying the two conditions

$$m_1+\cdots+m_h\ge h\quad\text{for all } h=1,2,\ldots,k-1,\quad\text{and} \tag{2.28}$$

$$m_1+\cdots+m_k=k. \tag{2.29}$$

Let us denote this set of k-tuples by $\mathcal{M}_k$. As we are going to outline in this section, it is well known that the cardinality of $\mathcal{M}_k$ is given by the Catalan numbers $\frac{1}{k+1}\binom{2k}{k}$. (We refer the reader to Exercise 6.19 in [47] for 66 combinatorial interpretations of the Catalan numbers[3], out of which our k-tuples are item **w.**, modulo the substitutions $n=k$ and $a_i=m_i-1$.) Thus, in particular, these k-tuples are in bijection with planar binary trees (item **d.** in [47, Ex. 6.19]). For the convenience of the reader, we explain this bijection here in detail. (What we do, is, essentially, extract the appropriate restriction of the bijection in [47, Example 5.3.8], using the tree language of Loday [33].)

Recall, from [33, Sec. 1.5] or [9], that for any planar binary trees t,s, the tree t *over* s is defined as the grafting

$$t/s:=\begin{array}{ccc} t & & \\ & \backslash & \\ & & s\end{array}$$

of the root of t on the left-most leaf of s, and similarly the tree t *under* s is defined as the grafting

$$t\backslash s:=\begin{array}{ccc} & & s \\ & / & \\ t & & \end{array}$$

of the root of s on the right-most leaf of t. The operations *over* and *under* are two associative (non-commutative) operations with unit given by the "root tree" $|$. Moreover, any planar binary tree can be written as a monomial in $\curlyvee$, with respect to *over* and *under*.

[3]with some more recent ones appearing on Richard Stanley's WWW site `http://www-math.mit.edu/~rstan/`

Using this notation, the mapping from $\mathcal{M}_k$ to the set Y_k of planar binary trees with k internal vertices is given by the following algorithm.

Definition 2.16. For any $k \geq 1$, let $\Phi : \mathcal{M}_k \longrightarrow Y_k$ be the map defined by the following recursive algorithm. For any $m = (m_1, \dots, m_k) \in \mathcal{M}_k$,

1. if $m = (1)$, then set $\Phi(m) := \Upsilon$;
2. if $m = (m_1, \dots, m_l, m_{l+1}, \dots, m_k) \in \mathcal{M}_k$ is such that $(m_1, \dots, m_l) \in \mathcal{M}_l$ and $(m_{l+1}, \dots, m_k) \in \mathcal{M}_{k-l}$, then set
$$\Phi(m) := \Phi(m_1, \dots, m_l) \backslash \Phi(m_{l+1}, \dots, m_k);$$
3. if $m = (m_1, \dots, m_{k-1}, 0) \in \mathcal{M}_k$ is not decomposable in sub-tuples as in 2., then $m_1 > 1$; in this case set
$$\Phi(m) := \Phi(m_1 - 1, \dots, m_{k-1}) / \Upsilon .$$

It is easy to show that if a k-tuple $m = (m_1, \dots, m_k)$ is not decomposable in sub-tuples as in 2., then it is of the form given in 3. In fact, by Eq. (2.29) we can write $m_1 + \cdots + m_{k-1} = k - m_k$ and from Eq. (2.28) we get $m_k \leq 1$. But $m_k = 1$ implies that $(m_k, \dots, m_{k-1}) \in \mathcal{M}_{k-l}$, hence m would be decomposable as in 2, a contradiction. Therefore $m_k = 0$, and $m = (m_1, \dots, m_{k_1}, 0)$. Furthermore, if $m_1 = 1$ the k-tuple is decomposable into $(m_1) = (1) \in \mathcal{M}_1$ and $(m_2, \dots, m_{k-1}, 0) \in \mathcal{M}_{k-1}$, which contradicts again our original assumption. We must therefore have $m_1 > 1$ in this case.

To give an example, consider the sequence $m = (4, 0, 1, 0, 0, 2, 1, 0) \in \mathcal{M}_8$. This 8-tuple is decomposable into the two indecomposable tuples $(4, 0, 1, 0, 0) \in \mathcal{M}_5$ and $(2, 1, 0) \in \mathcal{M}_3$, so $\Phi(4, 0, 1, 0, 0, 2, 1, 0) = \Phi(4, 0, 1, 0, 0) \backslash \Phi(2, 1, 0)$. We now apply Step 3. of the algorithm in Definition 2.16 to the two terms on the right-hand side separately:

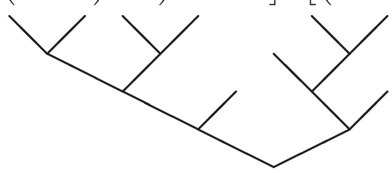
$$\begin{aligned}
\Phi(4,0,1,0,0) &= \Phi(3,0,1,0)/\Upsilon = \Phi(2,0,1)/\Upsilon/\Upsilon \\
&= (\Phi(2,0)\backslash\Phi(1))/\Upsilon/\Upsilon = \Big(\Big(\Phi(1)/\Upsilon\Big)\backslash\Upsilon\Big)/\Upsilon/\Upsilon \\
&= \Big(\Big(\Upsilon/\Upsilon\Big)\backslash\Upsilon\Big)/\Upsilon/\Upsilon ,
\end{aligned}$$

and

$$\Phi(2,1,0) = \Phi(1,1)/\Upsilon = (\Phi(1)\backslash\Phi(1))/\Upsilon = \Big(\Upsilon\backslash\Upsilon\Big)/\Upsilon .$$

In conclusion,

$$\Phi(4,0,1,0,0,2,1,0) = \Big[\Big(\Big(\Upsilon/\Upsilon\Big)\backslash\Upsilon\Big)/\Upsilon/\Upsilon\Big]\backslash\Big[\Big(\Upsilon\backslash\Upsilon\Big)/\Upsilon\Big]$$

$=$ 

The inverse mapping is given by the algorithm described below.

Definition 2.17. For any $k \geq 1$, let $\Psi : Y_k \longrightarrow \mathcal{M}_k$ be the map defined by the following recursive algorithm. For any $t \in Y_k$,

1. if $t = \Upsilon$, then set $\Psi(\Upsilon) := (1)$;
2. if $t = t_1 \backslash t_2$, then set $\Psi(t) := (\Psi(t_1), \Psi(t_2))$;
3. if $t = t_1/\Upsilon$, then set $\Psi(t) := (m_1 + 1, \dots, m_k, 0)$, where $(m_1, \dots, m_k) = \Psi(t_1)$.

It is easy to see that always exactly one of the cases 1., 2., or 3. applies.

3 Co-action and semi-direct co-product of the Hopf algebras

Since the group G^{dif} of formal diffeomorphisms acts by composition on the group G^{inv} of invertible series, one can consider the semi-direct product $G^{\mathrm{dif}} \ltimes G^{\mathrm{inv}}$ of the two groups. If we consider series with non-commutative coefficients, of course the semi-direct product $G^{\mathrm{dif}} \ltimes G^{\mathrm{inv}}(A)$ is still a group, while the semi-direct product $G^{\mathrm{dif}}(A) \ltimes G^{\mathrm{inv}}(A)$ is not anymore a group, because $G^{\mathrm{dif}}(A)$ itself is not a group.

In this section, we show that the dual construction on the co-ordinate rings still makes sense on the non-commutative algebras. It produces a Hopf algebra $\mathbb{C}(G^{\mathrm{dif}}) \ltimes \mathcal{H}^{\mathrm{inv}}$ which is neither commutative nor co-commutative in the case corresponding to the semi-direct product group, and an algebra $\mathcal{H}^{\mathrm{dif}} \ltimes \mathcal{H}^{\mathrm{inv}}$ which is also a co-algebra but not a bi-algebra in the more general case.

