Hopf algebras and renormalization in physics

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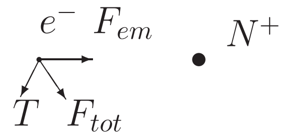
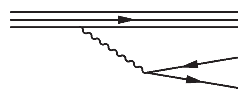
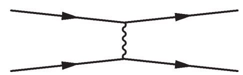
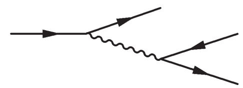
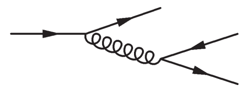
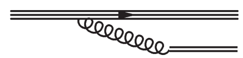
Contents

Physics	Matter (particles)	Forces (fields)	Interactions
classical: certainty	• galaxies • planets, stars	• gravitational (acts on mass, int. 10^{-40})	macro: position and velocity
	• cosmic rays	• weak (acts on “flavour”, int. 10^{-5})	
	• molecule = group of atomes	• residual electromagnetic (chemical link = exchange of electrons)	
	• atom = kernel + electrons	• electromagnetic (acts on electric charge, int. 10^{-2})	
	• X rays		
quantum:	• kernel = group of nucleons	• residual strong	micro: position <u>or</u> velocity
uncertainty	• γ rays		$n \xrightarrow{W^-} p + e^- + \bar{\nu}_e$
particles = fields	• nucleon = group of 3 quarks of type u, d (proton $p = uud$, neutron $n = udd$)	• strong (acts on “colour”, int. 1)	
	• quark (never saw isolated)	• strong force confinement	

Particles	**Fermions**	**Bosons**	**Feynman graphs**
elementary particles = quantum fields	• leptons : $\binom{e^-}{\nu_e}$ $\binom{\mu}{\nu_\mu}$ $\binom{\tau}{\nu_\tau}$ (mass?, charge, 3 flavours) • quarks : $\binom{u}{d}$ $\binom{c}{s}$ $\binom{t}{b}$ (mass, charge, 3 flavours, 3 colours = red, blu, green)	• photon γ (QED) • W^+, W^-, Z^0 (EW) • gluon g ? (QCD) • graviton ? • Higgs H^+, H^-, H^0 ? ..... (spontaneous symmetry break)	$e^- + e^- \xrightarrow{\gamma} e^- + e^-$  $\mu^- \xrightarrow{W^-} \nu_\mu + e^- + \bar{\nu}_e$  $u \xrightarrow{g} d + u + \bar{d}$ 
hadrons = groups of quarks	• baryons (3 quarks) $p = uud$, $n = udd$, $\Delta^{++} = uuu$...	• mesons (2 quarks) $\pi^+ = u\bar{d}$, $\pi^- = \bar{u}d$...	$\Delta^{++} \xrightarrow{g} p + \pi^+$ 

Quantum = (canonical + path integrals) quantization of classical

Fields	**Observables**	**Measures**
classical	functionals F of field φ	values $F(\varphi) \in \mathbb{R}$
quantum	self-adjoint operators O on states $v \in$ Hilbert	expectation value $v^t O v \in \mathbb{R}$ enough: $G(x^\mu - y^\mu)$ = probability from x^μ to y^μ where $x^\mu \in$ Minkowski $= \mathbb{R}^4$ with metric $(-1,1,1,1)$, $\mu = 0,1,2,3$

From classical to quantum

Lagrangian $\mathcal{L}(\varphi, \partial_\mu \varphi) = \mathcal{L}_{free} + \mathcal{L}_{int}$ $\Longrightarrow$ Euler equation $\dfrac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = 0$ $\Longrightarrow$

$\mathcal{L}_{int} = 0$ $\Longrightarrow$

$$\varphi_0(t,x) = \int \frac{1}{\sqrt{2E_p}} \left(a_p e^{i(px-\omega_p t)} + a_p^* e^{-i(px-\omega_p t)} \right) \frac{\mathrm{d}^3 p}{(2\pi)^3} \qquad \text{(wave)}$$

classical: $a_p, a_p^* =$ numbers $\in \mathbb{R}$ or $\mathbb{C}$

quantum: $a_p, a_p^* =$ annihilation and creation operators

$\mathcal{L}_{int} = j\ \varphi$
$j =$ source field $\Longrightarrow$

$$\varphi(x^\mu) = \varphi_0(x^\mu) + \int G_0(x^\mu - y^\mu) j(y^\mu) \mathrm{d}^4 y^\mu$$

classical: $G_0(x^\mu - y^\mu) =$ Green function (resolvant)

quantum: $G(x^\mu - y^\mu) = G_0(x^\mu - y^\mu)$!

