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Renormalization of QED with planar binary trees

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Abstract. The renormalized photon and electron propagators are expanded over planar binary trees. Explicit recurrence solutions are given for the terms of these expansions. In the case of massless Quantum Electrodynamics (QED), the relation between renormalized and bare expansions is given in terms of a Hopf algebra structure. For massive quenched QED, the relation between renormalized and bare expansions is given explicitly.

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1 Introduction

Wightman commented [1]: “Renormalization Theory has a history of egregious errors by distinguished savants. It has a justified reputation of perversity; a method that works up to 13-th order in the perturbation series fails in the 14-th order.” Although renormalization theory is often considered to be well understood, it is still a difficult subject plagued with considerable combinatorial complexity.

However, renormalization is not just a recipe to extract a finite part from an infinite integral, since it has a deep physical meaning. It was a guide to elaborate the theories of weak and strong interactions. It can be used to build consistent Lagrangians: the minimal coupling Lagrangian of scalar electrodynamics misses a quartic term which is reintroduced by renormalization [2]. Renormalization is also linked to the irreversibility of the macroscopic Universe [3]. Other arguments in favor of renormalization have been given by Jackiw [4]. Moreover, thanks to the work of Kreimer and Connes [5–9], renormalization has become a beautiful mathematical theory.

There is a class of physicists who think that a quantum field theory is entirely contained in its Feynman diagrams. Since two members of this class, Veltman and ’t Hooft, have been recently awarded the Nobel prize in physics, it seems rather out of place to try to formulate quantum field theory without Feynman diagrams. Nevertheless, we shall pursue our attempt to build such a theory by studying how renormalization can be implemented in the framework of planar binary trees.

The typical equations of QED are ‘implicit’ functional derivative equations. In [10], the tree expansion method enabled us to establish an explicit recursive solution for the photon and electron propagators. For renormalized QED including mass renormalization the functional equations change drastically, and it is not a priori clear that a similar explicit recursive solution can be given. In this paper, we obtain an explicit recursive solution for renormalized QED. In other words, the terms of the tree expansion for the renormalized photon and electron propagators are written as an integral of renormalized terms for smaller trees. In the case of quenched QED, this recursive equation is transformed into an explicit relation between the renormalized and bare terms. Finally, we exhibit a Hopf algebra that encodes the renormalization of massless QED. This Hopf algebra is neither commutative nor cocommutative.

The plan of the paper is the following. We first introduce the problem of renormalization from different points of view, then we give the essential equations for the renormalization of QED. The tree expansion method is presented in detail, and used to derive the recursive equations for the renormalized electron and photon propagators. Then detailed relations between the renormalized and bare propagators of quenched QED are given. Finally the Hopf algebra of massless QED is described. A first appendix gives some proofs, a second one gives the relation between renormalized and bare propagators for massless QED. A last appendix expresses the smallest planar binary trees as a sum of Feynman diagrams.

A planar binary tree can be considered as a sum of Feynman diagrams (see appendix 3). To some extent, this sum is natural. For instance, all the diagrams obtained by permutations of the photon external lines of a Feynman diagrams cannot be distinguished by an experiment. Therefore, it is natural to regroup all these diagrams under a single symbol. However, when renormalization comes into play, each Feynman diagrams is renormalized by a number of Feynman subdiagrams multiplied by scalar counterterms. It is not a priori clear that these subdiagrams and counterterms can be grouped into the same trees as the original Feynman diagrams. The main point of this paper is to show that this is possible for QED. In other words, we provide an algebraic structure on planar binary trees which is compatible with renormalization.

In this paper, we consider only the renormalization of ultraviolet divergences, and we assume that infrared divergences are regularized, for instance by introducing a photon mass.

2 Renormalization

An enormous literature has been devoted to the renormalization theory. The reader is deferred to [11–13] and references therein for historical and conceptual aspects of renormalization. Here we shall concentrate on its technical aspects. Renormalization theory can be considered from at least three different points of view: the Dyson-Salam method, the extension of distributions and the product of distributions.

2.1 The Dyson point of view

According to the first point of view, perturbative quantum field theory yields divergent integrals in Fourier space, and renormalization is a technique intended to extract a finite part from them. A picture of how this could be achieved was first given by Dyson in 1949 [14] and Salam [15,16] in 1951. Explicit formulas were proposed by Bogoliubov and Parasiuk [17] and finally proved by Hepp [18,19]. This method is general, in the sense that it can be used for any quantum field theory, whether renormalizable or not.

To understand this renormalization process, it is very useful to treat a one-dimensional toy model of overlapping divergences proposed by Kreimer [5]. Let

$$f(x,y,c) = \frac{x}{x+c}\frac{1}{x+y}\frac{y}{y+c}.$$

We want to give a meaning to the integral

$$I(c) = \int_1^\infty \mathrm{d}x \int_1^\infty \mathrm{d}y f(x,y,c).$$

Power counting is applied as follows. If we substitute λx to x and take the limit $\lambda \to \infty$, we see that $I(c)$ varies as $\lambda^0 = 1$, and the integral is logarithmically divergent for x. Similarly, it is logarithmically divergent for y. If both x and y are multiplied by λ, the integral $I(c)$ varies as λ^1 in the limit $\lambda \to \infty$. Then, $I(c)$ is linearly divergent for the variables x, y. In the Dyson-Salam renormalization scheme, we first fix y in $f(x,y,c)$, we keep the part of $f(x,y,c)$ which does not depend on x (i.e. $y/(y+c)$) and we take the value of the rest (i.e. $x/((x+c)(x+y))$) at $c=0$ and $y=0$ (i.e. $1/x$). The product of these two factors (i.e. $y/(x(y+c))$) is called a counterterm and is subracted from $f(x,y,c)$ to remove the logarithmic divergence for x. This procedure produces the term

$$f(x,y,c) - \frac{y}{x(y+c)} = -\frac{y}{y+c}\frac{xy+xc+yc}{x(x+c)(x+y)},$$

which is now convergent for the integral over x (it varies as λ^{-1} by power counting). If we make the same subtraction while fixing the variable x we obtain, subtracting both counterterms:

$$\begin{aligned} g(x,y,c) &= f(x,y,c) - \frac{y}{x(y+c)} - \frac{x}{y(x+c)} \\ &= -\frac{x^2y^2 + xy^3 + yx^3 + cy^3 + cx^3 + cyx^2 + cxy^2}{x(x+c)(x+y)(y+c)y}. \end{aligned}$$

The result is disappointing, because $g(x,y,c)$ is now linearly divergent if x is multiplied by λ, if y is multiplied by λ and if x and y are both multiplied by λ. In other words, $g(x,y,c)$ is still more divergent than $f(x,y,c)$. The miracle happens when we subtract the global linear divergence of $g(x,y,c)$. The final term

$$\begin{aligned} \bar{f}(x,y,c) &= g(x,y,c) - g(x,y,0) - c\frac{\partial g(x,y,0)}{\partial c} \\ &= -c^2 \frac{xy+cx+cy}{x(x+c)(x+y)(y+c)y}, \end{aligned}$$

is now absolutely convergent for x, for y and for x, y.

2.2 The extension of distributions

From a mathematical point of view, renormalization theory can be considered as the problem of extending a distribution to a larger domain.

The standard example is $1/x$. If $\phi(x)$ is a test function that vanishes at 0, then

$$\int_{-\infty}^{\infty} \mathrm{d}x \frac{\phi(x)}{x}$$

exists. The question is how is it possible to extend this distribution to general test functions. The existence of this extension is ensured by the Hahn-Banach theorem [20] and a formula for such extensions is

$$\int_{|x|<a} \mathrm{d}x \frac{\phi(x)-\phi(0)}{x} + \int_{|x|>a} \mathrm{d}x \frac{\phi(x)}{x},$$

for any positive parameter a. Hence various extensions are possible, that are parametrized by a. Notice that the difference between two such integrals for a and a' is $(\log a' - \log a)\phi(0)$. Therefore, as distributions, two extensions of $1/x$ differ by $\log \Lambda \delta(x)$ for some Λ. The peculiarity of quantum field theory is that Λ can be determined by experiment.

The mathematical conditions for the existence of such an extension were investigated by Malgrange [21] and Estrada [22].

This extension method can be used to calculate, in some cases, the product of two distributions. For instance, by Fourier transform, it can be shown that $\delta(x-a)\delta(x) = \delta(a)$, for $a \neq 0$. However, if $a = 0$, the Fourier transform of the product diverges. This is exactly the same type of divergence that is met in the usual presentation of renormalization. The product of distributions $\delta(x)^2$ is zero for $x \neq 0$, thus a possible extension is $\delta(x)^2 = C\delta(x)$, where C is a constant determined by experiment.

In quantum field theory, causality, Poincaré invariance and unitarity were used by Stuekelberg and coll. to provide a prescription to carry out this extension [23–26]. Bogoliubov and coll. systematized this construction [17, 27–29], which took its final form with Epstein and Glaser [30]. Nowadays, the extension method is called the "causal approach", and the case of QED is treated in detail in Ref.[31].

A (correct) proof of the validity of Bogoliubov's method was finally given by Hepp [18] in 1966 and by Zimmermann [32] in 1969.

Recently, the causal approach has been reintepreted in terms of microlocal analysis [33]. This enabled these authors to adapt the causal approach to quantum field theory in curved spacetime.

An up to date and clear presentation of the causal approach can be found in Ref.[34].

2.3 The product of distributions

The most radical approach to renormalization would be to define a product of distributions, which could lead to a nonlinear theory of distributions. Schwartz has shown that this is impossible in general [35], but the notion of distribution can be extended to a more general kind of functions which can be multiplied. For a comparison with experimental results, we must project these new functions back onto standard distributions.

This approach was investigated by various authors [36–44]. The main drawback of these new generalized functions is that they lead to very intricate calculations. For instance, it is not difficult to show that [45]

$$\left(\frac{1}{x}\right)^2 = \frac{1}{x^2} + \pi^2\delta(x)^2,$$

but the computation of $(1/x)^3$ is already intractable. To understand this striking identity, we start from the continuous function $f(x) = x\log|x| - x$, which defines a distribution by $\int dx f(x)\phi(x)$ for any test function $\phi(x)$. Then the distribution $1/x$ is defined as d^2f/dx^2, the distribution $1/x^2$ is $-d^3f/dx^3$, and $(1/x)^2$ is the product of the distribution $1/x$ with itself.

In spite of their complexity, these new generalized functions have found some applications in physics [46–49]. For instance, a definite value could be given to the curvature of a cone at its apex [47].

Notice that, as for the extension of distributions, microlocal analysis is taking a growing importance in the study of the new generalized functions [50].

3 Renormalization of QED

QED was renormalized to all orders by Dyson [14]. We can now interpret his prescriptions in the framework of the Schwinger equations. It is standard to define free, bare and renormalized propagators. The free electron Green function $S^0(q)$ is the Green function for an electron without electromagnetic interaction. The bare electron Green function $S(q)$ is the Green function for an electron with electromagetic interaction, but without renormalization. In the perturbation expansion of $S(q)$, all terms (except the first one) are infinite. The renormalized Green function $\bar{S}(q)$ is the Green function for an electron with electromagnetic interaction, after renormalization. Similarly we define $D^0(q)$, $D(q)$ and $\bar{D}(q)$ as the free, bare and renormalized photon Green functions. [1]

3.1 The free propagators

The free electron propagator is

$$S^0(q) = (\gamma\cdot q - m + i\epsilon)^{-1}.$$

The scalar product is defined by

$$\gamma\cdot q = \sum_{\lambda\mu}\gamma^\lambda g_{\lambda\mu} q^\mu,$$

where the pseudo-metric tensor $g_{\lambda\mu}$ is

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

All the electron propagators $S^0(q)$, $S(q)$ and $\bar{S}(q)$ are 4×4 complex matrix functions of the 4-vector q. If I is the 2×2 identity matrix and σ_x, σ_y, σ_z the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the Dirac matrices can be written $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$$\gamma^1 = \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}.$$

The free photon propagator $D^0(q)$ is a complex 4×4 matrix with components $D^0_{\mu\nu}(q)$ defined by

$$D^0_{\mu\nu}(q) = -\frac{g_{\mu\nu}}{q^2 + i\epsilon} + (1 - 1/\xi)\frac{q_\mu q_\nu}{(q^2 + i\epsilon)^2}.$$

[1] Strictly speaking, a propagator is a one-particle Green function, but in this paper propagator and Green function will be used indifferently. For simplicity, fermions (electrons+positrons) are called electrons.

