The Hopf algebra of tree-expanded series

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1. Motivation: Renormalization in QFT

Renormalization group: $\boxed{G^{\mathrm{ren}} = G^{\mathrm{dif}} \ltimes G^{\mathrm{inv}}(A)}$ where

$$G^{\mathrm{inv}}(A) = \left\{ f(x) = \sum_{n=0}^{\infty} f_n\, x^n, \quad f_0 = 1, f_n \in A \right\}$$

$$A = \mathbb{C} \text{ for } \Phi^3, \quad A = M_4(\mathbb{C}) \text{ for QED}$$

Group of invertible series with pointwise product (abelian $\Leftrightarrow$ A commutative).

$$G^{\mathrm{dif}} = \left\{ \varphi(x) = \sum_{n=0}^{\infty} \varphi_n\, x^{n+1}, \quad \varphi_0 = 1, \varphi_n \in \mathbb{C} \right\}$$

Group of formal diffeomorphisms with composition.

$$G^{\mathrm{inv}} \times G^{\mathrm{dif}} \longrightarrow G^{\mathrm{inv}}$$

Right action by composition.

Renormalization action:

$$\boxed{\begin{array}{l} G^{\mathrm{inv}}(A) \times G^{\mathrm{ren}} \longrightarrow G^{\mathrm{inv}}(A), \\ f(x) \mapsto f^{\mathrm{ren}}(x) = f(\varphi(x)) \cdot g(x) \end{array}}$$

Feynman graphs, trees, integers:

$$\begin{array}{ccccc} \mathcal{F} & \twoheadrightarrow & Y & \overset{\text{order}}{\twoheadrightarrow} & \mathbb{N} \\ \mathbb{C}[[x]]_{\mathcal{F}} & \hookrightarrow & \mathbb{C}[[x]]_Y & \hookrightarrow & \mathbb{C}[[x]] \\ \text{if } f_n = \sum_{|\Gamma|=n} f(\Gamma) & & \text{if } f_n = \sum_{|t|=n} f(t) & & f(x) = \sum_{n=0}^{\infty} f_n\, x^n \end{array}$$

Meaning of BPHZ formula:

The renormalization groups are known through recursive computations on the coefficients $f(\Gamma)$.

Conclusion: Feynman graphs = natural *local coordinates* of QFT.

G^{ren} = group of *characters* for the *coordinate ring.*

What happens if A is *noncommutative*?

2. Dual Hopf algebras and coactions

For the Φ^3 theory, $A = \mathbb{C}$ is commutative:

Renormalization Hopf algebra:

$$\boxed{\begin{gathered}\mathcal{H}^{\mathrm{ren}} := \mathcal{H}^{\mathrm{dif}}_{ab} \ltimes \mathcal{H}^{\mathrm{inv}}_{ab} \\ \Delta^{\mathrm{ren}}(a \otimes b) = \Delta^{\mathrm{dif}}(a)\ \big[(\delta \otimes \mathrm{Id})\Delta^{\mathrm{inv}}(b)\big]\end{gathered}}$$

semidirect or smash coproduct, where

$$\mathcal{H}^{\mathrm{inv}}_{ab} = \mathrm{Fun}(G^{\mathrm{inv}}(\mathbb{C}), \mathbb{C})) \cong \mathbb{C}[b_1, b_2, \ldots]$$
$$b_n(f) = f_n = \tfrac{1}{n!} \tfrac{d^n f(0)}{dx^n}$$

commutative Hopf algebra with dual coproduct Δ^{inv};

$$\mathcal{H}^{\mathrm{dif}}_{ab} = \mathrm{Fun}(G^{\mathrm{dif}}, \mathbb{C})) \cong \mathbb{C}[a_1, a_2, \ldots]$$
$$a_n(\varphi) = \varphi_n = \tfrac{1}{(n+1)!} \tfrac{d^{n+1}\varphi(0)}{dx^{n+1}}$$

commutative Hopf algebra with dual coproduct Δ^{dif};

$$\delta : \mathcal{H}^{\mathrm{inv}}_{ab} \longrightarrow \mathcal{H}^{\mathrm{inv}}_{ab} \otimes \mathcal{H}^{\mathrm{dif}}_{ab}$$

dual right coaction.