$\mathcal{L}_{int} = g\ \varphi^k$
$g =$ coupling constant $\Longrightarrow$

classical: perturbative solutions in g

quantum: perturbative series in g indexed by Feynman graphs

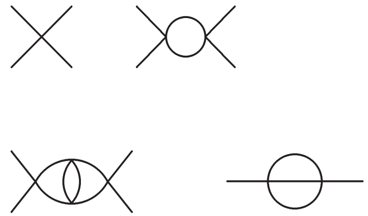
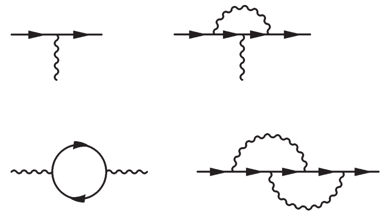
$$G(x^\mu - y^\mu) = \sum_{n \geq 0} G_n(x^\mu - y^\mu)\ g^n = \sum_n \sum_{|\Gamma| = n} U_{x^\mu - y^\mu}(\Gamma)\ g^n = \sum_\Gamma U_{x^\mu - y^\mu}(\Gamma)\ g^{|\Gamma|}$$

$U_{x^\mu - y^\mu}(\Gamma) =$ amplitude (integral) of Feynman graph Γ with n loops

Free theories

Field	Free Lagrangian	Euler equation	Green function on momentum p^μ
$\phi(x^\mu) \in \mathbb{C}$ boson (spin 0, mass m)	$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu\phi^2 - \frac{1}{2}m^2\phi^2$	Klein-Gordon: $(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2)\phi = 0$	$G_0(p) = \frac{i}{p^2-m^2+i\epsilon} \in \mathbb{C}$
$\psi(x^\mu) \in \mathbb{C}^4$ fermion (spin $\frac{1}{2}$, mass m)	$\mathcal{L}_{Dir} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$ γ^μ = 4 × 4 Dirac matrices	Dirac: $(i\gamma^\mu\partial_\mu - m)\psi = 0$	$S_0(p) = \frac{i}{\gamma^\mu p_\mu - m + i\epsilon} \in M_4(\mathbb{C})$
$A^\mu(x^\nu) \in \mathbb{C}^4$ boson (spin 1, mass 0)	$\mathcal{L}_{Max} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)^2$ $-\frac{\lambda}{2}(\partial_\nu A^\mu)^2$ λ = mass parameter	Maxwell: $\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu)$ $+\lambda\partial_\mu\partial^\nu A^\mu = 0$	$D_0(p) = \frac{-ig_{\mu\nu}}{p^2+i\epsilon} + i\frac{\lambda-1}{\lambda}\frac{p_\mu p_\nu}{(p^2+i\epsilon)^2}$ $\in M_4(\mathbb{C})$

Interacting theories

Interacting theory	**Lagrangian**	**Feynman graphs**	**Green function** = series
ϕ^4	$\mathcal{L}_{KG}(\phi) - \frac{1}{4!}\, g\, \phi^4$		$G(p) = \sum_{\Gamma \in \phi^4} U_p(\Gamma)\ g^{\lvert\Gamma\rvert}$
QED = abelian Gauge	$\mathcal{L}_{Dir}(\psi) + \mathcal{L}_{Max}(A^\mu) - e\, \bar{\psi}\gamma^\mu\psi A_\mu$		$S(p) = \sum_{\Gamma\ \text{fermion}} U_p^e(\Gamma)\ e^{2\lvert\Gamma\rvert}$ $D_{\mu\nu}(p) = \sum_{\Gamma\ \text{boson}} U_p^\gamma(\Gamma)\ e^{2\lvert\Gamma\rvert}$

More: ϕ^3, scalar QED, QCD = non-abelian Gauge, Yukawa, ...

Feynman graph = graph Γ with
- arrows depending on the fields
- valence of vertices depending on the interaction term

Feynman amplitude = integral $U(\Gamma)$ computed from the graph

Divergent Feynman integrals

1PI Feynman graph $\quad = \quad$ graph Γ without bridges

Feynman rule: The Feynman amplitude is multiplicative with respect to junction of graphs

1PI graphs: $\Longrightarrow$ $U_p^e\Big($  $\Big) = S_0(p)\ (-ie)\ \int D_{0,\mu\nu}(k)\gamma^\nu S_0(p-k)\gamma^\mu \frac{\mathrm{d}^4k}{(2\pi)^4}\ (-ie)\ S_0(p)$

connected graphs: $\Longrightarrow$ $U_p^e\Big($  $\Big) = U_p^e\Big($  $\Big)\ S_0(p)^{-1}\ U_p^e\Big($  $\Big)$

Problems: $\boxed{U(\Gamma) = \infty \ \ !}$ $\Longrightarrow$ find finite $R(\Gamma)$

In particular: - each cycle in Γ gives a divergent integral,
- each cycle in a cycle gives a subdivergency in a divergency.