The term $1/\xi$ was introduced by Heisenberg to make $D^0(q)$ non singular. The Green function used in classical electrodynamics is $D^{0T}(q)$ defined as

$$D^{0T}_{\mu\nu}(q) = -\frac{g_{\mu\nu}}{q^2+i\epsilon} + \frac{q_\mu q_\nu}{(q^2+i\epsilon)^2}.$$

A tensor $T_{\mu\nu}(q)$ such that $q^\mu T_{\mu\nu}(q)=0$ is called transverse. It can be checked that $D^{0T}_{\mu\nu}(q)$ is transverse, and non singular in the space of transverse tensors.

Up to an eventual factor i, the expressions for the free Green functions $S^0(q)$ and $D^0_{\mu\nu}(q)$ are standard (see, e.g. Ref.[2] p.93 and p.36, Ref.[51] p.184 and p.190, Ref.[52] p.218 and p.253, for a complete description and a derivation).

3.2 The bare propagators

The Schwinger equations for bare electron and photon propagators were given by Bogoliubov and Shirkov [29] and transformed into the following integral equations in [10].

$$\begin{aligned} S(q) &= S^0(q) \\ &\quad + ie_0^2 S^0(q)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda D_{\lambda\lambda'}(p)\frac{\delta S(q-p)}{e_0\delta A^0_{\lambda'}(p)}, \quad (1)\\ D_{\mu\nu}(q) &= D^0_{\mu\nu}(q) \\ &\quad - ie_0^2 D^0_{\mu\lambda}(q)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\frac{\delta S(p)}{e_0\delta A^0_{\lambda'}(-q)}\big]D_{\lambda'\nu}(q), \quad (2)\end{aligned}$$

where $A^0_\lambda(p)$ is an external electromagnetic field and $\frac{\delta S(q)}{\delta A^0_\lambda(p)}$ is the functional derivative evaluated at $A^0_\lambda(p)=0$.

The longitudinal part of the bare photon Green function is not modified by the interaction [2], and $D_{\lambda\mu}(q)$ can be written as the sum of its transverse and its longitudinal parts:

$$D_{\lambda\mu}(q) = D^T_{\lambda\mu}(q) - \frac{1}{\xi}\frac{q_\lambda q_\mu}{(q^2+i\epsilon)^2}, \quad (3)$$

where $D^T_{\lambda\mu}(q)$ is transverse. In Eq.(2), the photon propagator $D_{\lambda'\nu}(q)$ is not integrated, it just multiplies the integral. This is not very convenient and it will be useful to introduce the bare vacuum polarization, denoted $\Pi_{\lambda\mu}(q)$ and defined by

$$[D^{-1}]_{\lambda\mu}(q) = (q_\lambda q_\mu - q^2 g_{\lambda\mu}) - \xi q_\lambda q_\mu + \Pi_{\lambda\mu}(q). \quad (4)$$

The vacuum polarization tensor $\Pi_{\lambda\mu}(q)$ is transverse [2]. If we multiply (4) by (3), we obtain

$$g_\lambda^{\ \nu} - \frac{q_\lambda q^\nu}{q^2} = \big((q_\lambda q_\mu - q^2 g_{\lambda\mu}) + \Pi_{\lambda\mu}(q)\big)D^{T\mu\nu}(q). \quad (5)$$

The left-hand side of Eq.(5) is the projector onto the transverse tensors.

We show in section 13.2 that

$$\Pi^{\lambda\mu}(q) = ie_0^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\frac{\delta S(p)}{e_0\delta A^0_\mu(-q)}\big]. \quad (6)$$

3.3 Renormalized propagators

To obtain the Schwinger equations for the renormalized propagators, the best is to start from the renormalized Lagrangian, and to follow the steps given by Bogoliubov and Shirkov [29], Itzykson and Zuber [2] or Rochev [53]. However, to give an idea of the result, we introduce some of Dyson's recipes.

The longitudinal part of the photon Green function is not modified by renormalization [2], and the renormalized photon propagator can be decomposed as

$$\bar D_{\lambda\mu}(q) = \bar D^T_{\lambda\mu}(q) - \frac{1}{\xi}\frac{q_\lambda q_\mu}{(q^2+i\epsilon)^2}, \quad (7)$$

where $\bar D^T_{\lambda\mu}(q)$ is transverse.

Then, we introduce Dyson's relation between renormalized and bare Green functions (Ref.[2] p.414):

$$\bar S(q)Z_2 = S(q), \quad (8)$$
$$Z_3\bar D^T_{\mu\nu}(q) = D^T_{\mu\nu}(q), \quad (9)$$
$$Z_3 e_0^2 = e^2, \quad (10)$$
$$m_0 = m - \delta m, \quad (11)$$

where Z_2 and Z_3 are (infinite) scalars independent of q, and e is the renormalized charge. Equation (10) was conjectured by Dyson [14] and proved by Ward ([2], p.413). Finally, the external field A^0_λ is renormalized as A_λ, so that

$$e_0 A^0_\lambda = eA_\lambda. \quad (12)$$

To introduce the mass renormalization, we must start from the differential form of the Schwinger equation for the bare electron propagator, where we reintroduce the external field, for later convenience,

$$\begin{aligned}&[i\gamma\cdot\partial - m_0 - e_0\gamma\cdot A^0(x)]S(x,y;A^0) = \delta(x-y) + \\ &\qquad ie_0^2\int d^4z\,\gamma^\mu D_{\mu\rho}(x,z;A^0)\frac{\delta S(x,y;A^0)}{\delta e_0 A^0_\rho(z)}. \quad (13)\end{aligned}$$

Dyson showed that relations (8-12), are valid in the presence of an external field. We use them in Eq.(13) to obtain

$$\begin{aligned}&[i\gamma\cdot\partial - m]\bar S(x,y;A)Z_2 = \delta(x-y) - \delta m S(x,y;A)Z_2 \\ &\qquad + ie^2\int d^4z\,\gamma^\mu \bar D_{\mu\rho}(x,z;A)\frac{\delta S(x,y;A)}{\delta eA_\rho(z)}Z_2. \quad (14)\end{aligned}$$

In Eq.(14), we have changed the gauge parameter ξ_0 of $D_{\mu\rho}(x,z;A^0)$ into $\xi = Z_3\xi_0$ (see Ref.[2] p. 414).

If we multiply Eq.(14) by

$$S^0(z,y;A) = [i\gamma\cdot\partial - m - e\gamma\cdot A]^{-1} \quad (15)$$

and integrate over x we obtain the integral Schwinger equation for the renormalized electron propagator,

$$\begin{aligned}\bar S(x,y;A)Z_2 &= S^0(x,y;A) \\ &\quad + ie^2\int d^4z d^4z'\,S^0(x,z;A)\gamma^\lambda\bar D_{\lambda\lambda'}(z,z';A)\frac{\delta\bar S(z,y;A)}{e\delta A_{\lambda'}(z')}Z_2 \\ &\quad - \delta m\int d^4z S^0(x,z;A)\bar S(z,y;A)Z_2. \quad (16)\end{aligned}$$

In Eq.(16), we put $A = 0$ and we Fourier transform to find

$$\bar{S}(q)Z_2 = S^0(q) - \delta m S^0(q)\bar{S}(q)Z_2 + ie^2 S^0(q)\int \frac{\mathrm{d}^4 p}{(2\pi)^4}\gamma^\lambda \bar{D}_{\lambda\lambda'}(p)\frac{\delta\bar{S}(q-p)}{e\delta A_{\lambda'}(p)}Z_2. \quad (17)$$

This equations was given by Bogoliubov and Shirkov [29], as well as by Itzykson and Zuber ([2], p.481), except for the mass counterterm δm which was apparently overlooked by these authors. A complete derivation can be found in [53] (notice that his δm is our $Z_2\delta m$).

To obtain a convenient Schwinger equation for the renormalized photon propagator, we must introduce the renormalized vacuum polarization $\bar{\Pi}_{\lambda\mu}(q)$, defined by

$$[\bar{D}^{-1}]_{\lambda\mu}(q) = (q_\lambda q_\mu - q^2 g_{\lambda\mu}) - \xi q_\lambda q_\mu + \bar{\Pi}_{\lambda\mu}(q). \quad (18)$$

It may be useful to compare these definitions to those of Itzykson and Zuber [2]: $\bar{D}_{\mu\nu} = -i\bar{G}_{\mu\nu}$, $\bar{\Pi}_{\mu\nu} = -i\bar{\omega}_{\mu\nu}$.

If we multiply (18) by (7), we obtain

$$g_\lambda^{\ \nu} - \frac{q_\lambda q^\nu}{q^2} = \big((q_\lambda q_\mu - q^2 g_{\lambda\mu}) + \bar{\Pi}_{\lambda\mu}(q)\big)\bar{D}^{T\,\mu\nu}(q). \quad (19)$$

If we compare Eqs.(5) and (19), and use (9), we find

$$q_\lambda q_\mu - q^2 g_{\lambda\mu} + \bar{\Pi}_{\lambda\mu}(q) = Z_3(q_\lambda q_\mu - q^2 g_{\lambda\mu} + \Pi_{\lambda\mu}(q)). \quad (20)$$

Therefore, using Eqs. (8) and (12)

$$\Pi^{\lambda\mu}(q) = ie_0^2 Z_2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda \frac{\delta\bar{S}(p)}{e\delta A_\mu(-q)}\big].$$

Introducing this equation into (20), and using Eq.(10) we obtain

$$\bar{\Pi}_{\lambda\mu}(q) = (Z_3 - 1)(q_\lambda q_\mu - q^2 g_{\lambda\mu}) + Z_2 ie^2 \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\mathrm{tr}\big[\gamma_\lambda \frac{\delta\bar{S}(p)}{e\delta A^\mu(-q)}\big]. \quad (21)$$

Equations (17) and (21) will be the bases of recursive expressions for the renormalized electron and photon propagators.

4 Tree expansion of propagators

For the convenience of the reader, and because the notation of Ref. [10] as been modified [2], we recall the description of photon and electron propagators in terms of planar binary trees.

[2] There are no longer trees with black or white roots. The color of the root is now indicated by the function φ itself. $\varphi(t;q)$ corresponds to a tree with a black root, $\varphi_{\mu\nu}(t;q)$ to a tree with a white root. This notation is more elegant than the one used in [10].

4.1 Trees and propagators

The main trick of Ref.[10] was to write each propagator as a sum indexed by planar binary trees, to be defined in the next section.

The bare electron Green function in Fourier space, $S(q)$ is written as a sum over planar binary trees t

$$S(q) = \sum_t e_0^{2|t|}\varphi^0(t;q). \quad (22)$$

Here e_0 is the bare electron charge (i.e. the electron charge before renormalization). The fact that the expansion is over e_0^2 (and not e_0) was justified in Ref.[10]. Similarly, the renormalized electron Green function is expanded over planar binary trees

$$\bar{S}(q) = \sum_t e^{2|t|}\bar{\varphi}^0(t;q). \quad (23)$$

In Eq.(23) e is the renormalized (finite) electron charge.