Renormalization coaction:

$$\boxed{\begin{aligned}&\delta^{\mathrm{ren}} : \mathcal{H}^{\mathrm{inv}}_{ab} \longrightarrow \mathcal{H}^{\mathrm{inv}}_{ab} \otimes \mathcal{H}^{\mathrm{ren}}, \\ &\delta^{\mathrm{ren}} b = (\delta \otimes \mathrm{Id})\Delta^{\mathrm{inv}}(b)\end{aligned}}$$

Question: $G^{\mathrm{ren}}_{\mathcal{F}} \hookrightarrow G^{\mathrm{ren}} \quad \Rightarrow \quad \mathcal{H}^{\mathrm{ren}} \twoheadrightarrow \mathcal{H}^{\mathrm{ren}}_{\mathcal{F}}$,
hence the renormalization coproduct is defined for $\sum_{|\Gamma|=n} \Gamma \in \mathcal{H}^{\mathrm{ren}}_{\mathcal{F}}$.

Can this coproduct be defined on each Γ?

Meaning of Connes-Kreimer theorem:

For Φ^3, the renormalization Hopf algebra exists at the level of Feynman graphs:
given by $\mathcal{H}^{\mathrm{CK}} = \mathbb{C}[1\mathrm{PI}\ \Gamma \in \mathcal{F}]$ with coproduct

$$\Delta^{\mathrm{CK}}\Gamma = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\substack{\gamma_1, \ldots, \gamma_l \subset \Gamma \\ \gamma_i \cap \gamma_j = \emptyset}} \Gamma/\gamma_1 \ldots \gamma_l \otimes \gamma_1 \cdots \gamma_l.$$

3. QED Hopf algebras on planar binary trees

For QED, $A = M_4(\mathbb{C})$ is noncommutative:

Simple Lemma.

> If $\mathcal{H}^{\text{inv}} = \mathbb{C}\langle b_n, n \in \mathbb{N}\rangle$ is the Hopf algebra with
>
> $$\Delta^{\text{inv}} b_n = b_n \otimes 1 + 1 \otimes b_n + \sum_{m=1}^{n-1} b_{n-m} \otimes b_m,$$
>
> then $G^{\text{inv}}(A) \cong \text{Hom}_{Alg}(\mathcal{H}^{\text{inv}}, A)$ and $\mathcal{H}^{\text{inv}} \cong \text{Fun}(G^{\text{inv}}(A), A)$.

Then $\mathcal{H}^{\text{ren}} := \mathcal{H}^{\text{dif}}_{ab} \ltimes \mathcal{H}^{\text{inv}}$ is a noncommutative Hopf algebra.

Question: As before, can the renormalization coproduct be defined on each Feynman graph or on each tree representing QED?

Theorem. [Brouder-F.]