$g, e, m, ... \neq$ measured values !
call them *bare*: $g_0, e_0, m_0, ...$

$\Longrightarrow$ find g_0, e_0, m_0 from the *effective*: g, e, m

Dyson renormalization formulas

Bare Green functions:

$$\boxed{G(p;g_0) = \sum_\Gamma U_p(\Gamma)\; g_0^{|\Gamma|} = \sum_n G_n(p)\; g_0^n \quad \text{with} \quad G_n(p) = \sum_{|\Gamma|=n} U_p(\Gamma)}$$

same for $S(p;e_0^2)$ and $D_{\mu\nu}(p;e_0^2)$ $\Rightarrow$ fine structure constant $\alpha_0 = \frac{e_0^2}{4\pi}$

Renormalized Green functions:

$$\boxed{\bar{G}(p;g) = \sum_\Gamma R_p(\Gamma)\; g^{|\Gamma|} = \sum_n \bar{G}_n(p)\; g^n \quad \text{with} \quad \bar{G}_n(p) = \sum_{|\Gamma|=n} R_p(\Gamma)}$$

same for $\bar{S}(p;e^2)$ and $\bar{D}_{\mu\nu}(p;e^2)$ $\Rightarrow$ $\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}$

Dyson and Ward:

$$\bar{G}(g) = G(g_0)\; Z^{-1/2}(g) \qquad \text{with} \qquad g_0(g) = g\; Z^{-1}(g)$$

$$\left\{\begin{aligned} \bar{S}(\alpha) &= S(\alpha_0)\; Z_2^{-1}(\alpha) \\ \bar{D}^T_{\mu\nu}(\alpha) &= D^T_{\mu\nu}(\alpha_0)\; Z_3^{-1}(\alpha) \end{aligned}\right. \qquad \text{with} \qquad \alpha_0(\alpha) = \alpha\; Z_3^{-1}(\alpha)$$

Renormalization factors: $Z = 1 + \mathcal{O}(g^2), \quad Z_3 = 1 - \mathcal{O}(\alpha), \quad Z_2 = 1 + \mathcal{O}(\alpha)$

Coupling constants: $g_0 = g + g\mathcal{O}(g^2), \quad \alpha_0 = \alpha - \alpha\mathcal{O}(\alpha)$

Renormalization group $=$ $\boxed{\text{coupling constants} \ltimes \text{renormalization factors}}$ acts on Green functions: $G(p;g_0) \mapsto \bar{G}(p;g)$

Groups of formal diffeomorphisms and invertible series

Group of invertible series with product:

$$G^{\mathrm{inv}}(A) = \left\{ f(x) = 1 + \sum_{n=1}^{\infty} f_n\, x^n,\ f_n \in A \right\}$$

abelian $\Leftrightarrow$ A commutative
$A = \mathbb{C}$ for Φ^3, Φ^4,
$A = M_4(\mathbb{C})$ for QED

Group of diffeomorphisms with composition:

$$G^{\mathrm{dif}} = \left\{ \varphi(x) = x + \sum_{n=1}^{\infty} \varphi_n\, x^{n+1}, \quad \varphi_n \in \mathbb{C} \right\}$$

Right action by composition:

$$G^{\mathrm{inv}} \times G^{\mathrm{dif}} \longrightarrow G^{\mathrm{inv}}$$
$$(f, \varphi) \mapsto f(\varphi)$$

$\Longrightarrow$ **Semi-direct product:** $G^{\mathrm{dif}} \ltimes G^{\mathrm{inv}} := G^{\mathrm{ren}}$

Renormalization action:

$$\begin{array}{ccccc} G^{\mathrm{inv}} & \times & G^{\mathrm{ren}} & \longrightarrow & G^{\mathrm{inv}}, \\ f(x) & \times & (\varphi(x), g(x)) & \mapsto & f^{\mathrm{ren}}(x) = f(\varphi(x)) \cdot g(x) \end{array}$$

For bosons (ϕ, A_μ): $G^{\mathrm{ren}} = \left\{ (\varphi(x), \frac{\varphi(x)}{x}) \in G^{\mathrm{dif}} \ltimes G^{\mathrm{inv}} \right\} \cong G^{\mathrm{dif}} \Longrightarrow f^{\mathrm{ren}}(x) = f(\varphi(x)) \cdot \frac{\varphi(x)}{x}$

QFT subgroups:

$$G_{graphs} \Longleftrightarrow f_n = \sum_{\Gamma} f(\Gamma)$$

$G_{graphs} \subset G$

for QED also:

$$G_{trees} \Longleftrightarrow f_n = \sum_{t} f(t)$$

with $f(t) = \sum_{\Gamma} f(\Gamma)$

$G_{graphs} \subset G_{trees} \subset G$

BPHZ renormalization formula

To use Dyson formulas, need renormalization Z factors explicitely.