The bare and renormalized photon Green functions are written as

$$D_{\mu\nu}(q) = \sum_t e_0^{2|t|}\varphi^0_{\mu\nu}(t;q),$$
$$\bar{D}_{\mu\nu}(q) = \sum_t e^{2|t|}\bar{\varphi}^0_{\mu\nu}(t;q). \quad (24)$$

For the renormalization of the photon Green function and the vacuum polarization, it will be necessary to distinguish the photon Green function $D_{\mu\nu}(q)$ and the transverse photon Green function $D^T_{\mu\nu}(q)$. Since all terms $\varphi^0_{\mu\nu}(t;q)$ and $\bar{\varphi}^0_{\mu\nu}(t;q)$ are transverse for $t \neq \shortmid$, the transverse renormalized propagator is

$$\bar{D}^T_{\mu\nu}(q) = \varphi^T_{\mu\nu}(\shortmid;q) + \sum_{|t|>0} e^{2|t|}\bar{\varphi}^0_{\mu\nu}(t;q),$$

where $\varphi^T_{\mu\nu}(\shortmid;q) = D^{0T}_{\mu\nu}(q)$.

The bare and renormalized vacuum polarization are expanded similarly:

$$\Pi_{\lambda\mu}(q) = \sum_{|t|>0} e_0^{2|t|}\psi^0_{\lambda\mu}(t;q), \quad (25)$$
$$\bar{\Pi}_{\lambda\mu}(q) = \sum_{|t|>0} e^{2|t|}\bar{\psi}^0_{\lambda\mu}(t;q). \quad (26)$$

For later convenience, we finally define

$$\psi^0_{\lambda\mu}(\shortmid;q) = q_\lambda q_\mu - q^2 g_{\lambda\mu},$$

so that

$$[D^{0^{-1}}]_{\lambda\mu}(q) = (q_\lambda q_\mu - q^2 g_{\lambda\mu}) - \xi q_\lambda q_\mu \quad (27)$$
$$= \psi^0_{\lambda\mu}(\shortmid;q) - \xi q_\lambda q_\mu. \quad (28)$$

A few identities will be useful in the sequel

$$D^0_{\lambda\mu}(q)(q^\mu q_\nu - q^2 g^\mu_{\ \nu}) = -q^2 D^{0T}_{\lambda\nu}(q),$$
$$\psi^0_{\lambda\lambda'}(\shortmid;q)D^{0\lambda'\mu'}(q)\psi^0_{\mu'\mu}(\shortmid;q) = \psi^0_{\lambda\mu}(\shortmid;q),$$
$$D^0_{\lambda\lambda'}(q)\psi^{0\lambda'\mu'}(\shortmid;q)D^0_{\mu'\mu}(q) = D^{0T}_{\lambda\mu}(q).$$

Notice that $-q^2 D^{0T}_{\lambda\nu}(q)$ is the projector onto the transverse tensors.

The tree representation of the photon propagator enjoys the following property [10]

$$\varphi^0_{\mu\nu}(t_l \vee t_r; q) = \varphi^0_{\mu\lambda}(| \vee t_r; q)[(D^0)^{-1}]^{\lambda\lambda'}(q)\, \varphi^0_{\lambda'\nu}(t_l; q).$$

This equality is non trivial only if $t_l \neq |$. But then $\varphi^0_{\mu\lambda}(| \vee t_r; q)$ and $\varphi^0_{\lambda'\nu}(t_l; q)$ are transverse [10], this cancels the term proportional to ξ and we can write

$$\varphi^0_{\mu\nu}(t_l \vee t_r; q) = \varphi^0_{\mu\lambda}(| \vee t_r; q)\psi^{0\lambda\lambda'}(|; q)\varphi^0_{\lambda'\nu}(t_l; q) \quad (29)$$

The fact that $\varphi^0_{\mu\nu}(t_l \vee t_r; q)$ is a product for $t_l \neq |$ can be checked in appendix 3.

As shown in section 13.2, $\psi^0_{\mu\nu}(t_l \vee t_r) = 0$ if $t_l \neq |$ and

$$\psi^0_{\mu\nu}(| \vee t_r; q) = -\psi^0_{\mu\lambda}(|; q)\varphi^{0\lambda\lambda'}(| \vee t_r; q)\psi^0_{\lambda'\nu}(|; q) \quad (30)$$

4.2 Trees and renormalization constants

According to Dyson's multiplicative renormalization of QED, we define three renormalization constants Z_2, Z_3 and δm. We expand these constants over planar binary trees

$$Z_2 = \sum_t e^{2|t|}\zeta_2(t), \quad \text{with} \quad \zeta_2(|) = 1, \quad (31)$$

$$Z_3 = \sum_t e^{2|t|}\zeta_3(t), \quad \text{with} \quad \zeta_3(|) = 1, \quad (32)$$

$$\delta m = \sum_t e^{2|t|}\zeta_m(t), \quad \text{with} \quad \zeta_m(|) = 0. \quad (33)$$

These expansions will be the basis of the tree by tree renormalization of QED.

4.3 Planar binary trees

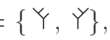
A planar binary tree is a tree with a designated vertex called the root. To follow the notation of Loday and Ronco [54], we write the root vertex as $|$. The other vertices are not explicitly drawn, but they are at the ends of each edge, which are

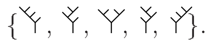
or /. The trees are binary because each vertex has either zero or two children. They are planar because  is different from . The planar binary trees have an odd number of vertices and for each tree t we define $|t|$ as the integer such that t has $2|t|+1$ vertices. In other words, $|t|$ is the number of internal vertices. We call Y_n the set of planar binary trees t with such that $|t| = n$.

The planar binary trees with up to 7 vertices are

$Y_0 = \{|\}$,

$Y_1 = \{Y\}$,

$Y_2 =$ ,

$Y_3 =$ .

We denote Y the set of all planar binary trees

$$Y = \bigcup_{n=0}^{\infty} Y_n.$$

Finally we consider the operation of *grafting* two trees, $\vee : Y_p \times Y_q \longrightarrow Y_{p+q+1}$, by which the roots of two trees t_1 and t_2 are joined into a new vertex that becomes the root of the tree $t = t_1 \vee t_2$, cf [54]. For instance

$Y \vee Y =$ . (34)

It is clear that any tree t, except the 0-tree $|$, is the grafting of two uniquely determined trees t_l and t_r with orders $|t_l|, |t_r| \leq |t| - 1$.

4.4 Recursive equations for bare propagators

In Ref.[10], we have obtained recursive relations for $\varphi(t)$ and $\varphi_{\lambda\mu}(t)$.

For the electron propagator, $\varphi(t)$ satisfies the recursive relation

$$\begin{aligned}\varphi^n(t; q; \{\lambda, p\}_{1,n}) &= S^0(q)\gamma^{\lambda_1} \\ &\times \varphi^{n-1}(t; q+p_1; \{\lambda, p\}_{2,n}) \\ &+ i\sum_{k=0}^{n} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} S^0(q)\gamma^{\lambda}\varphi^k_{\lambda\lambda'}(t_l; p; \{\lambda, p\}_{1,k}) \\ &\times \varphi^{n-k+1}_{\Sigma}(t_r; q-p; \lambda', p+P_k, \{\lambda, p\}_{k+1,n}), \quad (35)\end{aligned}$$

where we have noted $P_k = p_1 + \cdots + p_k$, $(P_0 = 0)$ and $\{\lambda, p\}_{1,n} = \lambda_1, p_1, \ldots, \lambda_n, p_n$. The initial data are

$$\begin{aligned}\varphi^0(|; q) &= S^0(q), \\ \varphi^1(|; q; \lambda_1, p_1) &= S^0(q)\gamma^{\lambda_1}S^0(q+p_1), \\ \varphi^n(|; q; \{\lambda, p\}_{1,n}) &= S^0(q)\gamma^{\lambda_1}S^0(q+p_1)\gamma^{\lambda_2}\cdots\gamma^{\lambda_n} \\ &\quad S^0(q+p_1+\cdots+p_n). \quad (36)\end{aligned}$$

The symbol $\varphi^{n+1}_{\Sigma}(t; q; z, \{z\}_{1,n})$ is defined as the sum of n terms, where the first variable $z = (\lambda, p)$ is exchanged in turn with all the variables $z_i = (\lambda_i, p_i)$.

$$\begin{aligned}\varphi^{n+1}_{\Sigma}(t; q; z, \{z\}_{1,n}) &= \varphi^{n+1}(t; q; z, \{z\}_{1,n}) \\ &+ \varphi^{n+1}(z_1, z, \{z\}_{2,n}) + \cdots + \varphi^{n+1}(\{z\}_{1,n-1}, z, z_n) \\ &+ \varphi^{n+1}(\{z\}_{1,n}, z).\end{aligned}$$

For the photon propagator, $\varphi_{\mu\nu}(t)$ satisfies the recursive relation

$$\begin{aligned}\varphi^n_{\mu\nu}(t; q; \{\lambda, p\}_{1,n}) &= -i\sum_{k=0}^{n}\int \frac{\mathrm{d}^4 p}{(2\pi)^4} D^0_{\mu\lambda}(q) \\ &\times \mathrm{tr}\big[\gamma^{\lambda}\varphi^{k+1}_{\Sigma}(t_r; p; \lambda', -q-P_k, \{\lambda, p\}_{1,k})\big] \\ &\times \varphi^{n-k}_{\lambda'\nu}(t_l; q+P_k; \{\lambda, p\}_{k+1,n}), \quad (37)\end{aligned}$$

with the initial data

$$\begin{aligned}\varphi^0_{\mu\nu}(|; q) &= D^0_{\mu\nu}(q), \\ \varphi^n_{\mu\nu}(|; q; \{\lambda, p\}_{1,n}) &= 0 \quad \text{for} \quad n \geq 1.\end{aligned}$$

In this paper, we use a non symmetrized $\varphi^n(|)$. However, for the validity of Ward-Takahashi identities, it would be necessary to handle symmetrized expressions

$$\frac{1}{n!}\sum_{\sigma\in\mathcal{S}_n}\varphi^n\big(|;q;\lambda_{\sigma(1)},p_{\sigma(1)},\ldots,\lambda_{\sigma(n)},p_{\sigma(n)}\big).$$

Such expressions seem complicated. It is possible that recent string-like methods help handling them [55,56].

4.5 The pruning operator

In this section, we introduce the pruning operator P which will prove very useful to obtain a recursive expression for renormalized propagators. If t is a tree, $P(t)$ is a sum of $n(t)$ terms of the form $u_j\otimes v_j$, where u_j and v_j are planar binary trees. More mathematically

$$P(t)=\sum_{j=1}^{n(t)}u_j\otimes v_j. \tag{38}$$

Before we fully define $P(t)$, we want to show why it is useful. If, for each tree t, $\varphi(t)$ and $\psi(t)$ are 4×4 complex matrices, we call the convolution of φ and ψ the quantity

$$(\varphi\star\psi)(t)=\sum_{i=1}^{n(t)}\varphi(u_i)\psi(v_i).$$

The main property of this convolution was established in Ref.[10]. If

$$X(\lambda)=\sum_t\lambda^{|t|}x(t)\quad\text{and}\quad Y(\lambda)=\sum_t\lambda^{|t|}y(t),$$

with $x(|)=y(|)=0$, then

$$X(\lambda)Y(\lambda)=\sum_t\lambda^{|t|}(x\star y)(t).$$

In other words, the pruning operator and the convolution enable us to multiply series indexed by planar binary trees.

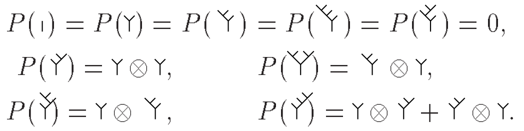
This nice property justifies the trouble of introducing $P(t)$. First, $n(t)$, the number of terms in Eq.(38), is defined by $n(|)=0$ and

$$\begin{aligned}&n(t)=0\quad\text{if}\quad t=t_l\vee|,\\&n(t)=n(t_r)+1\quad\text{if}\quad t=t_l\vee t_r,\quad t_r\neq|.\end{aligned}\tag{39}$$

Finally, $P(t)$ is determined recursively by $P(|)=0$ and

$$\begin{aligned}&P(t)=0\quad\text{if}\quad t=t_l\vee|,\\&P(t)=(t_l\vee|)\otimes t_r+\sum_{j=1}^{n(t_r)}(t_l\vee u_j)\otimes v_j\\&\qquad\qquad\text{if}\quad t=t_l\vee t_r,\quad t_r\neq|.\end{aligned}\tag{40}$$

The trees u_j and v_j in Eq.(40) are generated by Eq.(38) for $t=t_r$. For instance



We show in the appendix that the pruning operator is coassociative, that is

$$(P\otimes id)\otimes P=(id\otimes P)\otimes P. \tag{41}$$

Therefore the convolution is associative.