3.1 Action and semi-direct product of the groups of series

The composition $f \circ g$ of two invertible (formal power) series is not a formal power series, because the constant term $(f \circ g)_0 = \sum_{n=0}^{\infty} f_n (g_0)^n$ is an infinite sum. However, an invertible series f can be composed with a formal diffeomorphisms φ, and the result $f \circ \varphi$ is again an invertible series. Moreover, we have $(f \circ \varphi) \circ \psi = f \circ (\varphi \circ \psi)$. In other words, the composition $\circ : G^{\mathrm{inv}} \times G^{\mathrm{dif}} \longrightarrow G^{\mathrm{inv}}$ is a natural right action of G^{dif} on G^{inv}. Furthermore, this action commutes with the group structure of G^{inv}, in the sense that

$$(fg) \circ \varphi = (f \circ \varphi)\ (g \circ \varphi). \tag{3.1}$$

In such a situation, we can consider the semi-direct product $G^{\mathrm{dif}} \ltimes G^{\mathrm{inv}}$, which is the group defined on the direct product $G^{\mathrm{dif}} \times G^{\mathrm{inv}}$ by the law

$$(\varphi, f) \cdot_{\ltimes} (\psi, g) := \big(\varphi \circ \psi, (f \circ \varphi) g\big). \tag{3.2}$$

In the dual context, on the co-ordinate rings, we have the co-action $\delta^{\mathrm{dif}} : \mathbb{C}(G^{\mathrm{inv}}) \longrightarrow \mathbb{C}(G^{\mathrm{inv}}) \otimes \mathbb{C}(G^{\mathrm{dif}})$ which satisfies $\langle b_n, f \circ \varphi \rangle = \langle \delta^{\mathrm{dif}} b_n, f \otimes \varphi \rangle$, where the b_n's are the generators of $\mathbb{C}(G^{\mathrm{inv}})$ defined in Section 1.1. As we did in Section 2.1 for the composition of formal diffeomorphisms, we can compactly encode the co-action δ^{dif} on the generators b_n by introducing the generating series

$$B(x) = 1 + \sum_{n=1}^{\infty} b_n x^n = \sum_{n=0}^{\infty} b_n x^n \quad (b_0 := 1), \tag{3.3}$$

so that $\varphi(x) = \langle B(x), \varphi \rangle$. As for the co-product of $\mathbb{C}(G^{\mathrm{dif}})$, the co-action is then given by the formal residue

$$\delta^{\mathrm{dif}} B(x) = \langle z^{-1} \rangle B(z) \otimes \frac{1}{z - A(x)}, \tag{3.4}$$

where $A(x)$ is the generating series of the generators of $\mathbb{C}(G^{\mathrm{dif}})$ as in (2.5). The reader should note that we can also describe the co-product Δ^{inv} dual to the product (1.2) directly on the generating series. It has the simple expression

$$\Delta^{\mathrm{inv}} B(x) = B(x) \otimes B(x). \tag{3.5}$$

The co-action δ^{dif} allows us to construct the co-product on the co-ordinate ring $\mathbb{C}(G^{\mathrm{dif}} \ltimes G^{\mathrm{inv}})$ of the semi-direct product group. As an algebra, we have $\mathbb{C}(G^{\mathrm{dif}} \ltimes G^{\mathrm{inv}}) \cong \mathbb{C}(G^{\mathrm{dif}}) \otimes \mathbb{C}(G^{\mathrm{inv}})$, and the co-product dual to the product (3.2) is

$$\Delta^{\ltimes}(a \otimes b) = \Delta^{\mathrm{dif}}(a)\ \big[(\delta^{\mathrm{dif}} \otimes \mathrm{Id}) \Delta^{\mathrm{inv}}(b)\big], \qquad a \in \mathbb{C}(G^{\mathrm{dif}}),\ b \in \mathbb{C}(G^{\mathrm{inv}}). \tag{3.6}$$

3.2 Dual non-commutative co-action and semi-direct co-product

The previous discussion can be repeated for the group $G^{\mathrm{inv}}(\mathcal{A})$ of invertible series with coefficients in $\mathcal{A}$, and for its dual non-commutative Hopf algebra $\mathcal{H}^{\mathrm{inv}} = \mathbb{C}\langle b_1, b_2, \ldots \rangle$. If we adopt the definition (3.3) of the non-commutative generating series $B(x)$, and if $A(x)$ denotes the non-commutative generating series for the generators of the Hopf algebra $\mathcal{H}^{\mathrm{dif}}$, then we can still use formula (3.4) to define a co-action $\delta^{\mathrm{dif}} : \mathcal{H}^{\mathrm{inv}} \longrightarrow \mathcal{H}^{\mathrm{inv}} \otimes \mathcal{H}^{\mathrm{dif}}$ of $\mathcal{H}^{\mathrm{dif}}$ on $\mathcal{H}^{\mathrm{inv}}$.

Lemma 3.1. *The explicit expression of the co-action δ^{dif} on the generators of $\mathcal{H}^{\mathrm{inv}}$ is:*

$$\delta^{\mathrm{dif}} b_n = \sum_{k=0}^{n} b_k \otimes Q_{n-k}^{(k-1)}(a), \quad n \geq 0,$$

where we use the identification $b_0 = 1$, and where the polynomials $Q_m^{(k)}(a)$ are the polynomials from Definition 2.5.

Proof. We compute the explicit expression for the co-action by applying it to $B(x)$:

$$\begin{aligned} \delta^{\mathrm{dif}} B(x) &= \langle z^{-1} \rangle \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} b_k \otimes A(x)^n z^{k-n-1} = \sum_{n=0}^{\infty} b_n \otimes A(x)^n \\ &= \sum_{n=0}^{\infty} x^n \Big(\sum_{k=0}^{n} b_k \otimes Q_{n-k}^{(k-1)}(a) \Big). \end{aligned} \tag{3.7}$$

This proves the formula in the statement of the lemma. □

The first few values of the co-action on the generators b_i are

$$\begin{aligned} \delta^{\mathrm{dif}} b_1 &= b_1 \otimes 1, \\ \delta^{\mathrm{dif}} b_2 &= b_2 \otimes 1 + b_1 \otimes a_1, \\ \delta^{\mathrm{dif}} b_3 &= b_3 \otimes 1 + 2b_2 \otimes a_1 + b_1 \otimes a_2. \end{aligned}$$

Lemma 3.2. *The map $\delta^{\mathrm{dif}} : \mathcal{H}^{\mathrm{inv}} \longrightarrow \mathcal{H}^{\mathrm{inv}} \otimes \mathcal{H}^{\mathrm{dif}}$ is a co-action with respect to Δ^{dif}, that is*

$$(\delta^{\mathrm{dif}} \otimes \mathrm{Id}) \delta^{\mathrm{dif}} = (\mathrm{Id} \otimes \Delta^{\mathrm{dif}}) \delta^{\mathrm{dif}}.$$

Proof. It suffices to prove this equality for the generators b_n, or, equivalently, it suffices to prove it for the generating series $B(x)$: the left-hand side, by formula (3.4), is

$$\begin{aligned} (\delta^{\mathrm{dif}} \otimes \mathrm{Id}) \delta^{\mathrm{dif}} B(x) &= \langle {z_1}^{-1} \rangle \delta^{\mathrm{dif}} B(z_1) \otimes \frac{1}{z_1 - A(x)} \\ &= \langle {z_1}^{-1} \rangle \langle {z_2}^{-1} \rangle B(z_2) \otimes \frac{1}{z_2 - A(z_1)} \otimes \frac{1}{z_1 - A(x)}, \end{aligned}$$

and the right-hand side, by formulae (3.4) and (2.20), is

$$\begin{aligned} (\mathrm{Id} \otimes \Delta^{\mathrm{dif}}) \delta^{\mathrm{dif}} B(x) &= \langle {z_2}^{-1} \rangle B(z_2) \otimes \Delta^{\mathrm{dif}} \big(\frac{1}{z_2 - A(x)} \big) \\ &= \langle {z_2}^{-1} \rangle \langle {z_1}^{-1} \rangle B(z_2) \otimes \frac{1}{z_2 - A(z_1)} \otimes \frac{1}{z_1 - A(x)}. \end{aligned}$$

Thus, the co-action is co-associative. □

Lemma 3.3. *The Hopf algebra $\mathcal{H}^{\mathrm{inv}}$ is a co-algebra comodule over $\mathcal{H}^{\mathrm{dif}}$ (in the sense of [39]), that is*

$$(\Delta^{\mathrm{inv}} \otimes \mathrm{Id})\ \delta^{\mathrm{dif}} = (\mathrm{Id} \otimes \mathrm{Id} \otimes m)(\mathrm{Id} \otimes \tau \otimes \mathrm{Id})(\delta^{\mathrm{dif}} \otimes \delta^{\mathrm{dif}})\ \Delta^{\mathrm{inv}},$$

where τ is the twist operator $\tau(u \otimes v) = v \otimes u$ and m denotes the multiplication in $\mathcal{H}^{\mathrm{dif}}$.