Bogoliubov, Parasiuk, Hepp, Zimmermann:

$$\boxed{\begin{aligned} R_p(\Gamma) &= U_p(\Gamma) + \ C_p(\Gamma) + \sum_{\substack{\text{1PI} \ \gamma_1,\ldots,\gamma_l \subset \Gamma \\ \gamma_i \cap \gamma_j = \emptyset}} U_p(\Gamma/\gamma_1 \ldots \gamma_l)\ C_{p_1}(\gamma_1) \cdots C_{p_l}(\gamma_l), \\ C_p(\Gamma) &= -T^{deg(\Gamma)}_{\text{fixed } p} \left(U_p(\Gamma) + \sum_{\substack{\text{1PI} \ \gamma_1,\ldots,\gamma_l \subset \Gamma \\ \gamma_i \cap \gamma_j = \emptyset}} U_p(\Gamma/\gamma_1 \ldots \gamma_l)\ C_{p_1}(\gamma_1) \cdots C_{p_l}(\gamma_l) \right). \end{aligned}}$$

Here, the 1PI subgraphs γ's contain all the cycles which give subdivergencies.

Then: $\quad Z(g) = \sum_{\text{1PI } \Gamma} C(\Gamma)\ g^{|\Gamma|} \quad$ and $\quad Z_2(e^2) = \sum_{\text{1PI } \Gamma} C^e(\Gamma)\ e^{2|\Gamma|}, \quad Z_3(e^2) = \sum_{\text{1PI } \Gamma} C^\gamma(\Gamma)\ e^{2|\Gamma|}.$

$\Longrightarrow$ Dyson global formulas not enough, need computations on Feynman graphs!

Hopf algebras of formal diffeomorphisms and invertible series

Toy model $A = \mathbb{C}$: group G, $G \cong \mathrm{Hom}_{Alg}(\mathbb{C}(G), \mathbb{C})$ $\Longleftrightarrow$ coordinate ring $\mathbb{C}(G) := \mathrm{Fun}(G, \mathbb{C})$ = commutative Hopf algebra

Hopf algebra of invertible series:

$$\boxed{\begin{array}{c} \mathbb{C}(G^{\mathrm{inv}}) \cong \mathbb{C}[b_1, b_2, ...] \\[2ex] \Delta^{\mathrm{inv}} b_n = \displaystyle\sum_{m=0}^{n} b_{n-m} \otimes b_m \end{array}}$$

$$b_n(f) = f_n = \frac{1}{n!}\frac{d^n f(0)}{dx^n}$$

$$\langle \Delta^{\mathrm{inv}}(b_n), f \times g \rangle = b_n(f \cdot g)$$

Hopf algebra of diffeomorphisms (Faà di Bruno):

$$\boxed{\begin{array}{c} \mathbb{C}(G^{\mathrm{dif}}) \cong \mathbb{C}[a_1, a_2, ...] \\[2ex] \Delta^{\mathrm{dif}} a_n = \displaystyle\sum_{m=0}^{n} a_{n-m} \otimes \text{polynomial} \end{array}}$$

$$a_n(\varphi) = \varphi_n = \frac{1}{(n+1)!}\frac{d^{n+1}\varphi(0)}{dx^{n+1}}$$

$$\langle \Delta^{\mathrm{dif}}(a_n), \varphi \times \psi \rangle = a_n(\varphi \circ \psi)$$

Right coaction:

$$\delta : \mathbb{C}(G^{\mathrm{inv}}) \longrightarrow \mathbb{C}(G^{\mathrm{inv}}) \otimes \mathbb{C}(G^{\mathrm{dif}})$$

$$\langle \delta(b_n), f \times \varphi \rangle = b_n(f \circ \varphi)$$

$\Longrightarrow$ **Semi-direct coproduct:**

$$\boxed{\begin{array}{c} \mathcal{H}^{\mathrm{ren}} := \mathbb{C}(G^{\mathrm{dif}}) \ltimes \mathbb{C}(G^{\mathrm{inv}}) \\[2ex] \Delta^{\mathrm{ren}}(a_m \otimes b_n) = \Delta^{\mathrm{dif}}(a_m) \left[(\delta \otimes \mathrm{Id}) \Delta^{\mathrm{inv}}(b_n) \right] \end{array}}$$

Renormalization coaction:

$$\boxed{\delta^{\mathrm{ren}} : \mathbb{C}(G^{\mathrm{inv}}) \longrightarrow \mathbb{C}(G^{\mathrm{inv}}) \otimes \mathcal{H}^{\mathrm{ren}}, \qquad \delta^{\mathrm{ren}} b_n = (\delta \otimes \mathrm{Id}) \Delta^{\mathrm{inv}}(b_n)}$$

For bosons:

$$\boxed{\mathcal{H}^{\mathrm{ren}} \cong \mathbb{C}(G^{\mathrm{dif}}) \quad \text{and} \quad \delta^{\mathrm{ren}} b_n = \Delta^{\mathrm{dif}} b_n}$$

! ! !