We consider on trees the structure of an associative algebra $T(Y)$ given by the (non commutative) tensor product, $T(Y)=Y\oplus Y^{\otimes 2}\oplus Y^{\otimes 3}\oplus\ldots$, and we set the root $|$ as the unit: $|\otimes t=t\otimes|=t$. Then we extend P to $T(Y)$ as a multiplicative map, but P does not preserve the unit, since $P(|)$ is not equal to $|\otimes|$. We can define a coproduct

$$\begin{aligned}\Delta^P t&=|\otimes t+P(t)+t\otimes|,\\\Delta^P|&=|\otimes|.\end{aligned}$$

This Δ^P is the coproduct of a Hopf algebra over planar binary trees. Its antipode is given by the recursive formula

$$S_\star(t)=-t-(\mathrm{Id}\star S_\star)(t)=-t-(S_\star\star\mathrm{Id})(t), \tag{42}$$

for $t\neq|$, and $S_\star(|)=|$.

To define the convolution of $x(t)$ and $y(t)$, we needed the condition $x(|)=y(|)=0$. When this condition is not satisfied, we have two solutions. The first solution is to isolate the root, so that

$$\begin{aligned}X(\lambda)Y(\lambda)&=x(|)y(|)+(X(\lambda)-x(|))y(|)\\&\quad+x(|)(Y(\lambda)-y(|)+(X(\lambda)-x(|))(Y(\lambda)-y(|))\\&=x(|)y(|)+x(|)\sum_{|t|>0}\lambda^{|t|}y(t)\\&\quad+\sum_{|t|>0}\lambda^{|t|}x(t)y(|)+\sum_{|t|>0}\lambda^{|t|}(x\star y)(t).\end{aligned}$$

The second solution is to use the coproduct Δ^P. Thus, we define the convolution ∇ by:

$$(x\nabla y)(t)=\sum_j x(u_j)y(v_j),\ \text{where}\ \Delta^P(t)=\sum_j u_j\otimes v_j.$$

With this alternative convolution, the equality

$$X(\lambda)Y(\lambda)=\sum_t\lambda^{|t|}(x\nabla y)(t) \tag{43}$$

is satisfied even if x or y is not zero on the root.

In our final formulas, we prefer to use the convolution $\star$ because its ensures the recursivity of the expressions (the trees in $(x\star y)(t)$ are strictly smaller than t).

A last point of notation. If φ and ψ depend on other arguments, we leave them inside φ and ψ. For example

$$(\varphi(q)\star\psi(q))(t)=\sum_{j=1}^{n(t)}\varphi(u_j;q)\psi(v_j;q).$$

4.6 The self-energy

As a first application of the convolution defined in the previous section, we introduce the tree-expansion for the electron self-energy.

The bare electron self-energy $\Sigma(q)$ is defined by:

$$S^{-1}(q) = \gamma\cdot q - m - \Sigma(q) = -\sum_t e_0^{2|t|}\psi^0(t;q),$$

where

$$\psi^0(|;q) = -(\gamma\cdot q - m) \quad \text{and} \quad \Sigma(q) = \sum_{|t|>0} e_0^{2|t|}\psi^0(t;q),$$

so that $\psi^0(|;q)\varphi^0(|;q) = -1$. The pruning operator is used to define the expansion of the bare self-energy over trees in terms of the expansion of the bare electron Green function over trees:

$$\psi^0(t) = \psi^0(|)\varphi^0(t)\psi^0(|) + \psi^0(|)(\varphi^0\star\psi^0)(t), \quad (44)$$
$$\varphi^0(t) = \varphi^0(|)\psi^0(t)\varphi^0(|) + (\varphi^0\star\psi^0)(t)\varphi^0(|). \quad (45)$$

In terms of the antipode, (44) can be rewritten

$$\psi^0(t) = -\psi^0(|;q)(\varphi^0(q)\circ S_\star)(t)\psi^0(|;q). \quad (46)$$

Similarly, for the renormalized self-energy, we have

$$\bar\psi^0(t) = -\psi^0(|;q)(\bar\varphi^0(q)\circ S_\star)(t)\psi^0(|;q). \quad (47)$$

We must give some detail concerning the meaning of expressions like $(\varphi^0(q)\circ S_\star)(t)$. Because of its definition (42), the antipode $S_\star$ acting on t generates a sum of products of trees. The action of $\varphi^0(q)$ on this sum is prolonged from its action on Y to an algebra homomorphism over $T(Y)$. In other words

$$\varphi^0(q)(t_1+t_2) = \varphi^0(t_1;q) + \varphi^0(t_2;q),$$
$$\varphi^0(q)(\lambda t) = \lambda\varphi^0(t;q).$$

For a product of trees, we do not want to simply multiply two Feynman diagrams for the electron propagator, we must cancel one of the free propators between them. Thus, the product is

$$\varphi^0(q)(t_1t_2) = -\varphi^0(t_1;q)\psi^0(|;q)\varphi^0(t_2;q). \quad (48)$$

This operation becomes clear if one tries it on some examples given in appendix 3. Since the (matrix) product on the right-hand side of Eq.(48) is not commutative, the algebra product on trees is not commutative either.

In the presence of an external field A, the definition (15) of $S^0(z,y;A)$ gives, after inversion and Fourier transform,

$$\psi^0(|;q;A) = -(\gamma\cdot q - m - e\gamma\cdot A(q)) \quad (49)$$

Thus we obtain, at $A^0=0$

$$\psi^0(|;q) = -(\gamma^\alpha q_\alpha - m),$$
$$\psi^1(|;q;\lambda,p) = \gamma^\lambda,$$
$$\psi^n(|;q;\{\lambda,p\}_{1,n}) = 0 \quad \text{for} \quad n>1.$$

The components of $\psi^0(t;q)$ for the other trees t are obtained by using the chain rule for the functional derivative of (44) with respect to $e_0A^0_{\lambda_i}(p_i)$, taken at $A^0=0$. For $n=1$, this gives the same result as in sect. 6.4 of [10].

Finally it is shown in section 13.3 that the bare self-energy can be calculated from the recursive equation

$$\psi^0(t;q) = i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda\varphi^0_{\lambda\lambda'}(t_l;p)g(t_r;q-p;\lambda',p), \quad (50)$$

where

$$g(t_r;q-p;\lambda',p) = -(\varphi^1(q-p;\lambda',p)\nabla\psi^0(q))(t_r)$$
$$= (\varphi^0(q)\nabla\psi^1(q-p;\lambda',p))(t_r).$$

4.7 The higher components $\varphi^n(t)$

For a complete recursive solution of the renormalized propagators, we must define the higher components $\bar\varphi^n(t)$ and $\bar\varphi^n_{\mu\nu}(t)$. As for the bare propagators (see Ref. [10]), they are defined as the functional derivative with respect to an external electromagnetic field.

Let us be more accurate concerning this external field. As noticed by Bogoliubov and Shirkov, Eq.(16) is not the Schwinger equation for QED with an external electromagnetic field, since the latter involves tadpole diagrams which are absent from Eq.(16). However, Eq.(16) is the Schwinger equation for a renormalizable theory (i.e. QED without tadpoles), and Dyson's relations (8-12) still hold.

In the real space, the bare and renormalized electron Green functions are expanded as

$$S(x,y;A) = \sum_t e_0^{2|t|}\varphi^0(t;x,y;A),$$
$$\bar S(x,y;A) = \sum_t e^{2|t|}\bar\varphi^0(t;x,y;A).$$

In these expressions, we do not distinguish between A and A^0 because A^0 comes always multiplied by e_0 and $e_0A^0 = eA$. On the root, we have $\varphi^0(|;x,y;A) = S^0(x,y;A)$.

The higher components of $\bar\varphi^0(t;x,y;A)$ must satisfy

$$\frac{\delta}{e\delta A_\lambda(z)}\varphi^n(t;x,y;\{\lambda,z\}_{1,n};A) =$$
$$\varphi^{n+1}_\Sigma(t;x,y;\lambda,z,\{\lambda,z\}_{1,n};A).$$

where the notation φ^{n+1}_Σ and $\{\lambda,z\}_{1,n}$ is defined in section 4.4.

Since our purpose is QED without external field, A is just used to take functional derivatives, and the higher components we actually need are

$$\varphi^n(t;x,y;\{\lambda,z\}_{1,n}) = \varphi^n(t;x,y;\{\lambda,z\}_{1,n};A)$$

for $A=0$.

At $A=0$, the theory becomes translational invariant, and a Fourier transform gives us

$$\frac{\delta}{e\delta A_\lambda(p)}\varphi^n(t;q;\{\lambda,p\}_{1,n}) = \varphi^{n+1}_\Sigma(t;q;\lambda,p,\{\lambda,p\}_{1,n}).$$

In the recursive equations for $\varphi(t)$ we meet products of propagators such as $\varphi^0(t_1;q)\varphi^0(t_2;q)$. In the real space, this gives a convolution of $\varphi^0(t_1;x,y)$ and $\varphi^0(t_2;x,y)$. We take the functional derivative of this convolution with respect to $A(z)$, we Fourier transform the result and we obtain, at $A=0$

$$\frac{\delta}{e\delta A_\lambda(p)}\big(\varphi^0(t_1;q)\varphi^0(t_2;q)\big) = \varphi^0(t_1;q)\varphi^1(t_2;q;\lambda,p) + \varphi^1(t_1;q;\lambda,p)\varphi^0(t_2;q+p).$$

This expression satisfies energy-momentum conservation.

To take the functional derivatives of the recursive equations for renormalized quantities, we need the independence of the renormalization constants with respect to the external field. There are various ways to prove this. For instance, the differential form of the Ward identity (Eq.(21) in Ref.[10] is

$$\frac{\partial\varphi^n(t;q;\{\lambda,p\}_{1,n})}{\partial q^\mu} = -\varphi_\Sigma^{n+1}(t;q;\mu,0,\{\lambda,p\}_{1,n}).$$

The Ward identity is also valid for the renormalized electron propagator [57], so that

$$\frac{\partial\bar\varphi^n(t;q;\{\lambda,p\}_{1,n})}{\partial q^\mu} = -\bar\varphi_\Sigma^{n+1}(t;q;\mu,0,\{\lambda,p\}_{1,n}).$$

From the definitions (46) and (47) of the bare and renormalized self-energies we obtain

$$\frac{\partial\psi^n(t;q;\{\lambda,p\}_{1,n})}{\partial q^\mu} = -\psi_\Sigma^{n+1}(t;q;\mu,0,\{\lambda,p\}_{1,n}),$$
$$\frac{\partial\bar\psi^n(t;q;\{\lambda,p\}_{1,n})}{\partial q^\mu} = -\bar\psi_\Sigma^{n+1}(t;q;\mu,0,\{\lambda,p\}_{1,n}).$$

Now we start from the relation between the renormalized and bare self-energies, for instance

$$\bar\psi^0(\Upsilon;q) = \psi^0(\Upsilon;q) - \zeta_2(\Upsilon)(\gamma\cdot q - m) - \zeta_m(\Upsilon). \quad (51)$$

On the one hand, we take the derivative of Eq.(51) with respect to q_μ and use the Ward identities (and the fact that $\zeta_2(\Upsilon)$ and $\zeta_m(\Upsilon)$ do not depend on q) to obtain

$$-\bar\psi^1(\Upsilon;q;\mu,0) = -\psi^1(\Upsilon;q;\mu,0) - \zeta_2(\Upsilon)\gamma^\mu. \quad (52)$$

On the other hand, we take the functional derivative of Eq.(51) at $A=0$ and we obtain

$$\bar\psi^1(\Upsilon;q;\mu,p) = \psi^1(\Upsilon;q;\mu,p) + \zeta_2(\Upsilon)\gamma^\mu - \zeta_2'(\Upsilon)(\gamma\cdot q - m) \quad (53)$$
$$-\zeta_m'(\Upsilon), \quad (54)$$

where $\zeta_2'(\Upsilon)$ and $\zeta_m'(\Upsilon)$ denote the derivative of $\zeta_2(\Upsilon)$ and $\zeta_m(\Upsilon)$ with respect to $A_\mu(p)$ at $A=0$. The term $\zeta_2(\Upsilon)\gamma^\mu$ comes from the functional derivative of Eq.(49).