Proof. We check this identity on the generating series $B(x)$. The co-product Δ^{inv} on the generating series is given by (3.5), and by (1.3) on each b_i, while the co-action is given by (3.7). Thus, we have

$$\begin{aligned}
(\Delta^{\text{inv}} \otimes \text{Id})\ \delta^{\text{dif}} B(x) &= (\Delta^{\text{inv}} \otimes \text{Id})\ \sum_{k=0}^{\infty} b_k \otimes A(x)^k \\
&= \sum_{k=0}^{\infty} \sum_{m+n=k} b_m \otimes b_n \otimes A(x)^k = \sum_{m,n=0}^{\infty} b_m \otimes b_n \otimes A(x)^{m+n} \\
&= (\text{Id} \otimes \text{Id} \otimes m)(\text{Id} \otimes \tau \otimes \text{Id})\big(\sum_{m=0}^{\infty} b_m \otimes A(x)^m \otimes \sum_{n=0}^{\infty} b_n \otimes A(x)^n \big) \\
&= (\text{Id} \otimes \text{Id} \otimes m)(\text{Id} \otimes \tau \otimes \text{Id})(\delta^{\text{dif}} B(x) \otimes \delta^{\text{dif}} B(x)) \\
&= (\text{Id} \otimes \text{Id} \otimes m)(\text{Id} \otimes \tau \otimes \text{Id})(\delta^{\text{dif}} \otimes \delta^{\text{dif}})\Delta^{\text{inv}} B(x).
\end{aligned}$$

□

By Molnar's results [39], in such a situation we can consider the semi-direct or smash co-product $\mathcal{H}^{\text{dif}} \ltimes \mathcal{H}^{\text{inv}}$ of Hopf algebras. This space is at the same time an algebra and a co-algebra, with co-product given by formula (3.6). However, the co-product is not an algebra homomorphism, because $\mathcal{H}^{\text{dif}}$ is not a commutative algebra, and therefore $\mathcal{H}^{\text{dif}} \ltimes \mathcal{H}^{\text{inv}}$ is not a bi-algebra. In order to obtain a Hopf algebra structure, we should consider the semi-direct co-product $\mathbb{C}(G^{\text{dif}}) \ltimes \mathcal{H}^{\text{inv}}$ constructed in the same way. Despite the commutativity of $\mathbb{C}(G^{\text{dif}})$ and the co-commutativity of $\mathcal{H}^{\text{inv}}$, the resulting Hopf algebra is neither commutative nor co-commutative.

4 Relation with the QED renormalization Hopf algebras

In [9] and [7], it was shown that the renormalization of the electron propagator in quantum electrodynamics can be described in terms of a semi-direct co-product Hopf algebra $\mathcal{H}^{\alpha} \ltimes \mathcal{H}^{e}$ on the set of rooted planar binary trees. Here, $\mathcal{H}^{\alpha}$ is a commutative Hopf algebra which represents the renormalization of the electric charge, while $\mathcal{H}^{e}$ is a Hopf algebra which represents the electron propagators, and which is neither commutative nor co-commutative. The renormalization is then a co-action of $\mathcal{H}^{\alpha} \ltimes \mathcal{H}^{e}$ on $\mathcal{H}^{e}$, obtained as a restriction of the co-product.

The Hopf algebra $\mathcal{H}^{\alpha}$ also describes the renormalization of the photon propagators, by means of a co-action of $\mathcal{H}^{\alpha}$ on the non-commutative Hopf algebra $\mathcal{H}^{\gamma}$ which represents the photon propagators. It turns out that the generators of $\mathcal{H}^{\gamma}$ are exactly all the elements of $\mathcal{H}^{\alpha}$, and that the co-action on the generators of $\mathcal{H}^{\gamma}$ coincides with the co-product in $\mathcal{H}^{\alpha}$. Since $\mathcal{H}^{\gamma}$ is a non-commutative algebra, a suitable restriction of this co-action can be interpreted as a co-product on a non-commutative extension $\widetilde{\mathcal{H}^{\alpha}}$ of the commutative Hopf algebra $\mathcal{H}^{\alpha}$.

In this section, we show that $\mathcal{H}^{\text{inv}}$ is a Hopf sub-algebra of both $\mathcal{H}^{e}$ and $\mathcal{H}^{\gamma}$, and that $\mathcal{H}^{\text{dif}}$ is a Hopf sub-algebra of $\widetilde{\mathcal{H}^{\alpha}}$. By construction, it follows also that the co-ordinate ring $\mathbb{C}(G^{\text{dif}})$ is a Hopf sub-algebra of $\mathcal{H}^{\alpha}$, and that the semi-direct co-product $\mathbb{C}(G^{\text{dif}}) \ltimes \mathcal{H}^{\text{inv}}$ of Section 3 is a Hopf sub-algebra of the QED Hopf algebra $\mathcal{H}^{\alpha} \ltimes \mathcal{H}^{e}$.

4.1 $\mathcal{H}^{\text{dif}}$ and the charge renormalization Hopf algebra $\widetilde{\mathcal{H}^{\alpha}}$ on trees

We start by recalling the definition of $\widetilde{\mathcal{H}^{\alpha}}$ from [9]. As in Section 2.4, denote by Y_n the set of rooted planar binary trees with n internal vertices, and by $Y_{\infty} = \bigcup_{n\geq 0} Y_n$ the union of all such trees. Finally, denote by $\widetilde{\mathcal{H}^{\alpha}} := \mathbb{C}Y_{\infty}$ the vector space spanned by all trees. Then $\widetilde{\mathcal{H}^{\alpha}}$ is a non-commutative unital algebra with the *over* product t/s of Section 2.4, and unit given by $|$. In particular, $\widetilde{\mathcal{H}^{\alpha}}$ is a graded and connected algebra, with graded components $(\widetilde{\mathcal{H}^{\alpha}})_n = \mathbb{C}Y_n$.

Moreover, $\widetilde{\mathcal{H}^{\alpha}}$ is a free algebra (i.e., a tensor algebra). To see this, we denote by $\vee : Y_n \times Y_m \longrightarrow Y_{n+m+1}$ the map which grafts two trees on a new root. If, for any $t \in Y_n$, we set $V(t) = | \vee t \in Y_{n+1}$,

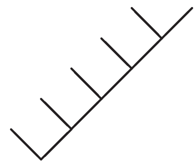


Figure 4.1: The right brush r_5

then $\widetilde{\mathcal{H}^\alpha}$ is isomorphic to the free algebra $\mathbb{C}\langle V(t), t \in Y_\infty\rangle$ generated by the trees $V(t)$. Indeed, any tree t can be decomposed as

$$t = t^l \vee t^r = t^l/V(t^r) = \cdots = V(t^{ll\ldots l})/V(t^{ll\ldots r})/\cdots/V(t^{lr})/V(t^r).$$

As usual, we identify $|$ with the element 1 in $\mathbb{C}\langle V(t), t \in Y_\infty\rangle$.

In [9], it was shown that $\widetilde{\mathcal{H}^\alpha}$ is a connected Hopf algebra which is neither commutative nor co-commutative. We recall briefly the explicit definitions. The co-product $\Delta^\alpha : \widetilde{\mathcal{H}^\alpha} \longrightarrow \widetilde{\mathcal{H}^\alpha} \otimes \widetilde{\mathcal{H}^\alpha}$ is defined recursively by the formulae

$$\begin{aligned}
\Delta^\alpha | &= | \otimes | \,, \\
\Delta^\alpha V(r) &= | \otimes V(r) + \delta^\alpha V(r), \\
\Delta^\alpha (r \vee s) &= \Delta^\alpha r / \Delta^\alpha V(s);
\end{aligned}$$

where $\delta^\alpha : \widetilde{\mathcal{H}^\alpha} \longrightarrow \widetilde{\mathcal{H}^\alpha} \otimes \widetilde{\mathcal{H}^\alpha}$ is the right co-action[4] of $\widetilde{\mathcal{H}^\alpha}$ on itself given by the recursive formulae

$$\begin{aligned}
\delta^\alpha | &= | \otimes | \,, \\
\delta^\alpha V(r) &= (V \otimes \mathrm{Id})\delta^\alpha(r), \\
\delta^\alpha (r \vee s) &= \Delta^\alpha r / \delta^\alpha(V(s)).
\end{aligned}$$

The co-unit $\varepsilon : \widetilde{\mathcal{H}^\alpha} \longrightarrow \mathbb{C}$ is the linear map which sends all the trees to 0, except for the "root tree" $|$ which is sent to 1. The antipode is defined by a standard recursive formula similar to the one in (2.19), since the algebra $\widetilde{\mathcal{H}^\alpha}$ is connected.

In the statement of the theorem below, we need one more notation: we write $|t|$ for the number of internal vertices of the tree t.

Theorem 4.1. *The map $\mathcal{H}^{\mathrm{dif}} \longrightarrow \widetilde{\mathcal{H}^\alpha}$ given by*

$$\Omega : a_n \mapsto t_n := \sum_{|t|=n} t,$$

and extended as a homomorphism of unital algebras, is an injective co-algebra homomorphism. In particular, the Hopf algebra $\mathcal{H}^{\mathrm{dif}}$ is a Hopf sub-algebra of $\widetilde{\mathcal{H}^\alpha}$.

Proof. To prove that Ω is injective, we consider a (non-commutative) polynomial $P(a)$ in the a_n's in the kernel of Ω, that is, we have $P(t) = 0$, where $P(t)$ is obtained from $P(a)$ by replacing a_n by t_n for all n and the product by the over product $/$. Obviously, $P(a)$ has no constant term, because otherwise we would trivially have $P(t) \neq 0$. Let us suppose that $P(a)$ is not identically zero. In that case, there exists a monomial $a_{i_1} a_{i_2} \cdots a_{i_k}$ which appears with non-zero coefficient in $P(a)$. Without loss of generality, we may assume that k is minimal. In particular, since $P(a)$ has no constant term, we have $k \geq 1$.