> 1) Charge renormalization given by the commutative Hopf algebra
>
> $\mathcal{H}^{\alpha} = \mathbb{C}[\curlyvee^{t}, t \in Y]$ with coproduct
>
> $$\Delta^{\alpha} \curlyvee^{t} = \sum \text{“what remains of } \curlyvee^{t}\text{”} \otimes \text{“}\backslash\text{-branches of } t\text{”}.$$
>
> 2) Electron renormalization given by the right coaction $\Delta^{e} := (\delta^{e} \otimes \text{Id})\Delta^{\text{inv}}_{e}$ of $\mathcal{H}^{\text{qed}}$ on $\mathcal{H}^{e}$, where
>
> • $\mathcal{H}^{e} = \mathbb{C}\langle Y\rangle/(1 - |)$ is a noncommutative Hopf algebra with coproduct Δ^{p}_{e} dual to the *under* product $t\backslash s = {}_{t}/^{s}$;
>
> • $\delta^{e} : \mathcal{H}^{e} \longrightarrow \mathcal{H}^{e} \otimes \mathcal{H}^{\alpha}$ coaction extended from $\delta \curlyvee^{t} = \Delta^{\alpha} \curlyvee^{t} - | \otimes \curlyvee^{t}$;
>
> • $\mathcal{H}^{\text{qed}} := \mathcal{H}^{\alpha} \ltimes \mathcal{H}^{e}$, with $\Delta^{\text{qed}} = \Delta^{\alpha} \times \Delta^{e}$, represents the electron renormalization group.
>
> 3) Photon renormalization given by the right coaction $\Delta^{\gamma} := m^{3}_{23}(\delta^{\gamma} \otimes \sigma)\Delta^{\text{inv}}_{\gamma}$ of $\mathcal{H}^{\alpha}$ on $\mathcal{H}^{\gamma}$, where
>
> • $\mathcal{H}^{\gamma} = \mathbb{C}\langle Y\rangle/(1 - |)$ is a noncommutative Hopf algebra with coproduct $\Delta^{\text{inv}}_{\gamma}$ dual to the *over* product $t/s = {}^{t}\backslash_{s}$;
>
> • $\delta^{\gamma} : \mathcal{H}^{\gamma} \longrightarrow \mathcal{H}^{\gamma} \otimes \mathcal{H}^{\alpha}$ coaction extended from δ;
>
> • projection $\mathcal{H}^{\alpha} \ltimes \mathcal{H}^{\gamma} \longrightarrow \mathcal{H}^{\alpha}$ of Hopf algebras induced by the 1-cocycle map $\sigma : \mathcal{H}^{\gamma} \longrightarrow \mathcal{H}^{\alpha}$ $\sigma(t_1 \dots t_n) = t_1 / \cdots / t_n$.

4. New noncommutative Hopf algebras

Remark: the set of series $\varphi(x) = \sum_n \varphi_n x^{n+1}$ is *not* a group if $\varphi_n \in$ noncommutative A, because the the composition is no more associative.

Theorem. [Brouder-F.]

1) On the noncommutative algebra $\mathcal{H}^{\mathrm{dif}} = \mathbb{C}\langle a_n, n \in \mathbb{N}\rangle$, the coproduct

$$\Delta^{\mathrm{dif}} a_n = a_n \otimes 1 + 1 \otimes a_n + \sum_{m=1}^{n-1} a_{n-m} \otimes \sum_{k=1}^{m} \binom{n}{k} \sum_{\substack{m_1+\cdots+m_k=m \\ m_1>0,\ldots,m_k>0}} a_{m_1}\cdots a_{m_k}$$

is coassociative.

2) The antipode is explicitly given by

$$S^{\mathrm{dif}} a_n \;=\; \sum_{k=0}^{n-1} (-1)^{k+1} \sum_{\substack{n_0+n_1+\cdots+n_k=n \\ n_0,\ldots,n_k>0}} \lambda(n_1,\ldots,n_k)\; a_{n_0} a_{n_1}\cdots a_{n_k},$$

with coefficients $\quad \lambda(n_1,\ldots,n_k) = \sum_{(m_1,\ldots,m_k)} \binom{n_1+1}{m_1}\cdots\binom{n_k+1}{m_k},$

where the sum runs over the k-tuples of non-negative integers such that $\sum_{i=1}^{l} m_i \leq l$ for all $l = 1,\ldots,k-1$, and $\sum_{i=1}^{k} m_i = k$.

Theorem. [Brouder-F.]

$\mathcal{H}^{\alpha}$ extends naturally to a noncommutative Hopf algebra

$$\widetilde{\mathcal{H}^{\alpha}} := \mathbb{C}\langle \curlyvee^{t}, t \in Y\rangle$$

with the same coproduct on generators.