If we take the value $p=0$ in Eq.(54) and compare with Eq.(52) we obtain $\zeta_2'(\Upsilon)=0$ and $\zeta_m'(\Upsilon)=0$. Further differentiation shows that $\zeta_2^{(n)}(\Upsilon)=0$ and $\zeta_m^{(n)}(\Upsilon)=0$. We can apply this proof to any tree t, once the subdivergences have been subtracted. A similar proof can be given for $\zeta_3(t)$, using Fury's theorem instead of Ward identities. This proof assumes that the renormalization conditions do not depend on A (e.g. minimal subtraction).

More physically, $S(x,y;A)$ has the same singular structure at $x=y$ as $S^0(x,y)$, except for logarithmic terms that are integrable. Thus, the renormalization constants are determined by $S^0(x,y)$.

This method of functional derivatives avoids the usual renormalization of vertex diagrams. Renormalizing propagators is sufficient.

5 The renormalized electron propagator

In this section, we show that the recursive equation for the electron propagator is

$$\begin{aligned}\bar\varphi^0(t;q) &= \rho(t)\varphi^0(\mid;q) - \zeta_m(t)\varphi^0(\mid;q)^2 \\ &\quad -\varphi^0(\mid;q)(\zeta_m\star\bar\varphi^0(q))(t) \\ &\quad +i\varphi^0(\mid;q)\int\frac{d^4p}{(2\pi)^4}\gamma^\lambda\bar\varphi^0_{\lambda\lambda'}(t_l;p)\bar\varphi^1(t_r;q-p;\lambda',p), \end{aligned} \quad (55)$$

where $\rho(t) = -\zeta_2(t) - (\rho\star\zeta_2)(t) = \zeta_2\circ S_\star(t)$, starting at $\rho(\Upsilon) = -\zeta_2(\Upsilon)$.

It is natural to define a new quantity $\alpha(t;q)$ by $\alpha(t;q) = 0$ for $t = \mid$ and, for $t = t_l\vee t_r$,

$$\alpha(t;q) = ie^2S^0(q)\int\frac{d^4p}{(2\pi)^4}\gamma^\lambda\bar\varphi_{\lambda\lambda'}(t_l;p)\bar\varphi^1(t_r;q-p;\lambda',p).$$

Then we consider Eq.(17) and in the integral over p, we expand the photon propagator over trees t_l using Eq.(24) and the electron propagator over trees t_r using Eq.(23). We recognize a sum of $\alpha(t_l\vee t_r)$ and the integral becomes

$$ie^2S^0(q)\int\frac{d^4p}{(2\pi)^4}\gamma^\lambda\bar D_{\lambda\lambda'}(p)\frac{\delta\bar S(q-p)}{e\delta A_{\lambda'}(p)} = \sum_t\alpha(t;q).$$

In the other terms of Eq.(17), we expand $\bar S(q)$ and renormalization constants over trees using Eqs.(23), (31), (32) and (33). We replace products by convolutions ∇ according to Eq.(43) in all expressions except the integral and we obtain.

$$\sum_t e^{|t|}(\bar\varphi^0(q)\nabla\zeta_2)(t) = \varphi^0(\mid;q) + \sum_t e^{|t|}(\alpha(q)\nabla\zeta_2)(t) - \sum_t e^{|t|}\varphi^0(\mid;q)(\zeta_m\nabla\bar\varphi^0(q)\nabla\zeta_2)(t).$$

The bold step is now to identify the terms corresponding to a given tree t. This yields

$$\begin{aligned}(\bar\varphi^0(q)\nabla\zeta_2)(t) &= \varphi^0(\mid;q)\epsilon(t) + (\alpha(q)\nabla\zeta_2)(t) \\ &\quad -\varphi^0(\mid;q)(\zeta_m\nabla\bar\varphi^0(q)\nabla\zeta_2)(t), \end{aligned}\quad (56)$$

where $\epsilon(t)=1$ if $t=\mid$ and $\epsilon(t)=0$ otherwise.

To simplify this expression, we follow Kreimer [58] and compute $(\bar\varphi^0(q)\nabla\zeta_2\nabla\zeta_2\circ S_\star)(t)$. The basic property of the antipode is $Id\nabla S_\star = \mid\epsilon$, therefore

$$\zeta_2\nabla\zeta_2\circ S_\star = \zeta_2(Id\nabla S_\star) = \zeta_2\epsilon = \epsilon. \quad (57)$$

The associativity of ∇ is here crucial. On the one hand, $\big(\bar\varphi^0(q)\nabla(\zeta_2\nabla\zeta_2\circ S_\star)\big)(t)=\bar\varphi^0(q)(t)$ according to (57). On the other hand, we calculate $\big((\bar\varphi^0(q)\nabla\zeta_2)\nabla\zeta_2\circ S_\star)\big)$, where we replace $\bar\varphi^0(q)\nabla\zeta_2$ by the right-hand side of Eq.(56). Equation (57) gives us

$$\big((\bar\varphi^0(q)\nabla\zeta_2)\nabla\zeta_2\circ S_\star)\big)=\zeta_2\circ S_\star(t)\varphi^0(\shortmid;q) \\ -\varphi^0(\shortmid;q)(\zeta_m\nabla\bar\varphi^0(q))(t)+\alpha(t).$$

From the associativity of ∇ and the definition of $\alpha(t;q)$ we obtain our final recursive equation (55) for the electron propagator.

The recursive equation is completed by the equation for the higher components of $\bar\varphi^0(t;q)$. If we take the functional derivative of Eq.(55), and make the same simplification as in Ref.[10]), we obtain

$$\begin{aligned}\bar\varphi^n(t;q;\{\lambda,p\}_{1,n})&=\varphi^0(\shortmid;q)\gamma^{\lambda_1}\bar\varphi^{n-1}(t;q;\{\lambda,p\}_{2,n})\\ &-\rho(t)\varphi^0(\shortmid;q)\delta_{n,0}-\zeta_m(t)\varphi^0(\shortmid;q)\bar\varphi^n(\shortmid;q;\{\lambda,p\}_{1,n})\\ &-\varphi^0(\shortmid;q)(\zeta_m\star\bar\varphi^n(q;\{\lambda,p\}_{1,n}))(t)\\ &+i\varphi^0(\shortmid;q)\sum_{k=0}^n\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda\bar\varphi^k_{\lambda\lambda'}(t_l;p;\{\lambda,p\}_{1,k})\\ &\times\bar\varphi^{n-k+1}_\Sigma(t_r;q-p;\lambda',p,\{\lambda,p\}_{k+1,n}).\end{aligned}\tag{58}$$

Another useful formula can be obtained by defining

$$\begin{aligned}\bar f^0(t;q)&=(\bar\varphi^0(q)\nabla\zeta_2)(t) &(59)\\ &=\bar\varphi^0(t;q)+(\bar\varphi^0(q)\star\zeta_2)(t)+\varphi^0(\shortmid;q)\zeta_2(t),\\ \bar f^1(t;q;\lambda',p)&=(\bar\varphi^1(q;\lambda',p)\nabla\zeta_2)(t). &(60)\end{aligned}$$

With this notation Eq.(56) is rewritten

$$\bar f^0(t;q)=-\zeta_m(t)\varphi^0(\shortmid;q)^2-\varphi^0(\shortmid;q)(\zeta_m\star\bar f^0(q))(t) \\ +i\varphi^0(\shortmid;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda\bar\varphi^0_{\lambda\lambda'}(t_l;p)\bar f^1(t_r;q-p;\lambda',p).\tag{61}$$

The higher components $\bar f^n(t;q)$ are obtained by functional derivative of (61), as explained in section 4.7. The recursive equation for $\bar f^n(t;q)$ is the same as Eq.(58), where φ is replaced by f and the term $\rho(t)\varphi^0(\shortmid;q)\delta_{n,0}$ is suppressed.

6 The renormalized photon propagator

Bogoliubov and Shirkov [29] have shown that the renormalization of two Feynman diagrams linked by a single photon (or electron) line is obtained by an independent renormalization of each of the two subgraphs. In our language, this means that the renormalized form of (29) is

$$\bar\varphi^0_{\mu\nu}(t_l\vee t_r;q)=\bar\varphi^0_{\mu\lambda}(\shortmid\vee t_r;q)\psi^{0\lambda\lambda'}(\shortmid;q)\bar\varphi^0_{\lambda'\nu}(t_l;q)\tag{62}$$

Therefore, all trees for the photon propagator can be renormalized once we have renormalized the special trees $\shortmid\vee t_r$. Now we show that the recursive equation for the renormalized photon term $\bar\varphi^0_{\mu\lambda}(\shortmid\vee t_r;q)$ is

$$\bar\varphi^0_{\mu\nu}(\shortmid\vee t;q)=-\zeta_3(\shortmid\vee t)\varphi^T_{\mu\nu}(\shortmid;q) \\ -i\varphi^T_{\mu\lambda}(\shortmid;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\bar f^1(t;p;\lambda',-q)\big]\varphi^T_{\lambda'\nu}(\shortmid;q)\tag{63}$$

where $\bar f^1$ was defined in the previous section.

To prove this, we start from several remarks: we have $\bar\psi^0_{\mu\nu}(t_l\vee t_r)=0$ is $t_l\neq\shortmid$ and

$$\bar\psi^0_{\mu\nu}(\shortmid\vee t_r;q)=-\psi^0_{\mu\lambda}(\shortmid;q)\bar\varphi^{0\lambda\lambda'}(\shortmid\vee t_r;q)\psi^0_{\lambda'\nu}(\shortmid;q).$$

Because of this close analogy between photon propagator and vacuum polarization, we shall rewrite (21) as

$$\begin{aligned}\sum_t e^{2|t|}\bar\varphi^0_{\mu\nu}(\shortmid\vee t)&=-\varphi^T_{\mu\lambda}(\shortmid;q)\bar\Pi^{\lambda\lambda'}(q)\varphi^T_{\lambda'\nu}(\shortmid;q)\\ &=-(Z_3-1)\varphi^T_{\lambda\mu}(\shortmid;q)\\ &-Z_2ie^2\varphi^T_{\mu\lambda}(\shortmid;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\frac{\delta\bar S(p)}{e\delta A_{\lambda'}(-q)}\big]\varphi^T_{\lambda'\nu}(\shortmid;q).\end{aligned}$$

We rewrite this expression to isolate the root components:

$$\begin{aligned}\sum_t\bar\varphi^0_{\mu\nu}(\shortmid\vee t)&=-(Z_3-1)\varphi^T_{\lambda\mu}(\shortmid;q)\\ &-ie^2\varphi^T_{\mu\lambda}(\shortmid;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\frac{\delta\bar S(p)}{e\delta A_{\lambda'}(-q)}\big]\varphi^T_{\lambda'\nu}(\shortmid;q)\\ &+ie^2\varphi^T_{\mu\lambda}(\shortmid;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\frac{\delta\bar S(p)}{e\delta A_{\lambda'}(-q)}\big](Z_2-1)\\ &\times\varphi^T_{\lambda'\nu}(\shortmid;q).\end{aligned}$$

We expand all quantities over trees, using

$$\frac{\delta\bar S(p)}{e\delta A_{\lambda'}(-q)}=\sum_t e^{2|t|}\bar\varphi^1(t;p;\lambda',-q),$$

and we multiply through the pruning operator. Then we identify the terms corresponding to the same tree and we obtain

$$\begin{aligned}\bar\varphi^0_{\mu\nu}(\shortmid\vee t;q)&=-\zeta_3(\shortmid\vee t)\varphi^T_{\mu\nu}(\shortmid;q)+\zeta_2(t)\varphi^0_{\mu\nu}(\Upsilon;q)\\ &-i\varphi^T_{\mu\lambda}(\shortmid;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\bar\varphi^1(t;p;\lambda',-q)\big]\varphi^T_{\lambda'\nu}(\shortmid;q)\\ &-i\varphi^T_{\mu\lambda}(\shortmid;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda(\bar\varphi^1\star\zeta_2)(t;p;\lambda',-q)\big]\varphi^T_{\lambda'\nu}(\shortmid;q).\end{aligned}$$

From the definition (60) for $\bar f^1$, we can rewrite this expression as our recursive equation (63).