Le us now consider the image of this monomial under Ω,

$$t_{i_1}/t_{i_2}/\cdots/t_{i_k}. \tag{4.1}$$

For any i, in the expansion of t_i, there appears the "right brush" r_i, which, by definition, is the planar binary tree consisting of i internal nodes, all of which (except for the root) are right descendants of another

[4]A right co-action δ^α of $\widetilde{\mathcal{H}^\alpha}$ on itself satisfies the co-associativity condition $(\delta^\alpha \otimes \mathrm{Id})\delta^\alpha = (\mathrm{Id} \otimes \Delta^\alpha)\delta^\alpha$.

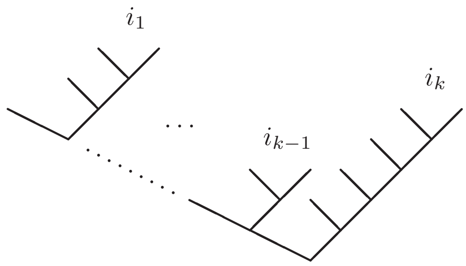


Figure 4.2: The over product $r_{i_1}/r_{i_2}/\cdots/r_{i_k}$

internal node. (See Figure 4.1 for an illustration of the right brush r_5). Hence, in the expansion of the monomial (4.1), there appears the over product $r_{i_1}/r_{i_2}/\cdots/r_{i_k}$ (see Figure 4.2). As is not difficult to see, this tree cannot appear in the expansion of any other monomial $t_{j_1}/t_{j_2}/\cdots/t_{j_l}$ with $l \geq k$. Therefore, it cannot cancel out in the expansion of $P(t)$, a contradiction.

To prove that $\mathcal{H}^{\mathrm{dif}}$ is a sub-co-algebra of $\widetilde{\mathcal{H}^{\alpha}}$, we only need to show that $\Delta^{\alpha} t_n = (\Omega \otimes \Omega)\Delta^{\mathrm{dif}} a_n$. To do this, we actually prove also that $\mathcal{H}^{\mathrm{dif}}$ is a right sub-co-module of $\widetilde{\mathcal{H}^{\alpha}}$, where the right co-action of $\mathcal{H}^{\mathrm{dif}}$ on itself is induced by the natural right co-action $\delta^{\mathrm{dif}} : \mathcal{H}^{\mathrm{inv}} \longrightarrow \mathcal{H}^{\mathrm{inv}} \otimes \mathcal{H}^{\mathrm{dif}}$ defined in Section 3.2, via the co-module homomorphism $\mathcal{H}^{\mathrm{dif}} \longrightarrow \mathcal{H}^{\mathrm{inv}}$, $a_n \mapsto b_n$. To be precise, we are going to show that

$$\Delta^{\alpha}(t_n) = \sum_{k=0}^{n} t_k \otimes Q_{n-k}^{(k)}(t), \tag{4.2}$$

$$\delta^{\alpha}(t_n) = \sum_{k=0}^{n} t_k \otimes Q_{n-k}^{(k-1)}(t), \tag{4.3}$$

where the polynomial $Q_n^{(m)}(t)$ is as in Definition 2.5, only that the a_n's get replaced by the t_n's and the product by the over product $/$. By Lemmas 2.9 and 3.1, this would accomplish the proof that $\mathcal{H}^{\mathrm{dif}}$ is a sub-co-algebra of $\widetilde{\mathcal{H}^{\alpha}}$.

We prove these two claims simultaneously by induction on n. To simplify notation in the following calculations, we shall from now on omit the symbol $/$ in the *over* product x/y of the elements x and y of $\widetilde{\mathcal{H}^{\alpha}}$ and write simply xy.

For $n = 0$ we have

$$\Delta^{\alpha}(\,|\,) = |\otimes|\,,$$
$$\delta^{\alpha}(\,|\,) = |\otimes|\,,$$

while for $n = 1$ we have

$$\Delta^{\alpha}(\curlyvee) = \curlyvee \otimes | + | \otimes \curlyvee\,,$$
$$\delta^{\alpha}(\curlyvee) = \curlyvee \otimes |\,.$$

So, formulae (4.2) and (4.3) hold for $n = 0$ and $n = 1$ because $t_0 = |$ and $t_1 = \curlyvee$. Now we suppose that they hold up to a fixed $n \geq 1$, and we show that they hold for $n + 1$.

For this induction step, we repeatedly need an expansion formula for t_{n+1}. If we decompose $t = r \vee s$ into a left tree r and a right tree s, with $|t| = |r| + |s| + 1$, then we have $t = r/V(s) = rV(s)$, and thus

$$t_{n+1} = \sum_{\substack{0 \leq |r|,|s| \leq n \\ |r|+|s|=n}} rV(s) = \sum_{m=0}^{n} t_{n-m} V(t_m). \tag{4.4}$$

Induction step for (4.2). Using the definitions of Δ^α and δ^α, for any $m \le n$ we have

$$\begin{aligned}\Delta^\alpha V(t_m) &= \sum_{|t|=m} \Delta^\alpha V(t) = \sum_{|t|=m} \mid\ \otimes V(t) + \sum_{|t|=m} \delta^\alpha V(t)\\ &= \sum_{|t|=m} \mid\ \otimes V(t) + \sum_{|t|=m} (V \otimes \mathrm{Id})\delta^\alpha(t)\\ &= \mid\ \otimes V(t_m) + \sum_{k=0}^{m} V(t_k) \otimes Q^{(k-1)}_{m-k}(t). \end{aligned}\tag{4.5}$$

Therefore, using Eq. (4.4), the definition of Δ^α, and Eqs. (4.2) and (4.5), we get

$$\begin{aligned}\Delta^\alpha(t_{n+1}) &= \sum_{m=0}^{n} \Delta^\alpha(t_{n-m})\Delta^\alpha V(t_m)\\ &= \sum_{m=0}^{n}\sum_{k=0}^{n-m} t_k \otimes Q^{(k)}_{n-m-k}(t)V(t_m) + \sum_{m=0}^{n}\sum_{k=0}^{n-m}\sum_{l=0}^{m} t_k V(t_l) \otimes Q^{(k)}_{n-m-k}(t)Q^{(l-1)}_{m-l}(t). \end{aligned}\tag{4.6}$$

We simplify the two terms on the right-hand side separately. In the first term, we use the recurrence for $Q^{(k)}_{n-m-k}(t)$ given in Lemma 2.6, Eq. (4.4), and then again Lemma 2.6, to obtain

$$\begin{aligned}\sum_{m=0}^{n}\sum_{k=0}^{n-m} t_k \otimes Q^{(k)}_{n-m-k}(t)V(t_m) &= \sum_{m=0}^{n}\sum_{k=0}^{n-m} t_k \otimes \sum_{l=0}^{n-m-k} Q^{(k-1)}_{n-m-k-l}(t)t_l V(t_m)\\ &= \sum_{k=0}^{n}\sum_{p=0}^{n-k} t_k \otimes Q^{(k-1)}_{n-k-p}(t)\sum_{l=0}^{p} t_l V(t_{p-l})\\ &= \sum_{k=0}^{n}\sum_{p=0}^{n-k} t_k \otimes Q^{(k-1)}_{n-k-p}(t)t_{p+1}\\ &= \sum_{k=0}^{n} t_k \otimes \sum_{i=0}^{n-k+1} Q^{(k-1)}_{n-k-i+1}(t)t_i - \sum_{k=0}^{n} t_k \otimes Q^{(k-1)}_{n-k+1}(t)\\ &= \sum_{k=0}^{n} t_k \otimes Q^{(k)}_{n-k+1}(t) - \sum_{k=0}^{n} t_k \otimes Q^{(k-1)}_{n-k+1}(t). \end{aligned}\tag{4.7}$$

On the other hand, to the second term in (4.6) we apply the quadratic identity satisfied by the $Q^{(m)}_n(t)$'s proved in Lemma 2.7 and (4.4), to obtain

$$\begin{aligned}\sum_{k=0}^{n}\sum_{l=0}^{n-k} t_k V(t_l) \otimes \left(\sum_{m=l}^{n-k} Q^{(k)}_{n-m-k}(t)Q^{(l-1)}_{m-l}(t)\right) &= \sum_{k=0}^{n}\sum_{l=0}^{n-k} t_k V(t_l) \otimes Q^{(k+l)}_{n-k-l}(t)\\ &= \sum_{p=0}^{n}\sum_{l=0}^{p} t_{p-l} V(t_l) \otimes Q^{(p)}_{n-p}(t)\\ &= \sum_{p=0}^{n} t_{p+1} \otimes Q^{(p)}_{n-p}(t)\\ &= \sum_{p=0}^{n+1} t_p \otimes Q^{(p-1)}_{n-p+1}(t). \end{aligned}\tag{4.8}$$

By summing the two expressions (4.7) and (4.8), we arrive exactly at (4.2) with n replaced by $n+1$, because $Q^{(n)}_0(t) = Q^{(n+1)}_0(t)$.