Hopf algebra on Feynman graphs

Question: Since $G^{\text{ren}}_{graphs} \hookrightarrow G^{\text{ren}} \quad \Longrightarrow \quad \mathcal{H}^{\text{ren}} \longrightarrow \mathcal{H}^{\text{ren}}_{graphs} := \mathbb{C}[\text{1PI } \ \Gamma]$ via $a_n, b_n \mapsto \sum_{|\Gamma|=n} \Gamma$,

can we define the renormalization coproduct on each Γ?

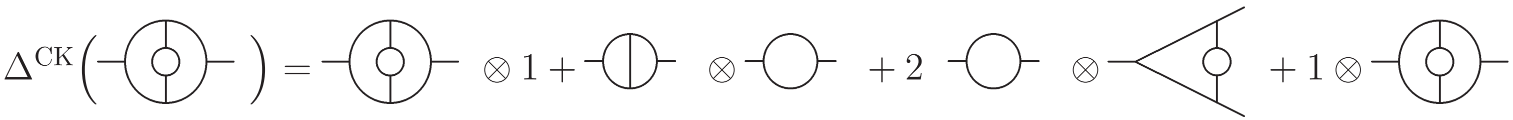
Theorem. [Connes-Kreimer] For the scalar theory ϕ^3:

1) $\mathcal{H}^{CK} = \mathbb{C}[\text{1PI } \ \Gamma]$ is a commutative and connected graded Hopf algebra, with coproduct

$$\Delta^{\text{CK}}\Gamma = \Gamma \otimes 1 + \sum_{\substack{\text{1PI } \ \gamma_1,\dots,\gamma_l \subset \Gamma \\ \gamma_i \cap \gamma_j = \emptyset}} \Gamma/(\gamma_1 \dots \gamma_l) \ \otimes \ (\gamma_1 \cdots \gamma_l) \quad + 1 \otimes \Gamma.$$

2) The BPHZ formula is equivalent to the coproduct Δ^{CK}: $\quad R(\Gamma) = \langle U \otimes C, \Delta^{\text{CK}}\Gamma \rangle$.

3) The group of characters $G^{CK} := \text{Hom}_{Alg}(\mathcal{H}^{CK}, \mathbb{C})$ is the renormalization group.

Example: 

Conclusions: 1) Feynman graphs = natural *local coordinates* in QFT (= basis for the algebra of functions)

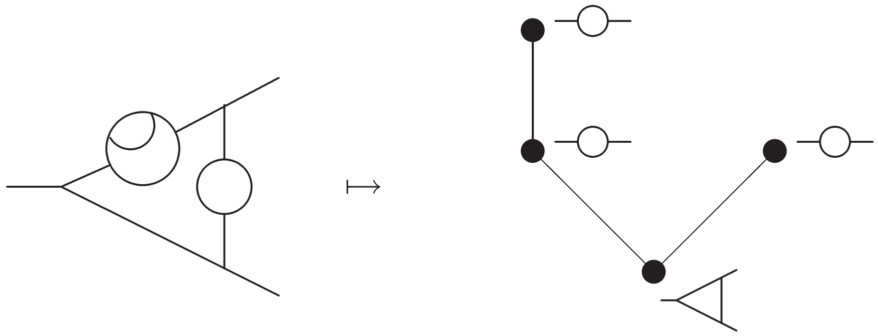
2) By Feynman rules: connected graph = junction of 1PI graphs

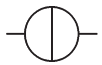
junction = disjoint union = free product

3) Green functions = characters of the algebra of connected Feynman graphs.

Hopf algebra on rooted trees

Hierarchy of divergences: 1PI Feynman graphs $\longrightarrow$ Rooted trees decorated with simple divergencies



Problem for overlapping divergences  : need the difference of trees $\Rightarrow$ forests

Theorem. [Kreimer] For the scalar theory ϕ^3:

1) $\mathcal{H}_{\mathrm{R}} = \mathbb{C}[T \text{ rooted trees}]$ is a commutative and connected graded Hopf algebra, with coproduct

$$\Delta T = T \otimes 1 + \sum_{\substack{\text{admissible}\\ \text{cuts}}} \text{“what remains of } T\text{”} \otimes \text{“branches of } T\text{”} \quad + 1 \otimes T.$$

2) The BPHZ formula is equivalent to the coproduct Δ^{K}: $\quad R(T) = \langle U \otimes C, \Delta^{\mathrm{K}} T \rangle.$

Alternative algebras for QED

For QED, f_n and $f(\Gamma) \in A = M_4(\mathbb{C})$ non-commutative: $\mathrm{Fun}(G^{\mathrm{inv}}(A), \mathbb{C}) \neq \mathbb{C}[b_n]$ or $\mathbb{C}[1\mathrm{PI}\ \Gamma]$!