The higher components are obtained very simply by taking the functional derivative of Eq.(63). Since $\varphi^T_{\mu\nu}(\shortmid;q)$ is independent of the external field, we obtain

$$\bar\varphi^n_{\mu\nu}(\shortmid\vee t;q;\{\lambda,p\}_{1,n})=-i\varphi^T_{\mu\lambda}(\shortmid;q) \\ \times\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\bar f^{n+1}_\Sigma(t;p;\lambda',-q,\{\lambda,p\}_{1,n})\big]\varphi^T_{\lambda'\nu}(\shortmid;q).$$

For the other trees, we use

$$\bar\varphi^n_{\mu\nu}(t_l\vee t_r;q;\{\lambda,p\}_{1,n}) = \sum_{k=0}^n \bar\varphi^k_{\mu\lambda}(\mid\vee t_r;q;\{\lambda,p\}_{1,k}) \times\psi^{0\lambda\lambda'}(\mid;q)\bar\varphi^{n-k}_{\lambda'\nu}(t_l;q;\{\lambda,p\}_{k+1,n}).$$

6.1 Properties of renormalized photon propagator

From (61) and (63) we can deduce that the renormalized photon propagator does not depend on any $\zeta_2(t)$. In fact, we shall prove that $\bar f^0(t;q)$, $\bar f^1(t;q;\lambda,p)$ and $\bar\varphi^0_{\mu\nu}(t;q)$ do not depend on any $\zeta_2(t')$. To do this, we reintroduce a non-zero external field A. The property is clearly true for $t=\mid$. If it is true for all trees with $|t|<N$, let us take a tree with $|t|=N$. Because of (61), $\bar f^0(t;q)$ does not depend on any $\zeta_2(t')$. Since $\bar f^1(t;q;\lambda,p)$ is obtained by a functional derivative of $\bar f^0(t;q)$ with respect to eA, it does not depend on any $\zeta_2(t')$ either (eA does not depend on any $\zeta_2(t')$). If $t=\mid\vee t_r$, because of (63), $\bar\varphi^0_{\mu\nu}(\mid\vee t_r;q)$ does not depend on any $\zeta_2(t')$ since none of the terms on the right hand side do. Finally, if t is not of the form $t=\mid\vee t_r$, it is of the form $t=t_l\vee t_r$, and $\bar\varphi^0_{\mu\nu}(t_l\vee t_r;q)$ is obtained from $\bar\varphi^0_{\mu\nu}(\mid\vee t_r;q)$ and $\bar\varphi^0_{\mu\nu}(t_l;q)$, which do not depend on any $\zeta_2(t')$.

With the same reasoning, we see that $\bar f^0(t;q)$ and $\bar\varphi^0_{\mu\nu}(t;q)$ are independent of the gauge parameter ξ for $t\neq\mid$.

7 Electron self-energy

To calculate the electron self-energy, we start from (55) that we rewrite

$$\bar\varphi^0(t;q) = \rho(t)\varphi^0(\mid;q) - \zeta_m(t)\varphi^0(\mid;q)^2 + \alpha(t;q) -\varphi^0(\mid;q)(\zeta_m\star\bar\varphi^0(q))(t).$$

The self-energy is obtained by introducing the last equation into (44). This gives us

$$\begin{aligned}\bar\psi^0(t;q) =& -\rho(t)\psi^0(\mid;q) - \zeta_m(t) + \psi^0(\mid;q)\alpha(t;q)\psi^0(\mid;q)\\ &-(\rho\star\bar\psi^0(q))(t) + (\zeta_m\star\bar\varphi^0(q))(t)\psi^0(\mid;q)\\ &+\varphi^0(\mid;q)(\zeta_m\star\bar\psi^0(q))(t) + (\zeta_m\star\bar\varphi^0(q)\star\bar\psi^0(q))(t)\\ &+\psi^0(\mid;q)(\alpha(q)\star\bar\psi^0(q))(t).\end{aligned}$$

If we factorize ζ_m and use (45), the expression reduces to

$$\begin{aligned}\bar\psi^0(t;q) =& -\rho(t)\psi^0(\mid;q) - \zeta_m(t) + \psi^0(\mid;q)\alpha(t;q)\psi^0(\mid;q)\\ &-(\rho\star\bar\psi^0(q))(t) + \psi^0(\mid;q)(\alpha(q)\star\bar\psi^0(q))(t).\end{aligned}$$

From the definition of $\alpha(t;q)$ and the result of section 13.3, we obtain

$$\begin{aligned}\bar\psi^0(t;q) =& -\rho(t)\psi^0(\mid;q) - \zeta_m(t) - (\rho\star\bar\psi^0(q))(t)\\ &+i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda\bar\varphi^0_{\lambda\lambda'}(t_l;p)\bar g(t_r;q-p;\lambda',p),\end{aligned}$$

where

$$\begin{aligned}\bar g(t_r;q-p;\lambda',p) &= -(\bar\varphi^1(q-p;\lambda',p)\nabla\bar\psi^0(q))(t_r) \quad (64)\\ &= (\bar\varphi^0(q)\nabla\bar\psi^1(q-p;\lambda',p))(t_r). \quad (65)\end{aligned}$$

It can be shown recursively that the only term proportional to $\psi^0(\mid;q)$ in $-\rho(t)\psi^0(\mid;q)-(\rho\star\bar\psi^0(q))(t)$ is $\zeta_2(t)\psi^0(\mid;q)$.

8 Renormalization conditions

Renormalization conditions are the conditions used to give a unique value to the renormalized quantities. As noticed by Itzykson and Zuber [2], there is some freedom in the choice of these conditions. However, it is necessary that the conditions are satisfied for the root. In other words, the renormalization conditions can be non-zero only for the root. The physical meaning of these renormalization conditions can be found in any textbook on quantum field theory. In particular, the relation between the renormalization conditions and the introduction of experimental quantities into the theory is discussed at length in Refs. [2,11].

For instance, mass shell renormalization conditions for QED [2] p.413 could be translated into (for $t\neq\mid$)

$$\begin{aligned}\bar\psi^0(t;q)|_{\gamma^\mu q_\mu=m} &= 0,\\ \bar\psi^1(t;q;\lambda,0)|_{\gamma^\mu q_\mu=m} &= 0,\\ \bar\omega(t;q)|_{q=0} &= 0.\end{aligned}$$

The strange prescription $\gamma^\mu q_\mu=m$ means that the quantities $\bar\psi^0(t;q)$ and $\bar\psi^1(t;q;\lambda,0)$ must be multiplied by $u(q)$ on the right where $u(q)$ is a solution of the Dirac equation $(\gamma^\mu q_\mu - m)u(q)=0$. This amounts to replacing q^2 by m^2 and $\gamma^\mu q_\mu$ by m in the analytic expressions for $\bar\psi^0(t;q)$ and $\bar\psi^1(t;q;\lambda,0)$, when they are available.

For the photon, we define $\bar\omega(q^2)$ by

$$[\bar D^{-1}]_{\lambda\mu}(q) = (q_\lambda q_\mu - q^2 g_{\lambda\mu})(1+\bar\omega(q^2)) - \xi q_\lambda q_\mu. \quad (66)$$

This $\bar\omega(q^2)$ is the same as $\bar\omega_R(q^2)$ in Itzykson and Zuber [2].

For the photon, we know that $\psi^0_{\mu\nu}(t;q)$ is transverse. Therefore, we can define $\bar\omega(\mid\vee t;q)$ by

$$\psi^0_{\mu\nu}(\mid\vee t;q) = \psi^0_{\mu\nu}(\mid;q)\bar\omega(\mid\vee t;q).$$

Notice that $\sum_t\bar\omega(\mid\vee t;q)=\bar\omega(q)$, where $\bar\omega(q)$ was defined in (66). The renormalization condition on $\bar\omega(\mid\vee t;q)$ replaces a more complicated renormalization condition on $\psi^0_{\mu\nu}(t;q)$ (involving the third derivative of $\psi^0_{\mu\nu}(t;q)$ with respect to q).

To show how this works in practice, we use the identity

$$\bar\varphi^0_{\lambda\mu}(\mid\vee t;q) = -\bar\omega(\mid\vee t;q^2)\bar\varphi^T_{\lambda\mu}(\mid;q)$$

Now we assume that we have calculated the integral

$$h(q) = -\frac{i}{q^2}g^{\lambda\lambda'}\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\bar f^1(t;p;\lambda',-q)\big].$$

This integral is logarithmically divergent, and we must determine how to remove its divergence. We start from Eq.(63), that we multiply by $g^{\mu\nu}$ and sum over μ and ν. Using the fact that the integral over p is transverse, we obtain

$$3\bar{\omega}(\shortmid\vee t;q^2)=3\zeta_3(\shortmid\vee t)+h(q).$$

Since the renormalization condition is $\bar{\omega}(\shortmid\vee t;0)=0$, we just take $\zeta_3(\shortmid\vee t)=-h(0)/3$. This satisfies the renormalization condition and this determines $\zeta_3(\shortmid\vee t)$. For the electron propagator, the renormalization conditions determine $\zeta_2(t)$ and $\zeta_m(t)$.

In practice, other renormalization conditions are used, such as minimal subtraction. Once quantities have been made convergent by a renormalization condition, it is always possible to translate the results into different renormalization conditions, using the Hopf structure of renormalization [58]. The fact that the renormalization conditions can be composed and that the result does not depend on which condition is used first is a direct consequence of the coassociativity of the Hopf algebra of renormalization [58].

9 Massive quenched QED

Quenched QED is QED without vacuum insertion graphs. Quenched QED has been recently advocated and discussed by Broadhurst and coll. [59,60].

The Feynman diagrams describing the electron propagator of quenched QED have no fermion loop. Within the present approach, all Feynman diagrams of a given order for quenched QED are summed into one tree $\varphi^0(\Upsilon^n)$. The trees Υ^n are defined recursively by $\Upsilon^0=\shortmid$, $\Upsilon^n=\shortmid\vee\Upsilon^{n-1}$. In words, they are the trees without left branches. The Feynman diagrams corresponding to $\varphi^0(\Upsilon^n)$ for $n\le 3$ are given in appendix 3.