Induction step for (4.3). Using the definition of δ^α, for any $m \leq n$, we have

$$\begin{aligned} \delta^\alpha V(t_m) &= \sum_{|t|=m} \delta^\alpha V(t) = \sum_{|t|=m} (V \otimes \mathrm{Id})\delta^\alpha(t) \\ &= \sum_{k=0}^{m} V(t_k) \otimes Q_{m-k}^{(k-1)}(t). \end{aligned} \tag{4.9}$$

Therefore, using Eq. (4.4), the definition of δ^α, Eqs. (4.2) and (4.9), we get

$$\begin{aligned} \delta^\alpha(t_{n+1}) &= \sum_{m=0}^{n} \Delta^\alpha(t_{n-m})\delta^\alpha V(t_m) \\ &= \sum_{m=0}^{n} \sum_{k=0}^{n-m} \sum_{l=0}^{m} t_k V(t_l) \otimes Q_{n-m-k}^{(k)}(t) Q_{m-l}^{(l-1)}(t) \\ &= \sum_{k=0}^{n} \sum_{l=0}^{n-k} t_k V(t_l) \otimes \left(\sum_{m=l}^{n-k} Q_{n-m-k}^{(k)}(t) Q_{m-l}^{(l-1)}(t) \right). \end{aligned}$$

We apply again the quadratic relation of Lemma 2.7 to the sum over m. This leads to

$$\begin{aligned} \delta^\alpha(t_{n+1}) &= \sum_{k=0}^{n} \sum_{l=0}^{n-k} t_k V(t_l) \otimes Q_{n-k-l}^{(k+l)}(t) \\ &= \sum_{p=0}^{n} \left(\sum_{l=0}^{p} t_{p-l} V(t_l) \right) \otimes Q_{n-p}^{(p)}(t). \end{aligned}$$

Finally, another application of the recurrence (4.4) yields

$$\delta^\alpha(t_{n+1}) = \sum_{p=0}^{n} t_{p+1} \otimes Q_{n-p}^{(p)}(t),$$

which, after replacing p by $k-1$, agrees with (4.3) with n replaced by $n+1$. $\square$

4.2 $\mathcal{H}^{\mathrm{inv}}$ and the propagator Hopf algebras $\mathcal{H}^e$ and $\mathcal{H}^\gamma$ on trees

Denote by $\mathcal{H}^e$ and $\mathcal{H}^\gamma$ the free associative algebras $\mathbb{C}\langle Y_\infty \rangle / (\,|\, - 1)$ on the set of all trees, where we identify the "root tree" $|$ with the unit. As we mentioned in Section 2.4, the *under* and the *over* products on trees are associative and have $|$ as a unit. Therefore their dual co-operations are co-unital and co-associative, and their multiplicative extensions on $\mathcal{H}^e$ and $\mathcal{H}^\gamma$, denoted respectively by Δ_e^p and Δ_γ^p, define two structures of a Hopf algebra on $\mathcal{H}^e$, respectively on $\mathcal{H}^\gamma$, which are neither commutative nor co-commutative.

The respective co-products can be defined on the generators $t = r \vee s$ in a recursive manner, by putting

$$\Delta_e^p(r \vee s) = 1 \otimes (r \vee s) + \sum (r \vee s_{(1)}) \otimes s_{(2)}, \quad \text{where } \sum s_{(1)} \otimes s_{(2)} = \Delta_e^p(s), \tag{4.10}$$

$$\Delta_\gamma^p(r \vee s) = (r \vee s) \otimes 1 + \sum r_{(1)} \otimes (r_{(2)} \vee s), \quad \text{where } \sum r_{(1)} \otimes r_{(2)} = \Delta_\gamma^p(r). \tag{4.11}$$

Here we use Sweedler's notation $\sum t_{(1)} \otimes t_{(2)}$ for both $\Delta_e^p(t)$ and $\Delta_\gamma^p(t)$ (cf. e.g. [1, p. 56]).

Proposition 4.2. *The map $\mathcal{H}^{\mathrm{inv}} \longrightarrow \mathcal{H}^e$ given by*

$$b_n \mapsto t_n := \sum_{|t|=n} t,$$

and extended as a homomorphism of unital algebras, is an injective co-algebra homomorphism. In particular, the Hopf algebra $\mathcal{H}^{\mathrm{inv}}$ is a Hopf sub-algebra of $\mathcal{H}^e$. The same is true if $\mathcal{H}^e$ is replaced by $\mathcal{H}^\gamma$ in these statements.

Proof. The reason why the map is injective is given in the proof of Theorem 4.1. To prove that $\mathcal{H}^{\mathrm{inv}}$ is a sub-co-algebra of $\mathcal{H}^e$, and of $\mathcal{H}^\gamma$ respectively, we need to show that $\Delta^p_e(t_n) = \Delta^p_\gamma(t_n) = \Delta^{\mathrm{inv}}(b_n)$, that is

$$\Delta^p_e(t_n) = \Delta^p_\gamma(t_n) = \sum_{k=0}^{n} t_k \otimes t_{n-k}. \tag{4.12}$$

We prove these identities by induction on n. Since the proof is the same for the two co-products Δ^p_e and Δ^p_γ, we only write it down for Δ^p_e. For $n = 0$ we have $\Delta^p_e(\,|\,) = \,|\, \otimes \,|\,$, and for $n = 1$ we have $\Delta^p_e(\,\mathsf{Y}\,) = \mathsf{Y} \otimes |\, + \,| \otimes \mathsf{Y}$. So, formula (4.12) holds for $n = 0$ and for $n = 1$ because $t_0 = \,|\,$ and $t_1 = \mathsf{Y}$. Now we suppose that it holds up to a fixed $n \geq 1$, and we show that it holds for $n + 1$.

Using the decomposition $t = r \vee s$ of a tree into its left and right components, we have

$$t_{n+1} = \sum_{m=0}^{n} \sum_{\substack{|s|=m \\ |r|=n-m}} r \vee s.$$

Moreover, by the induction hypothesis, we know that

$$\sum_{|s|=m} s_{(1)} \otimes s_{(2)} = \sum_{|s|=m} \Delta^p_e(s) = \Delta^p_e(t_m) = \sum_{k=0}^{m} t_{m-k} \otimes t_m.$$

Using these two identities, we obtain

$$\begin{aligned}
\Delta^p_e(t_{n+1}) &= \sum_{m=0}^{n} \sum_{\substack{|s|=m \\ |r|=n-m}} \Delta^p_e(r \vee s) \\
&= \,| \otimes \sum_{m=0}^{n} \sum_{\substack{|s|=m \\ |r|=n-m}} r \vee s + \sum_{m=0}^{n} \sum_{\substack{|s|=m \\ |r|=n-m}} (r \vee s_{(1)}) \otimes s_{(2)} \\
&= \,| \otimes t_{n+1} + \sum_{m=0}^{n} \sum_{k=0}^{m} (t_{n-m} \vee t_{m-k}) \otimes t_k \\
&= \sum_{k=0}^{n+1} t_{n-k+1} \otimes t_k.
\end{aligned}$$

□

In [9], it was shown that the co-action δ^α of $\mathcal{H}^\alpha$ on itself can be extended to two co-actions δ^e and δ^γ of $\mathcal{H}^\alpha$ on $\mathcal{H}^e$ and $\mathcal{H}^\gamma$, respectively. These allow one to define the semi-direct Hopf algebra $\mathcal{H}^\alpha \ltimes \mathcal{H}^e$ (also called smash Hopf algebra; cf. [38, 39]), which represents the renormalization Hopf algebras for quantum electrodynamics.

Corollary 4.3. *The semi-direct Hopf algebra $\mathbb{C}(G^{\mathrm{dif}}) \ltimes \mathcal{H}^{\mathrm{inv}}$ is a Hopf sub-algebra of the QED renormalization Hopf algebra $\mathcal{H}^\alpha \ltimes \mathcal{H}^e$.*

Proof. This follows from the fact that the maps $a_n \mapsto t_n$ and $b_n \mapsto t_n$ induce an injective algebra and co-algebra homomorphism from $\mathcal{H}^{\mathrm{dif}} \ltimes \mathcal{H}^{\mathrm{inv}}$ to $\widetilde{\mathcal{H}^\alpha} \ltimes \mathcal{H}^e$. To prove this, we use Theorem 4.1, Proposition 4.2, and we extend the co-action δ^{dif}, which corresponds to δ^α thanks to Eq. 4.3, to all suitable spaces. □

5 Relation with the renormalization functor

In [11] and [49], it was shown that renormalization in quantum field theory can be considered as a functor of bi-algebras. More precisely, if $\mathcal{B}$ is a bi-algebra and $T(\mathcal{B})^+ = \bigoplus_{n\geq 1} T^n(\mathcal{B})$, then the tensor algebra $T(T(\mathcal{B})^+)$ can be equipped with the structure of a bi-algebra which corresponds to the Epstein-Glaser renormalization of quantum field theories in the configuration space, cf. [14]. It was also shown that the renormalization of scalar quantum fields is ruled by the commutative version of this bi-algebra, realized on the double symmetric space $S(S(\mathcal{B})^+)$.