5. Table of groups and noncommutative Hopf algebras

Series expanded over:	**Groups** (= comm. Hopf alg.)	**Noncommutative Hopf algebras**
integers	$G^{\mathrm{dif}} \ltimes G^{\mathrm{inv}}$	$\mathcal{H}^{\mathrm{dif}}$ $\mathcal{H}^{\mathrm{dif}}_{ab} \ltimes \mathcal{H}^{\mathrm{inv}}$ [Brouder-F., QED]
rooted planar binary trees	?	$\widetilde{\mathcal{H}^{\alpha}} \cong \mathcal{H}^{LR}$ [Loday-Ronco] [Holtkamp, Foissy] $\mathcal{H}^{\mathrm{qed}} = \mathcal{H}^{\alpha} \ltimes \mathcal{H}^{e}$ [Brouder-F., QED]
(decorated, planar) rooted trees	$\mathcal{H}_R = \mathrm{Fun}(G_R)$ group known through its coordinate ring [Kreimer, Φ^3] or its enveloping algebra [Grossman-Larson] known a repr. on the space of actions [Krajewski]	$\mathcal{H}_{PRD}$ non-comm. version of $\mathcal{H}_R$ [Foissy] [Painate, Hoffman] operadic version [van der Laan]
Feynman graphs	$\mathcal{H}^{\mathrm{CK}} = \mathrm{Fun}(G^{CK})$ group known through its coordinate ring or its Lie algebra [Connes-Kreimer, Φ^3]	??

6. Groups of tree-expanded series

Formal symbols x^t for any tree $t \in Y$.

Electron invertible series:

$$G^e = \left\{ f(x) = \sum_{t\in Y} f(t)\ x^t,\ \ f(|) = 1, f(t) \in A \right\}$$

group with
the *under* product
$f(x)\backslash g(x) := \sum_{t,s\in Y} f(t)\ g(s)\ x^{t\backslash s}$.

Photon invertible series:

$$G^\gamma = \left\{ f(x) = \sum_{t\in Y} f(t)\ x^t,\ \ f(|) = 1, f(t) \in A \right\}$$

group with
the *over* product
$f(x)/g(x) := \sum_{t,s\in Y} f(t)\ g(s)\ x^{t/s}$.

Charge formal diffeomorphisms:

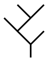
$$G^\alpha = \left\{ \varphi(x) = \sum_{t\in Y} \varphi(t) x^{t/\Upsilon},\ \ \varphi(|) = 1, \varphi(t) \in \mathbb{C} \right\}$$

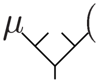
group with
the composition law
$(\varphi \circ \psi)(x) := \varphi(\psi(x))$

Composition: $\boxed{\psi(x)^t := \mu_t(\psi(x))}$

where μ_t is the monomial which describes t as a sequence of *over* and *under* products of Υs.

For instance:

 $= (\Upsilon\backslash\Upsilon)/\Upsilon$ hence μ  $(s) = (s\backslash s)/s,$

 $= (\Upsilon/\Upsilon)\backslash\Upsilon = \Upsilon/(\Upsilon\backslash\Upsilon)$ hence μ  $(s) = (s/s)\backslash s = s/(s\backslash s).$

Theorem. [F.]

1) The sets G^e, G^γ and G^α are non abelian groups.

2) Moreover, they represent the QED renormalization at the level of trees:

$$G^\alpha \cong \mathrm{Hom}_{Alg}(\mathcal{H}^\alpha, \mathbb{C}) \quad \text{and} \quad G^\alpha \ltimes G^e \cong \mathrm{Hom}_{Alg}(\mathcal{H}^{\mathrm{qed}}, A).$$

3) The order map $|\ \ | : Y \twoheadrightarrow \mathbb{N}$ induces the projections of groups

$$G^\alpha \twoheadrightarrow G^{\mathrm{dif}}_Y \subset G^{\mathrm{dif}}, \qquad G^\gamma \twoheadrightarrow G^{\mathrm{inv}}_Y \subset G^{\mathrm{inv}}, \qquad G^e \twoheadrightarrow G^{\mathrm{inv}}_Y \subset G^{\mathrm{inv}}.$$