The bare and renormalized electron propagators for quenched QED are written

$$S^Q(q)=\sum_{n=0}^{\infty}e_0^n\varphi^0(\Upsilon^n;q)\quad\text{and}\quad \bar{S}^Q(q)=\sum_{n=0}^{\infty}e^n\bar{\varphi}^0(\Upsilon^n;q).$$

The Schwinger equation for the renormalized electron propagator of quenched QED is

$$\begin{aligned}\bar{S}^Q(q)&=S^0(q)-(Z_2-1)\bar{S}^Q(q)-\delta m Z_2 S^0(q)\bar{S}^Q(q)\\&+ie^2Z_2S^0(q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda D^0_{\lambda\lambda'}(p)\frac{\delta\bar{S}^Q(q-p)}{e\delta A_{\lambda'}(p)}.\end{aligned}\qquad(67)$$

The relation we want to show is

$$\begin{aligned}\bar{\varphi}^0(\Upsilon^n;q)&=\varphi^0(\Upsilon^n;q)+\sum_{a=0}^{n-1}\sum_{k=0}^{n-a}\frac{1}{k!}\alpha_k(\Upsilon^{n-a})\frac{\partial^k\varphi^0(\Upsilon^a;q)}{\partial m^k}\\&=\sum_{k=0}^{n}\frac{1}{k!}\big(\alpha_k\nabla\frac{\partial^k\varphi^0(q)}{\partial m^k}\big)(\Upsilon^n),\end{aligned}\qquad(68)$$

where

$$\begin{aligned}\alpha_0&=\rho=-\zeta_2-\rho\star\zeta_2,\\\alpha_1&=-\zeta_m-\zeta_m\star\alpha_0,\\\alpha_k&=-\zeta_m\star\alpha_{k-1}\quad\text{if}\quad k\ge 2.\end{aligned}$$

Notice that $\alpha_k(\Upsilon^n)=0$ if $k>n$, because Υ^n cannot be split into more than n part by the pruning operator P.

To prove this, we start from the Schwinger equation (67) and transform it into a recurrence relation:

$$\begin{aligned}\bar{\varphi}^0(t;q)&=\rho(t)\varphi^0(\shortmid;q)-\zeta_m(t)\varphi^0(\shortmid;q)^2\\&-\varphi^0(\shortmid;q)(\zeta_m\star\bar{\varphi}^0(q))(t)\\&+iS^0(q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda D^0_{\lambda\lambda'}(p)\bar{\varphi}^1(t_r;q-p;\lambda',p),\end{aligned}\qquad(69)$$

where $\rho(t)=-\zeta_2(t)-(\rho\star\zeta_2)(t)=\zeta_2\circ S_\star(t)$, starting at $\rho(\Upsilon)=-\zeta_2(\Upsilon)$.

Using the definitions given in section 4.7, we can show that

$$\frac{\delta}{e\partial A_\lambda(p)}\frac{\partial}{\partial m}\varphi^0(\Upsilon^n;q)=\frac{\partial}{\partial m}\frac{\delta}{e\partial A_\lambda(p)}\varphi^0(\Upsilon^n;q),$$

and that

$$\begin{aligned}\frac{\partial^k}{\partial m^k}\varphi^0(\Upsilon^n;q)&=k\varphi^0(\shortmid;q)\frac{\partial^{k-1}}{\partial m^{k-1}}\varphi^0(\Upsilon^n;q)\\&+i\varphi^0(\shortmid;q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda D^0_{\lambda\lambda'}(p)\frac{\partial^k}{\partial m^k}\varphi^1(t_r;q-p;\lambda',p).\end{aligned}$$

From these identities, Eq.(68) follows easily by recursion.

The Feynman diagrams describing the photon propagator of quenched QED have a single electron loop, as can be seen in the diagrams for $\varphi^0_{\lambda\mu}(\Upsilon^n)$ in appendix 3. The bare and renormalized photon propagators are written

$$\begin{aligned}D^Q_{\lambda\mu}(q)&=\sum_{n=0}^{\infty}e_0^n\varphi^0_{\lambda\mu}(\Upsilon^n;q),\\\bar{D}^Q_{\lambda\mu}(q)&=\sum_{n=0}^{\infty}e^n\bar{\varphi}^0_{\lambda\mu}(\Upsilon^n;q).\end{aligned}$$

The relation we want to show is

$$\begin{aligned}\bar{\varphi}^0_{\lambda\mu}(\Upsilon^n;q)&=\varphi^0_{\lambda\mu}(\Upsilon^n;q)+\sum_{k=0}^{n-1}\sum_{a=1}^{n-k}\frac{1}{k!}\beta_k(\Upsilon^{n-a})\frac{\partial^k\varphi^0_{\lambda\mu}(\Upsilon^a;q)}{\partial m^k}\\&-\zeta_3(\Upsilon^n)\varphi^T_{\lambda\mu}(\shortmid;q),\end{aligned}\qquad(70)$$

where the scalars β_k are defined recursively

$$\begin{aligned}\beta_1&=-\zeta_m,\\\beta_k&=-\zeta_m\star\beta_{k-1}\quad k\ge 2,\end{aligned}$$

with $\beta_k(\Upsilon^n)=0$ if $k>n$. This can be written more compactly:

$$\bar{\varphi}^0_{\lambda\mu}(\Upsilon^n;q)=\exp\big[-\zeta_m\frac{\partial}{\partial m}\star\big]\varphi^0_{\lambda\mu}(\Upsilon^n;q)-\zeta_3(\Upsilon^n)\varphi^T_{\lambda\mu}(\shortmid;q).$$

To prove Eq.(70), we first use our previous result Eq.(68) to show that

$$\bar{f}^0(\curlyvee^n;q)=\varphi^0(\curlyvee^n;q)+\sum_{k=1}^{n}\sum_{a=0}^{n-k}\frac{1}{k!}\beta_k(\curlyvee^{n-a})\frac{\partial^k\varphi^0(\curlyvee^a;q)}{\partial m^k}.$$

Then we introduce this expression into Eq.(63), and the result follows from the relation

$$\frac{\partial^k}{\partial m^k}\varphi^0_{\mu\nu}(\curlyvee^{n+1};q)=-i\varphi^T_{\mu\lambda}(|;q)$$
$$\times\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\partial^k}{\partial m^k}\mathrm{tr}\big[\gamma^\lambda\bar{f}^1(\curlyvee^n;p;\lambda',-q)\big]\varphi^T_{\lambda'\nu}(|;q). \quad (71)$$

10 Hopf algebra for massless QED

In this section, we determine a coproduct from the recursive equations (55), (61) and (63) and the product law (62) for massless QED. The case of massless QED is much simpler because the mass is not renormalized. Thus, for all trees t, $\zeta_m(t)=0$.

This coproduct determines the renormalized propagators as a function of the unrenormalized ones. In this section, it will be useful to dinstinguish the electron and photon trees by the color of the root. A tree with a black root is written $t^\bullet$ and represents an electron propagator, a tree with a white root is written t° and represents a photon propagator. In a tree $t_l\vee t_r$, t_l is white and t_r is black. There are now two graftings operators $\vee$ and $\vee$, so that $t_l\vee t_r$ is a black tree and $t_l\vee t_r$ a white one.

Using a variation of Sweedler's notation, we write

$$\Delta t^\circ=\sum_{\Delta t^\circ}t^\circ_{(1)}\otimes t^\circ_{(2)}, \quad (72)$$

$$\Delta t^\bullet=\sum_{\Delta t^\bullet}t^\circ_{(1)}t^\bullet_{(1)}\otimes t^\bullet_{(2)}, \quad (73)$$

$$F(t^\bullet)=\sum_{F(t^\bullet)}t^\circ_{(1)}\otimes t^\bullet_{(2)}. \quad (74)$$

These equations mean that the coproduct of t° generates a sum of tensor products with one white tree on the left and one white tree on the right, the coproduct of $t^\bullet$ generates a sum of tensor products with one black tree and one white tree on the left and one black tree on the right, finally the coproduct $F(t)$ generates a sum of tensor products with one white tree on the left and one black tree on the right. These trees can eventually be the root, which is the unit element of the algebra (the root is neither white nor black, or both, as you wish).

To avoid products of white trees in Eqs.(72), (73) and (74), we took advantage of the fact that, according to Eq.(29), the $\varphi_{\mu\nu}$ of a white tree $t_l\vee t_r$ can be written as a product of $\varphi_{\mu\nu}(|\vee t_r)$ by $\varphi_{\mu\nu}(t_l)$. From Eq.(62), we also know that this property is compatible with renormalization. Therefore, we translate this property into an inner product over white trees. The product of two white trees s°/t° (read "s over t") is defined recursively by $s/|=s$ and $s/(t_l\vee t_r)=(s/t_l)\vee t_r$. In particular $t_l/(|\vee t_r)=t_l\vee t_r$, which is what we need. Surprisingly, this product has been used independently by Loday and Ronco in a completely different context [61].

The coproduct Δ acting on white and black trees is defined by the recursive equations

$$\Delta(|\vee t)=(|\vee t)\otimes|+\sum_{F(t)}t^\circ_{(1)}\otimes(|\vee t^\bullet_{(2)}), \quad (75)$$

$$\Delta(t_l\vee t_r)=(t_l\vee t_r)\otimes|+\sum_{\Delta t_l,\Delta t_r}(t^\circ_{l(1)}/t^\circ_{r(1)})t^\bullet_{r(1)}$$
$$\otimes(t^\circ_{l(2)}\vee t^\bullet_{r(2)}), \quad (76)$$

$$F(t_l\vee t_r)=\sum_{\Delta t_l,F(t_r)}(t^\circ_{l(1)}/t^\circ_{r(1)})\otimes(t^\circ_{l(2)}\vee t^\bullet_{r(2)}), \quad (77)$$

with the initial values $\Delta|=|\otimes|$ and $F(|)=|\otimes|$, and with the compatibility of the "over" product with renormalization: $\Delta(s^\circ/t^\circ)=\Delta t^\circ\Delta s^\circ$. In particular,

$$\Delta t_l\vee t_r=\Delta(|\vee t_r)\Delta t_l.$$

With this notation, we can now write the coproduct of a general white tree

$$\Delta(t_l\vee t_r)=\sum_{\Delta t_l}(t^\circ_{l(1)}\vee t_r)\otimes t^\circ_{l(2)}$$
$$+\sum_{\Delta t_l,F(t_r)}(t^\circ_{l(1)}/t^\circ_{r(1)})\otimes(t^\circ_{r(2)}\vee t^\bullet_{(2)}). \quad (78)$$

These preliminaries enable us to write the relation between renormalized and unrenormalized propagators as

$$\bar{\varphi}^0_{\mu\nu}(t^\circ;q)=\sum_{\Delta t^\circ}\zeta(t^\circ_{(1)})\varphi^0_{\mu\nu}(t^\circ_{(2)};q), \quad (79)$$

$$\bar{\varphi}^0(t^\bullet;q)=\sum_{\Delta t^\bullet}\zeta(t^\circ_{(1)})\zeta(t^\bullet_{(1)})\varphi^0(t^\bullet_{(2)};q), \quad (80)$$

$$\bar{f}(t;q)=\sum_{\Delta t^\bullet}\zeta(t^\circ_{(1)})f^0(t^\bullet_{(2)}). \quad (81)$$

The general counterterm ζ is a scalar over black and white trees defined by $\zeta(|)=1$ and

$$\zeta(t^\bullet)=\rho(t^\bullet), \quad (82)$$
$$\zeta(s^\circ/t^\circ)=\zeta(s^\circ)\zeta(t^\circ),$$
$$\zeta(|\vee t)=-\zeta_3(|\vee t).$$

In particular $\zeta(t_l\vee t_r)=-\zeta_3(|\vee t_r)\zeta(t_l)$. We recall that $\rho(t)=\zeta_2\circ S_\star(t)$. Equations (79) and (80) are given in expanded form in appendix 2 for trees up to order 3.