In this section, we show that $\mathcal{H}^{\mathrm{dif}}$ is isomorphic to the Hopf algebra obtained from $T(T(\mathcal{B})^+)$ when $\mathcal{B}$ is the trivial bi-algebra, by taking a certain quotient. In the first subsection, we recall the definition of the bi-algebra $T(T(\mathcal{B})^+)$ for a generic bi-algebra $\mathcal{B}$, and of its associated Hopf algebra. Then, in the second subsection, we define a bi-algebra $\mathcal{B}^{\mathrm{dif}}$, from which one can obtain the Hopf algebra $\mathcal{H}^{\mathrm{dif}}$ as a quotient. We finally prove the above isomorphism claim in the third subsection.

5.1 The bi-algebra $T(T(\mathcal{B})^+)$

We now introduce a bi-algebra structure on $T(T(\mathcal{B})^+)$ which occurred for the first time in [11]. However, the reader should be aware that our presentation here is the opposite of the one in [11]. Since we shall have to deal with two distinguished tensor products and two co-products, we fix some specific notation to avoid confusions.

Let $\mathcal{B}$ be a bi-algebra, and let x and y be elements of $\mathcal{B}$. We denote the product in $\mathcal{B}$ by $x \cdot y$, and we let $\Delta^{\mathcal{B}}(x) = \sum x_{(1)} \otimes x_{(2)}$ denote the co-product in $\mathcal{B}$, again using Sweedler's notation. In $T(\mathcal{B})^+$, we replace the tensor symbol by a comma, that is, given $a = x^1 \otimes \cdots \otimes x^n \in T^n(\mathcal{B}) \subseteq T(\mathcal{B})^+$, we write $a = (x^1, \ldots, x^n)$ instead. Furthermore, in $T(T(\mathcal{B})^+)$ we omit tensor symbols, that is, given an element $u = a^1 \otimes \cdots \otimes a^n \in T^n(T(\mathcal{B})^+) \subseteq T(T(\mathcal{B})^+)$, with $a^i \in T(\mathcal{B})^+$, we write $u = a^1 \ldots a^n$ instead.

On $T(T(\mathcal{B})^+)$ we consider the algebra structure given by the tensor product (whose symbol is omitted), and we define a co-product Δ recursively, starting from the co-product $\Delta^{\mathcal{B}}$ on $\mathcal{B}$, as follows.

First, for any $x \in \mathcal{B}$, we introduce three linear operators on $T(T(\mathcal{B})^+)$, A_x, B_x, and C_x, which correspond to the product by x in $\mathcal{B}$, $T(\mathcal{B})$, and $T(T(\mathcal{B})^+)$, respectively. More precisely, if $a = (x_1, x_2, \ldots, x_n) \in T^n(\mathcal{B}) \subset T^1(T(\mathcal{B})^+)$, we set

$$
\begin{aligned}
A_x(a) &= (x \cdot x_1, x_2, \ldots, x_n) \in T^n(\mathcal{B}) \subset T^1(T(\mathcal{B})^+), \\
B_x(a) &= (x, x_1, x_2, \ldots, x_n) \in T^{n+1}(\mathcal{B}) \subset T^1(T(\mathcal{B})^+), \\
C_x(a) &= (x)(x_1, x_2, \ldots, x_n) \in T^1(\mathcal{B}) \otimes T^n(\mathcal{B}) \subset T^2(T(\mathcal{B})^+),
\end{aligned}
$$

and if $u = a_1 a_2 \ldots a_n \in T^n(T(\mathcal{B})^+)$, the operators A_x, B_x and C_x act only on a_1:

$$
\begin{aligned}
A_x(u) &= A_x(a_1) a_2 \ldots a_n \in T^n(T(\mathcal{B})^+), \\
B_x(u) &= B_x(a_1) a_2 \ldots a_n \in T^n(T(\mathcal{B})^+), \\
C_x(u) &= C_x(a_1) a_2 \ldots a_n \in T^{n+1}(T(\mathcal{B})^+).
\end{aligned}
$$

In particular, any element $a = (x_1, x_2, \ldots, x_n)$ of $T(\mathcal{B})^+$ can be written as $a = B_{x_1}(a')$, with $a' = (x_2, \ldots, x_n)$. Now, the co-product in $T(T(\mathcal{B})^+)$ is defined recursively on the generators as follows:

$$
\Delta\big((x)\big) = \sum (x_{(1)}) \otimes (x_{(2)}), \tag{5.1}
$$

$$
\Delta(B_x(a)) = \sum (A_{x_{(1)}} \otimes B_{x_{(2)}} + B_{x_{(1)}} \otimes C_{x_{(2)}}) \Delta a, \tag{5.2}
$$

where $\sum x_{(1)} \otimes x_{(2)} = \Delta^{\mathcal{B}}(x)$. For example,

$$
\begin{aligned}
\Delta\big((x)\big) &= \sum (x_{(1)}) \otimes (x_{(2)}), \\
\Delta\big((x, y)\big) &= \sum (x_{(1)}, y_{(1)}) \otimes (x_{(2)})(y_{(2)}) + \sum (x_{(1)} \cdot y_{(1)}) \otimes (x_{(2)}, y_{(2)}), \\
\Delta\big((x, y, z)\big) &= \sum (x_{(1)}, y_{(1)}, z_{(1)}) \otimes (x_{(2)})(y_{(2)})(z_{(2)}) + \sum (x_{(1)}, y_{(1)} \cdot z_{(1)})) \otimes (x_{(2)})(y_{(2)}, z_{(2)}) \\
&\quad + \sum (x_{(1)} \cdot y_{(1)}, z_{(1)}) \otimes (x_{(2)}, y_{(2)})(z_{(2)}) + \sum (x_{(1)} \cdot y_{(1)} \cdot z_{(1)}) \otimes (x_{(2)}, y_{(2)}, z_{(2)}).
\end{aligned}
$$

The co-unit ε of $T(T(\mathcal{B})^+)$ is the algebra homomorphism $T(T(\mathcal{B})^+) \to \mathbb{C}$ whose restriction to $T(\mathcal{B})^+$ is given by $\varepsilon((x)) = \varepsilon_{\mathcal{B}}(x)$, for $x \in \mathcal{B}$, and $\varepsilon((x_1, \ldots, x_n)) = 0$ for $n \geq 1$.

In [11], it was proved that $T(T(\mathcal{B})^+)$ is a bi-algebra, and that one obtains a Hopf algebra structure on the quotient $T(T(\mathcal{B})^+)/\langle (x - \varepsilon_{\mathcal{B}}(x)1) \rangle$ by the bi-ideal generated by $(x - \varepsilon_{\mathcal{B}}(x)1)$.

5.2 The bi-algebra $\mathcal{B}^{\mathrm{dif}}$

Let $\mathcal{B}^{\mathrm{dif}} = \langle a_0, a_1, a_2, \ldots \rangle$ denote the free associative algebra on the variables $a_0, a_1, a_2, \ldots$. These are the same variables as those which generate $\mathcal{H}^{\mathrm{dif}}$, except that there is an extra variable a_0 which is different from 1. Then $\mathcal{B}^{\mathrm{dif}}$ is an associative unital algebra, and the formula of Lemma 2.9, in which the $Q_m^{(k)}(a)$ are the polynomials from Definition 2.5 but without the identification of a_0 with 1, defines a co-associative co-product Δ^{dif} on the generators of $\mathcal{B}^{\mathrm{dif}}$. The first few values of the co-product in $\mathcal{B}^{\mathrm{dif}}$ are:

$$\begin{aligned}
\Delta a_0 &= a_0 \otimes a_0, \\
\Delta a_1 &= a_0 \otimes a_1 + a_1 \otimes a_0^2, \\
\Delta a_2 &= a_0 \otimes a_2 + a_1 \otimes (a_0 a_1 + a_1 a_0) + a_2 \otimes a_0^3, \\
\Delta a_3 &= a_0 \otimes a_3 + a_1 \otimes (a_0 a_2 + a_2 a_0 + a_1^2) + a_2 \otimes (a_0^2 a_1 + a_0 a_1 a_0 + a_1 a_0^2) + a_3 \otimes a_0^4.
\end{aligned}$$

Furthermore, we can define a co-unit as the linear map $\varepsilon : \mathcal{B}^{\mathrm{dif}} \longrightarrow \mathbb{C}$ given by $\varepsilon(1) = 1$, $\varepsilon(a_0) = 1$ and $\varepsilon(a_n) = 0$ for $n \geq 1$. Thus, $\mathcal{B}^{\mathrm{dif}}$ becomes an associative bi-algebra. However, $\mathcal{B}^{\mathrm{dif}}$ is *not* a Hopf algebra, because the antipode cannot be defined on a_0.