1) Matrix elements: Basis Γ_{ij} for the matrix elements $f(\Gamma_{ij}) := (f(\Gamma))_{ij}$.
Then $\mathrm{Fun}(G^{\mathrm{inv}}(A), \mathbb{C}) = \mathbb{C}[1\mathrm{PI}\ \Gamma_{ij}]$, but junction $\neq$ free product!

2) Non-commutative characters: Green functions = "characters with values in A": $G^p \cong \mathrm{Hom}_{Alg}(\mathbb{C}\langle 1\mathrm{PI}\ \Gamma\rangle, A)$.
Then $\mathbb{C}\langle 1\mathrm{PI}\ \Gamma\rangle$ is an algebra, but not necessarily Hopf.

Simple Lemma.

$\mathcal{H}^{\mathrm{inv}} := \mathrm{Fun}(G^{\mathrm{inv}}(A), A) \cong \mathbb{C}\langle b_n, n \in \mathbb{N}\rangle$ is a Hopf algebra with

$$\Delta^{\mathrm{inv}} b_n = b_n \otimes 1 + 1 \otimes b_n + \sum_{m=1}^{n-1} b_{n-m} \otimes b_m,$$

and $G^{\mathrm{inv}}(A) \sim \mathrm{Hom}_{Alg}(\mathcal{H}^{\mathrm{inv}}, A)$. Moreover $\mathcal{H}^{\mathrm{inv}}_{ab} = \mathbb{C}(G^{\mathrm{inv}})$.

Then $\mathcal{H}^{\mathrm{ren}} := \mathbb{C}(G^{\mathrm{dif}}) \ltimes \mathcal{H}^{\mathrm{inv}}$ = non-comm. Hopf algebra $\Rightarrow$ electron renormalization with Δ^{ren}, δ^{ren}

Question: Since $G^{\mathrm{ren}}_{trees} \hookrightarrow G^{\mathrm{ren}} \implies \mathcal{H}^{\mathrm{ren}} \longrightarrow \mathcal{H}^{\mathrm{ren}}_{trees} := \mathbb{C}[t \in Y]$ via $a_n, b_n \mapsto \sum_{|t|=n} t$,
can we define the renormalization coproduct on each QED tree?

Hopf algebras on planar binary rooted trees

Theorem. [Brouder-F.]

1) **Charge renormalization:** commutative Hopf $\mathcal{H}^\alpha = \mathbb{C}[\curlyvee^{t}, t \in Y]$ with coproduct

$$\Delta^\alpha \curlyvee^{t} = \sum \text{“what remains of } \curlyvee^{t}\text{”} \otimes \text{“}\backslash\text{-branches of } t\text{”}.$$

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2) **Electron renormalization:** coaction $\Delta^e := (\delta^e \otimes \mathrm{Id})\Delta_e^{\mathrm{inv}}$ of $\mathcal{H}^{\mathrm{qed}}$ on $\mathcal{H}^e$, where

- $\mathcal{H}^e = \mathbb{C}\langle Y\rangle/(1 - \mid)$ is non-commutative Hopf with Δ_e^{inv} dual to product *under* $t\backslash s = t^{\diagup^{s}}$;
- $\delta^e : \mathcal{H}^e \longrightarrow \mathcal{H}^e \otimes \mathcal{H}^\alpha$ coaction extended from $\delta \curlyvee^{t} = \Delta^\alpha \curlyvee^{t} - \mid \otimes \curlyvee^{t}$;
- renormalization group: $\mathcal{H}^{\mathrm{qed}} := \mathcal{H}^\alpha \ltimes \mathcal{H}^e$ with $\Delta^{\mathrm{qed}} = \Delta^\alpha \times \Delta^e$.

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3) **Photon renormalization:** coaction $\Delta^\gamma := m_{23}^3(\delta^\gamma \otimes \sigma)\Delta_\gamma^{\mathrm{inv}}$ of $\mathcal{H}^\alpha$ on $\mathcal{H}^\gamma$, where

- $\mathcal{H}^\gamma = \mathbb{C}\langle Y\rangle/(1 - \mid)$ is non-commutative Hopf with $\Delta_\gamma^{\mathrm{inv}}$ dual to product *over* $t/s = {}^{t\diagdown}s$;
- $\delta^\gamma : \mathcal{H}^\gamma \longrightarrow \mathcal{H}^\gamma \otimes \mathcal{H}^\alpha$ coaction extended from δ;
- $\mathcal{H}^\alpha \ltimes \mathcal{H}^\gamma \twoheadrightarrow \mathcal{H}^\alpha$ induced by the 1-cocycle $\sigma : \mathcal{H}^\gamma \longrightarrow \mathcal{H}^\alpha$ $\sigma(t_1 \dots t_n) = t_1/\cdots/t_n$.