In order to obtain a Hopf algebra, we have to consider the quotient $\mathcal{B}^{\mathrm{dif}}/\langle a_0 - \varepsilon(a_0)1 \rangle$ of the bi-algebra $\mathcal{B}^{\mathrm{dif}}$ by the bi-ideal which identifies a_0 with the unit 1. What we obtain is precisely the Hopf algebra $\mathcal{H}^{\mathrm{dif}}$ of formal diffeomorphisms.

5.3 Recursive definition of the co-product Δ^{dif}

We are now ready to prove that the Hopf algebra $\mathcal{H}^{\mathrm{dif}}$ of formal diffeomorphism can be also obtained as the quotient $T(T(\mathcal{B})^+)/\langle (x - \varepsilon_{\mathcal{B}}(x)1) \rangle$ when $\mathcal{B}$ is the trivial bi-algebra.

Theorem 5.1. *Let $\mathcal{B} = \mathbb{C}1$ be the trivial bi-algebra with $1 \cdot 1 = 1$ and $\Delta^{\mathcal{B}}(1) = 1 \otimes 1$. Denote by $1^{\otimes n} = (1, \ldots, 1)$ the only element (up to a scalar) in $T^n(\mathcal{B})$. Then the algebra homomorphism*

$$\varphi : T(T(\mathbb{C}1)^+)/\langle (1) - \varepsilon(1)1 \rangle \longrightarrow \mathcal{H}^{\mathrm{dif}}, \quad \varphi(1^{\otimes n}) = a_{n-1},$$

is a Hopf algebra isomorphism.

Proof. Of course, the relation between $\mathcal{B}^{\mathrm{dif}}$ and $\mathcal{H}^{\mathrm{dif}}$ is the same as the relation between $T(T(\mathbb{C}1)^+)$ and $T(T(\mathbb{C}1)^+)/\langle (1) - \varepsilon(1)1 \rangle$, and the map φ can be lifted up to a map $\varphi : T(T(\mathbb{C}1)^+) \longrightarrow \mathcal{B}^{\mathrm{dif}}$. Therefore, it is enough to show that φ is an isomorphism of associative bi-algebras from $T(T(\mathbb{C})^+)$ to $\mathcal{B}^{\mathrm{dif}}$.

The map φ is clearly a bijection, with inverse map $\varphi^{-1}(a_n) = 1^{\otimes(n+1)}$. It is an algebra homomorphism by definition, so it only remains to prove that it is a co-algebra homomorphism, that is $\Delta^{\mathrm{dif}}(a_n) = (\varphi \otimes \varphi)\Delta(1^{\otimes(n+1)})$. Since Δ on $T(T(\mathbb{C}1)^+)$ is defined by the recurrence relations (5.1) and (5.2), with $x = 1$ and $\Delta^{\mathcal{B}}(1) = 1 \otimes 1$, we only need to show that Δ^{dif} satisfies the analogous recurrence relations. To do this, denote by A^φ, B^φ and C^φ the operators acting on $\mathcal{B}^{\mathrm{dif}}$ which arise canonically from the operators A_1, B_1 and C_1 on $T(T(\mathbb{C}1)^+)$ by using the map $\varphi : T(T(\mathbb{C}1)^+) \longrightarrow \mathcal{B}^{\mathrm{dif}}$. Explicitly, for any $u \in \mathcal{B}^{\mathrm{dif}}$, we have:

$$A^\varphi(u) = u, \quad B^\varphi(a_n u) = a_{n+1} u, \quad C^\varphi(u) = a_0 u. \tag{5.3}$$

We claim that the recurrence relation (5.2) in $\mathcal{B}^{\mathrm{dif}}$ becomes

$$\Delta^{\mathrm{dif}}(a_{n+1}) = (A^\varphi \otimes B^\varphi + B^\varphi \otimes C^\varphi)\Delta^{\mathrm{dif}}(a_n). \tag{5.4}$$

To see this, we apply B^φ to the identity $Q^{(k)}_{m-1}(a) = \sum_{l=0}^{m-1} a_l Q^{(k-1)}_{m-1-l}(a)$ of Lemma 2.6. This gives

$$\begin{aligned}
B^\varphi\big(Q^{(k)}_{m-1}\big) &= \sum_{i=0}^{m-1} B^\varphi\big(a_i Q^{(k-1)}_{m-1-i}\big) \\
&= \sum_{i=0}^{m-1} a_{i+1} Q^{(k-1)}_{m-1-i} = \sum_{i=1}^{m} a_i Q^{(k-1)}_{m-i} \\
&= Q^{(k)}_m - a_0 Q^{(k-1)}_m .
\end{aligned}$$

Using this identity for $m-1 = n-k$, the value of the operators A^φ, B^φ and C^φ computed in Eq. (5.3), and the Definition 2.5 of the polynomials $Q^{(k)}_m$, we obtain for the right-hand side of (5.4):

$$\begin{aligned}
(A^\varphi \otimes B^\varphi + B^\varphi \otimes C^\varphi)\Delta^{\mathrm{dif}}(a_n) &= \sum_{k=0}^{n} a_k \otimes B^\varphi(Q^{(k)}_{n-k}) + \sum_{k=0}^{n} a_{k+1} \otimes a_0 Q^{(k-1)}_{n-k} \\
&= \sum_{k=0}^{n} a_k \otimes Q^{(k)}_{n-k+1} - \sum_{k=0}^{n} a_k \otimes a_0 Q^{(k-1)}_{n-k+1} + \sum_{k=1}^{n+1} a_k \otimes a_0 Q^{(k-1)}_{n-k+1} \\
&= \sum_{k=0}^{n} a_k \otimes Q^{(k)}_{n-k+1} - a_0 \otimes a_0 Q^{(-1)}_{n+1} + a_{n+1} \otimes a_0 Q^{(n)}_0 \\
&= \sum_{k=0}^{n+1} a_k \otimes Q^{(k)}_{n+1-k} = \Delta^{\mathrm{dif}}(a_{n+1}).
\end{aligned}$$

This proves our claim. □

Note that formulae (5.3) and (5.4) provide a recursive definition for the co-product Δ^{dif} on $\mathcal{B}^{\mathrm{dif}}$.

6 Formal diffeomorphisms in several variables

In this section, we show how to extend the construction of the non-commutative Hopf algebra of formal diffeomorphisms to series with several variables. We content ourselves with considering the case of two variables only, as it serves well as an illustration, and as there is no substantial difference to the more general several variables case, except that notation becomes considerably more cumbersome.

Let G^{dif}_2 denote the set of pairs (α, β) of series

$$\begin{aligned}
\alpha(x,y) &= x + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \alpha_{nk}\ x^{n+1} y^k, \\
\beta(x,y) &= y + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \beta_{nk}\ x^{n} y^{k+1},
\end{aligned}$$

in two variables x and y, with complex coefficients α_{nk} and β_{nk}. Consider the composition $(\alpha,\beta)\circ(\mu,\nu) := (\alpha(\mu,\nu), \beta(\mu,\nu))$ of such pairs, which is defined as the pair of series

$$\begin{aligned}
\alpha(\mu,\nu) &= \mu(x,y) + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \alpha_{nk}\ \mu(x,y)^{n+1} \nu(x,y)^k, \\
\beta(\mu,\nu) &= \nu(x,y) + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \beta_{nk}\ \mu(x,y)^{n} \nu(x,y)^{k+1}.
\end{aligned}$$

The set G^{dif}_2 is a group. Its unit is given by the pair (x,y), while the inverse pair of a given pair can be found with the help the well-known Lagrange–Good formula (see e.g. [21]) in dimension 2.

Therefore, the co-ordinate ring $\mathbb{C}(G^{\mathrm{dif}}_2)$ is a commutative Hopf algebra, isomorphic to the polynomial ring $\mathbb{C}[a_1, a_2, \ldots; b_1, b_2, \ldots]$ in two infinite series of variables labelled by natural numbers.

To find a non-commutative version of this Hopf algebra, we proceed as for G^{dif} in Section 2.1. We consider the non-commutative algebra $\mathbb{C}\langle a_1, a_2, \ldots ; b_1, b_2, \ldots\rangle$ in two infinite series of variables, and we introduce the generating series

$$A(x,y) = x + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} a_{nk}\ x^{n+1}y^k,$$
$$B(x,y) = y + \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} b_{nk}\ x^n y^{k+1}.$$

The co-product is then defined through the double residues

$$\Delta A(x,y) = \langle {z_1}^{-1}\rangle\langle {z_2}^{-1}\rangle \frac{1}{(z_1 - A(x,y))}\frac{1}{(z_2 - B(x,y))} \otimes A(z_1,z_2),$$
$$\Delta B(x,y) = \langle {z_1}^{-1}\rangle\langle {z_2}^{-1}\rangle \frac{1}{(z_1 - A(x,y))}\frac{1}{(z_2 - B(x,y))} \otimes B(z_1,z_2).$$

Following the lines of Section 2.2, one can show that this co-product is co-associative, and, together with the standard co-unit, it gives $\mathbb{C}\langle a_1, a_2, \ldots ; b_1, b_2, \ldots\rangle$ the structure of a Hopf algebra which is neither commutative nor co-commutative.

The explicit form of the action of the co-product on the generators a_n and b_n can be found using the same arguments as those given in Section 2. We leave the details to the reader.

References

[1] E. Abe, *Hopf algebras*, Cambridge University Press, Cambridge, 1980.

[2] E. Bagan, M. Lavelle and D. McMullan, *Charges from dressed matter: Construction*, Ann. Phys. **282** (2000), 471–502.

[3] E. Bagan, M. Lavelle and D. McMullan, *Charges from dressed matter: Physics and renormalisation*, Ann. Phys. **282** (2000), 503–540.

[4] G. M. Bergman and A. O. Hausknecht, *Cogroups and co-rings in categories of associative algebras*, Math. Surveys and Monographs, vol. 45, Amer. Math. Soc., Providence, Rhode Island, 1996.

[5] J. Bordes, H. M. Chan and S. T. Tsou, *Circumstantial evidence for rotating mass matrix from fermion mass and mixing data*, Eur. Phys. J. C **27** (2003), 189–200.

[6] Ch. Brouder, *On the trees of quantum fields*, Eur. Phys. J. C **12** (2000), 535–549. `http://www.arxiv.org/abs/hep-th/9906111`

[7] Ch. Brouder and A. Frabetti, *Renormalization of QED with planar binary trees*, Eur. Phys. J. C **19** (2001), 715–741. `http://www.arxiv.org/abs/hep-th/0003202`.

[8] Ch. Brouder and A. Frabetti, *Noncommutative renormalization for massless QED*, preprint (2000); `http://www.arxiv.org/abs/hep-th/0011161`.

[9] Ch. Brouder and A. Frabetti, *QED Hopf algebras on planar binary trees*, J. Alg. **267** (2003), 298–322.

[10] Ch. Brouder and A. Frabetti, *Hopf algebra of renormalization for massive QED*, in preparation.

[11] Ch. Brouder and W. Schmitt, *Quantum groups and quantum field theory: III. Renormalisation*, preprint (2002); `http://www.arxiv.org/abs/hep-th/0210097`.

[12] A. Connes and D. Kreimer, *Renormalization in quantum field theory and the Riemann–Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem.* Comm. Math. Phys. **210** (2000), 249–273; `http://www.arxiv.org/abs/hep-th/9912092`.

[13] G. P. Egorychev, *Integral representation and the computation of combinatorial sums*, "Nauka" Sibirsk. Otdel. 1977, Novosibirsk; translated from the Russian: Translations of Mathematical Monographs, 59, Amer. Math. Soc., Providence, R. I., 1984.

[14] H. Epstein and V. Glaser, *The role of locality in perturbation theory*, Ann. Inst. Henri Poincaré **19** (1973), 211–295.

[15] F. Faà di Bruno, *Sullo sviluppo delle funzioni*, Ann. Sci. Mat. Fis., Roma **6** (1855), 479–480.

[16] A. Frabetti, *Groups of tree-expanded formal series*, in preparation.

[17] B. Fresse, *Cogroups in algebras over an operad are free algebras*, Comment. Math. Helv. **73** (1998), 637–676.

[18] F. Gavarini, *Poisson geometrical symmetries associated to non-commutative formal diffeomorphisms*, Comm. Math. Phys., to appear; `http://www.arxiv.org/abs/math.QA/0309163`.

[19] F. Gavarini, *The global quantum duality principle: theory, examples, and applications*, preprint (2003), `http://www.arxiv.org/abs/math.QA/0309163`.

[20] I. M. Gessel, *A noncommutative generalization and a q-analog of the Lagrange inversion formula*, Trans. Amer. Soc. **257** (1980), 455–482.

[21] I. M. Gessel, *A combinatorial proof of the multivariable Lagrange inversion formula*, J. Combin. Theory Ser. A **45** (1987), 178–196.

[22] R. L. Graham and D. E. Knuth and O. Patashnik, *Concrete Mathematics*, Eighth Printing. Addison–Wesley, Reading, Massachusetts, 1989.

[23] P. Henrici, *Die Lagrange–Bürmannsche Formel bei formalen Potenzreihen*, Jahresber. Deutsch. Math.-Verein. **86** (1984), 115–134.

[24] E. Hewitt and K. A. Ross, *Abstract harmonic analysis II*, Springer–Verlag, Berlin, 1970.

[25] G. Hochschild, *The structure of Lie groups*, Holden-Day, San Francisco, 1965.

[26] R. Holtkamp, *Comparison of Hopf algebras on trees*, Arch. Math. **80** (2003), 368–383.

[27] S. A. Joni and G. Rota, *Coalgebras and bialgebras in combinatorics*, Stud. Appl. Math. **61** (1979), 93–139.

[28] T. Kato, *Perturbation Theory of Linear Operators*, Springer–Verlag, Berlin, 1980.

[29] M. G. Krein, *A principle of duality for bicompact groups and quadratic block algebras* (in Russian), Doklady Akad. Nauk SSSR (N.S.) **69** (1949), 725–728.

[30] L. de Lagrange, *Nouvelle méthode pour résoudre des équations littérales par le moyen des séries*, Mém. Acad. Roy. Sci. Belles–Lettres de Berlin **24** (1770).

[31] S. Lang, *Algebra*, Addison–Wesley, New York, 1984.

[32] J.–L. Loday, *Künneth-style formula for the homology of Leibniz algebras*, Math. Z. **221** (1996), 41–47.

[33] J.–L. Loday, *Arithmetree*, J. Alg. **258** (2002), 275–309.

[34] J.–L. Loday, *Dialgebras and related operads*, Lect. Notes Math. **1763**, 7–66, Springer, Berlin, 2001; `http://www.arxiv.org/abs/math.QA/0102053`.

[35] J.–L. Loday and M. Ronco, *Algèbres de Hopf colibres*, C. R. Acad. Sci Paris **337** (2003), 153–158.

[36] J.–L. Loday and M. Ronco, *On the structure of cofree Hopf algebras*, preprint (2004), `http://www.arxiv.org/abs/math.QA/0405330`.

[37] J.–L. Loday and M. Ronco, *Hopf algebra of the Planr Binary Trees*, Adv. Math. **139** (1998), 293–309.

[38] S. Majid, *Foundations of Quantum Group Theory*, Cambridge University Press, Cambridge, 1995.

[39] R. K. Molnar, *Semi-direct products of Hopf algebras*, J. Alg. **47** (1977), 29–51.

[40] M. Paschke and R. Verch, *Local covariant quantum field theory over spectral geometries*, preprint (2004), `http://www.arxiv.org/abs/gr-qc/0405057`.

[41] G. Pinter, *The Hopf algebra structure of Connes and Kreimer in Epstein–Glaser renormalization*, Lett. Math. Phys. **54** (2000), 227–233.

[42] G. Pinter, *Finite renormalization in the Epstein–Glaser framework and renormalization of the S-matrix of* Φ^4*-theory*, Ann. Phys. **10** (2001), 333–363.

[43] M. Planck, *Eight Lectures on Theoretical Physics*, Dover, New York, 1998.

[44] M. Ronco, *A Milnor–Moore theorem for dendriform algebras*, C. R. Acad. Sci. Paris **332** (2001), 109–114.

[45] M. Ronco, *Eulerian idempotents and Milnor–Moore theorem for certain non-commutative Hopf algebras*, J. Alg. **254** (2002), 152–172.

[46] W. R. Schmitt, *Incidence Hopf algebras*, J. Pure Appl. Alg. **96** (1994), 299–330.

[47] R. P. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, Cambridge, 1999.

[48] T. Tannaka, *Über den Dualitätssatz der nichtkommutativen topologischen Gruppen*, Tohoku math. **45** (1939), 1–12.

[49] P. van der Laan, *Operads and the Hopf algebras of renormalization*, preprint (2003), `http://www.arxiv.org/abs/math-ph/0311013`.

[50] D. V. Voiculescu, K. J. Dykema, A. Nica, *Free Random Variables*, CRM Monograph Series, AMS 1992.