Remark: $\Delta^\gamma \equiv \Delta^\alpha$ on single trees $t \in Y$!

Feed back in mathematics

1) Combinatorial Hopf algebras: Foissy, Holtkamp, Brouder, F., Krattenthaler, Loday, Ronco, Grossmann, Larson, Hoffman, Painate...

2) Relation with operads: Chapoton, Livernet, Foissy, Holtkamp, van der Laan, Loday, Ronco,...

3) Combinatorial groups: invent group law on tree-expanded series of the form $f(x) = \sum_{t \in Y} f(t)\, x^t$. Interesting composition!

4) Non-commutative Hopf algebras and groups: look for new duality between groups and non-commutative Hopf algebras. Need a new coproduct with values in the *free product* of algebras. Bergman, Hausknecht, Fresse, F., Holtkamp,...

Feed back in physics

1) More computations and developpements: Kreimer, Broadhurst, Delbourgo, Ebrahimi-Fard, Bierenbaum,...

2) Hopf algebras everywhere: Connes, Kreimer's school, Pinter et al., Brouder, Fauser, F., Oeckl, Schmitt in QFT; Patras and Cassam-Chenai in quantum chemistry...

1) Combinatorial Hopf algebras

[Foissy, Holtkamp]

$$\mathcal{H}_{\mathrm{PRD}} = \mathbb{C}\langle T \text{ planar rooted decorated trees}\rangle$$

$$\Delta T = \sum_{\substack{\text{admissible}\\ \text{cuts}}} \text{"what remains of } T\text{"} \otimes \text{"branches of } T\text{"}$$

Hopf algebra
non-commutative version of $\mathcal{H}_{\mathrm{R}}$

[Brouder-F.]

$$\mathcal{H}^{\mathrm{dif}} = \mathbb{C}\langle a_n, n \in \mathbb{N}\rangle$$

$$\Delta^{\mathrm{dif}} a_n = \sum_{m=0}^{n} a_{n-m} \otimes \sum_{k=1}^{m} \binom{n}{k} \sum_{\substack{m_1+\cdots+m_k=m\\ m_1>0,\dots,m_k>0}} a_{m_1}\cdots a_{m_k}$$

Hopf algebra
non-commutative version of $\mathbb{C}(G^{\mathrm{dif}})$

[F.-Krattenthaler]

$$S^{\mathrm{dif}} a_n = \sum_{k=0}^{n-1} (-1)^{k+1} \sum_{\substack{n_1+\cdots+n_{k+1}=n\\ n_1,\dots,n_{k+1}>0}} \sum_{\substack{m_1+\cdots+m_k=k\\ m_1+\cdots+m_h\geq h\\ h=1,\dots,k-1}} \binom{n_1+1}{m_1}\cdots\binom{n_k+1}{m_k} a_{n_1}\cdots a_{n_k} a_{n_{k+1}}$$

explicit
non-commutative
antipode

[Brouder-F.]

$\mathcal{H}^{\alpha}$ extends naturally to $\widetilde{\mathcal{H}^{\alpha}} := \mathbb{C}\langle \curlyvee^{t}, t \in Y\rangle$

Hopf algebra analogue to $\mathcal{H}^{\mathrm{dif}}$
non-commutative version of $\mathcal{H}^{\alpha}$

2) Relation with operads

[Chapoton-Livernet] $\boxed{L := \mathrm{Prim}((\mathcal{H}_{\mathrm{R}})^*) \quad \Longrightarrow \quad \text{Lie algebra from the free } \textit{pre-Lie algebra} \text{ on one generator}}$

[Foissy, Holtkamp, Patricia?] $\boxed{\widetilde{\mathcal{H}^\alpha} \cong (\widetilde{\mathcal{H}^\alpha})^* \cong \mathcal{H}^{\mathrm{LR}}}$ related to the Loday-Ronco Hopf algebra $\Longrightarrow$ free *dendriform Hopf algebra*

[van der Laan]

$\mathcal{P}$ operad $\Longrightarrow$ $S(\oplus \mathcal{P}(n)_{S_n})$ commutative Hopf algebra

$\mathcal{P}$ non-Σ operad $\Longrightarrow$ $T(\oplus \mathcal{P}(n))$ Hopf algebra

operadic version

[van der Laan] $\boxed{\mathcal{F} \text{ operad of Feynman graphs } \Rightarrow S(\oplus \mathcal{F}(n)_{S_n}) = \mathcal{H}^{\mathrm{CK}}}$

3) Combinatorial groups

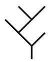
Tree-expanded series: formal symbols x^t , for any tree $t \in Y$.

Invertible series for the electron:
$$G^e := \left\{ f(x) = \sum_{t \in Y} f(t)\ x^t,\ f(|) = 1 \right\}$$
group with the product *under*
$$f(x) \backslash g(x) := \sum_{t,s \in Y} f(t)\ g(s)\ x^{t \backslash s}$$

Invertible series for the photon:
$$G^\gamma = \left\{ f(x) = \sum_{t \in Y} f(t)\ x^t,\ f(|) = 1 \right\}$$
group with the product *over*
$$f(x)/g(x) := \sum_{t,s \in Y} f(t)\ g(s)\ x^{t/s}$$

Diffemorphisms for the charge:
$$G^\alpha = \left\{ \varphi(x) = \sum_{t \in Y} \varphi(t) x^{\mathrm{Y} \backslash t},\ \varphi(|) = 1 \right\}$$
group with the composition law $(\varphi \circ \psi)(x) := \varphi(\psi(x))$

Composition:
$$\boxed{\begin{array}{l} \psi(x)^t := \mu_t(\psi(x)) \\ \Leftrightarrow \psi(x) \text{ in each vertex of } t \end{array}}$$
where μ_t is the monomial which describes t as a sequence of *over* and *under* products of Y

For instance:  $= (\mathrm{Y} \backslash \mathrm{Y})/\mathrm{Y}$ hence μ $(s) = (s \backslash s)/s$

Theorem. [F.]

1) The sets G^e, G^γ and G^α form non-abelian groups.

2) QED renormalization at tree-level: $G^\alpha \cong \mathrm{Hom}_{Alg}(\mathcal{H}^\alpha, \mathbb{C})$ and $G^\alpha \ltimes G^e \cong \mathrm{Hom}_{Alg}(\mathcal{H}^{\rm qed}, A)$

3) The "order" map $|\ | : Y \twoheadrightarrow \mathbb{N}$ induces group projections
$$G^\alpha \twoheadrightarrow G^{\rm dif}_{trees} \subset G^{\rm dif}, \qquad G^\gamma \twoheadrightarrow G^{\rm inv}_{trees} \subset G^{\rm inv}, \qquad G^e \twoheadrightarrow G^{\rm inv}_{trees} \subset G^{\rm inv}.$$

4) Non-commutative Hopf algebras and groups

Fact:

$G^{\mathrm{inv}}(A)$ still group if A non-commutative, and

$\mathcal{H}^{\mathrm{inv}} = \mathrm{Fun}(G^{\mathrm{inv}}(A), A) = \mathbb{C}\langle b_n, n \in \mathbb{N}\rangle$ non-commutative Hopf

Question: which duality " group $G \longleftrightarrow$ non-commutative Hopf $\mathcal{H}$ " ?

Answer: replace coproduct $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ with $\Delta_* : \mathcal{H} \longrightarrow \mathcal{H} \star \mathcal{H}$,

where $\star$ = free product, such that $T(U \oplus V) \cong T(U) \star T(V)$.

Call $\mathcal{H}_* = (\mathcal{H}, \Delta_*)$ and look for duality $G \cong \mathrm{Hom}_{Alg}(\mathcal{H}_*, A)$.

$\Rightarrow$ Co-groups in associative algebras: [Bergman-Hausknecht,Fresse]

Group of invertible series:

$\Delta_*^{\mathrm{inv}} : \mathcal{H}^{\mathrm{inv}} \longrightarrow \mathcal{H}^{\mathrm{inv}} \star \mathcal{H}^{\mathrm{inv}}$, $\Delta_*^{\mathrm{inv}}(b_n) = \Delta^{\mathrm{inv}}(b_n)$ well defined and co-associative!

Moreover $G^{\mathrm{inv}}(A) \cong \mathrm{Hom}_{Alg}(\mathcal{H}_*^{\mathrm{inv}}, A)$.

Group of diffeomorphisms:

$G^{\mathrm{dif}}(A)$ not a group, because $\circ$ not associative.

$\Delta_*^{\mathrm{dif}} : \mathcal{H}^{\mathrm{dif}} \longrightarrow \mathcal{H}^{\mathrm{dif}} \star \mathcal{H}^{\mathrm{dif}}$, $\Delta_*^{\mathrm{dif}}(a_n) = \Delta^{\mathrm{dif}}(a_n)$ well defined but not co-associative!

$\Rightarrow$ [Holtkamp] on trees.