We prove this recursively. From the list of appendix 2, Eqs.(79) and (80) are satisfied for all trees up to order 3. The same can be checked for Eq.(81). Assume that they are satisfied up for trees with $2N-1$ vertices. Take a tree with $2N+1$ vertices. Take first $|\vee t$, then use Eq.(81) for $\bar{f}^1$ in Eq.(63). This yields

$$\bar{\varphi}^0_{\mu\nu}(|\vee t;q)=-\zeta_3(|\vee t)\varphi^T_{\mu\nu}(|;q)$$
$$+\sum_{F(t)}\zeta(t_{(1)})\varphi^0_{\mu\nu}(|\vee t_{(2)}).$$

This is Eq.(79) for the coproduct defined by Eq.(75). If we take now $t_l \vee t_r$, where t_l and t_r have less than $2N+1$ vertices, we can expand $\bar{\varphi}^0_{\mu\nu}(\mid \vee t_r)$ and $\bar{\varphi}^0_{\mu\nu}(t_l)$ over unrenormalized terms. Then, using Eq.(62) we find

$$\begin{aligned}\bar{\varphi}^0_{\mu\nu}(t_l \vee t_r;q) &= \Big(-(\zeta_3(\mid \vee t_r)\varphi^T_{\mu\nu}(\mid;q) \\ &+ \sum_{F(t_r)} \zeta(t_{r(1)})\varphi^0_{\mu\nu}(\mid \vee t_{r(2)};q)\Big) \sum_{\Delta t_l} \zeta(t_{l(1)})\bar{\varphi}^0_{\mu\nu}(t_{l(2)};q) \\ &= \sum_{\Delta t_l} \zeta(t_{l(1)} \vee t_r)\bar{\varphi}^0_{\mu\nu}(t_{l(2)};q) \\ &+ \sum_{F(t_r)\Delta t_l} \zeta(t_{l(1)}/t_{r(1)})\bar{\varphi}^0_{\mu\nu}(t_{l(2)} \vee t_{r(2)};q).\end{aligned}$$

This is Eq.(79) with the coproduct defined in Eq.(78) and the correct ζ.

For the coproduct acting on electron trees, we start from the recursive equation (55) and we use the expansion (79) for $\bar{\varphi}^0_{\mu\nu}(t_l;q)$ and (80) for $\bar{\varphi}^0(t_r;q)$. This gives us

$$\begin{aligned}\bar{\varphi}^0(t_r \vee t_l;q) &= \rho(t_l \vee t_r)\varphi^0(\mid;q) \\ &+ \sum_{F(t_r)\Delta t_l} \zeta(t_{l(1)}/t^\circ_{r(1)})\zeta(t^\bullet_{r(1)})\bar{\varphi}^0(t_{l(2)} \vee t_{r(2)};q).\end{aligned}$$

The first term $\rho(t_l \vee t_r)$ is consistent with $\zeta(t_l \vee t_r)$ as defined in Eq.(82). And the other terms are consistent with Eq.(80) using the coproduct Eq.(76).

Exactly the same substitution leads to Eq.(81) using the coproduct Eq.(77).

In Ref.[62], it will be shown that Δ is coassociative and defines a Hopf algebra over the two-coloured planar binary trees.

11 Conclusion

The method of Schwinger equations has a number of advantages: operator-valued distributions are avoided, as well as indefinite norms and many of the difficult mathematical concepts of quantum field theory. As compared to Feynman diagrams, the method of planar binary trees is more compact and does not require symmetry factors. Moreover, our treatment of renormalization does not require a special treatment of the so-called overlapping divergences. Such a special treatment is even necessary for Kreimer's original method of renormalization by rooted trees [58].

The present work was much inspired by Refs.[5–7, 58] but it was carried out independently of the recent and fascinating results by Connes and Kreimer [8, 9]. Still, our Hopf algebra can probably be modified to fit into their general framework. We hope to explore this connection in a forthcoming publication.

We conclude this paper with a word of caution. We have not actually proved that each tree is renormalized by the Hopf algebra. In other words, we have not proved that all $\bar{\varphi}^0(t;q)$ and $\bar{\varphi}^0_{\lambda\mu}(t;q)$ are finite. However, we can give an argument that can eventually lead to a proof of finiteness. In the case of quenched QED, there is a single tree (Υ^n) at each order of the perturbation theory. Each order of perturbation theory of quenched QED is renormalized by the standard renormalization of Feynman diagrams, so we know that our $\bar{\varphi}^0(\Upsilon^n;q)$ and $\bar{\varphi}^0_{\lambda\mu}(t;q)$ are finite, since they are uniquely fixed by the Schwinger equations. The other trees are obtained from some Υ^n by insertion of renormalized photon propagators. Since these insertions do not introduce new divergences, and particularly no new overlapping divergence, we can conclude that all trees are finite. The point that is still not solved is the insertion of higher-order photon Green functions, such as the photon-photon scattering diagrams. Since these diagrams are finite, they probably do not spoil the convergence, but we could not prove this generally.

12 Aknowledgements

Ch. B warmly thanks Dirk Kreimer for his invitation to Mainz and his illuminating introduction to renormalization. We are very grateful to Dirk Kreimer and David Broadhurst for their constant help and support. We thankfully acknowledge the help and encouragement from Jean-Louis Loday, Muriel Livernet and Frédéric Chapoton. Finally, we want to stress that our understanding of renormalization has greatly benefited from the lectures by Alain Connes at the Collège de France. This is IPGP contribution #0000.

13 Appendix 1

This appendix contains some proofs.

13.1 Proof of (41) (the pruning operator is coassociative)

If $t = \mid$ we have $P(\mid) = 0$, so the identity (41) holds. So, suppose that $t \neq \mid$. The reason why the identity (41) holds for t is that applying the recursive definition of P, on the successive right grafters of t, at each step both sides of (41) coincide on the terms which do not involve $P(t')$ for the last right grafter t' considered. Of course, when we finally meet a right grafter t' such that $P(t') = 0$ we obtain the equality (41). We develop this idea formally.

Any tree $t \neq \mid$ can be written in a unique way as

$$t = t_1 \vee (t_2 \vee (\ldots \vee (t_n \vee \mid)\ldots)),$$

for some $n \leq |t| + 1$. In fact, it suffices to decompose the tree t into its left and right grafting trees, then to decompose successively the right trees as graftings of two new trees and pick up all their left sides, $t_1 := t_l$, $t_2 := (t_r)_l$, $t_3 := ((t_r)_r)_l$ and so on, until we meet an undecomposable right side $(\ldots((t_r)_r)\ldots)_r = \mid$. Since $|t| = |t_1| + |t_2| + \ldots + |t_n| + n - 1$ and $|t_i| \geq 0$ for all $i = 1, \ldots, n$, the procedure must finish for an $n \leq |t| + 1$.

Since $P(t_n \vee \mid) = 0$, we have

$$\begin{aligned}P(t_{n-1}\vee(t_n\vee |)) &= t_{n-1}\vee | \otimes t_n\vee |,\\ P(t_{n-2}\vee(t_{n-1}\vee(t_n\vee |))) &= t_{n-2}\vee | \otimes t_{n-1}\vee(t_n\vee |)\\ &\quad + t_{n-2}\vee(t_{n-1}\vee |)\otimes t_n\vee |\end{aligned}$$

and

$$\begin{aligned}&P(t_{n-3}\vee(t_{n-2}\vee(t_{n-1}\vee(t_n\vee |)))) =\\ &\qquad t_{n-3}\vee | \otimes t_{n-2}\vee(t_{n-1}\vee(t_n\vee |))\\ &\qquad + t_{n-3}\vee(t_{n-2}\vee |)\otimes t_{n-1}\vee(t_n\vee |)\\ &\qquad + t_{n-3}\vee(t_{n-2}\vee(t_{n-1}\vee |))\otimes t_n\vee |\end{aligned}$$

Thus, for the tree $t = t_1\vee(t_2\vee(\ldots\vee(t_n\vee |)\ldots))$, we obtain

$$\begin{aligned}P(t) = \sum_{i=1}^{n-1} & t_1\vee(t_2\vee(\ldots\vee(t_i\vee |)\ldots))\\ &\otimes t_{i+1}\vee(t_{i+2}\vee(\ldots\vee(t_n\vee |)\ldots)).\end{aligned}$$

Hence

$$\begin{aligned}&(P\otimes id)\circ P(t) =\\ &= \sum_{j=1}^{n-1} P(t_1\vee(t_2\vee(\ldots\vee(t_i\vee |)\ldots)))\\ &\quad \otimes t_{i+1}\vee(t_{i+2}\vee(\ldots\vee(t_n\vee |)\ldots))\\ &= \sum_{j=1}^{n-1}\sum_{i=1}^{j-1} t_1\vee(\ldots(t_i\vee |)\ldots)\\ &\quad \otimes t_{i+1}\vee(\ldots(t_j\vee |)\ldots)\otimes t_{j+1}\vee(\ldots(t_n\vee |)\ldots)\\ &= \sum_{1\le i<j\le n-1} t_1\vee(\ldots(t_i\vee |)\ldots)\\ &\quad \otimes t_{i+1}\vee(\ldots(t_j\vee |)\ldots)\otimes t_{j+1}\vee(\ldots(t_n\vee |)\ldots),\end{aligned}$$

and similarly

$$\begin{aligned}&(id\otimes P)\circ P(t) =\\ &= \sum_{k=1}^{n-1} t_1\vee(t_2\vee(\ldots\vee(t_k\vee |)\ldots))\\ &\quad \otimes P(t_{k+1}\vee(t_{k+2}\vee(\ldots\vee(t_n\vee |)\ldots)))\\ &= \sum_{k=1}^{n-1}\sum_{l=k+1}^{n-1} t_1\vee(\ldots(t_k\vee |)\ldots)\\ &\quad \otimes t_{k+1}\vee(\ldots(t_l\vee |)\ldots)\otimes t_{l+1}\vee(\ldots(t_n\vee |)\ldots)\\ &= \sum_{1\le k<l\le n-1} t_1(\ldots\vee(t_k\vee |)\ldots)\\ &\quad \otimes t_{k+1}\vee(\ldots(t_l\vee |)\ldots)\otimes t_{l+1}\vee(\ldots(t_n\vee |)\ldots).\end{aligned}$$

Hence the identity (41) holds for any tree.

13.2 Proof of (30) and (6)

We use (29) to prove (30).

$$\begin{aligned}D_{\mu\nu}(q) &= \sum_t e_0^{2|t|}\varphi^0_{\mu\nu}(t;q)\\ &= \varphi^0_{\mu\nu}(|;q) + \sum_{t_1t_2} e_0^{2|t_1|+2|t_2|+2}\varphi^0_{\mu\nu}(t_1\vee t_2;q)\\ &= \varphi^0_{\mu\nu}(|;q) + \sum_{t_1t_2} e_0^{2|t_1|+2|t_2|+2}\varphi^0_{\mu\lambda}(|\vee t_2;q)\\ &\quad \times\psi^{0\lambda\lambda'}(|;q)\varphi^0_{\lambda'\nu}(t_1;q)\\ &= \varphi^0_{\mu\nu}(|;q) + \sum_{t_2} e_0^{2|t_2|+2}\varphi^0_{\mu\lambda}(|\vee t_2;q)\\ &\quad \times\psi^{0\lambda\lambda'}(|;q)D_{\lambda'\nu}(q).\end{aligned}$$

We multiply the last equation by $D^{-1}(q)$ on the right and by ${D^0}^{-1}(q)$ on the left. This gives us

$$\begin{aligned}[{D^0}^{-1}]_{\mu\nu}(q) = [D^{-1}]_{\mu\nu}(q) + \sum_{t_2} & e_0^{2|t_2|+2}[{D^0}^{-1}]_{\mu\lambda}(q)\\ &\varphi^{0\lambda\lambda'}(|\vee t_2;q)\psi^0_{\lambda'\nu}(|;q).\end{aligned}$$

Since $\varphi^{0\lambda\lambda'}(|\vee t_2;q)$ is transverse, we can replace $[{D^0}^{-1}]_{\mu\lambda}(q)$ by $\psi^0_{\mu\lambda}(|;q)$ in the above equation.

Using 28 and the definition of $\Pi_{\mu\nu}(q)$ given in (25) we obtain

$$\Pi_{\mu\nu}(q) = -\sum_{t_2} e_0^{2|t_2|+2}\psi^0_{\mu\lambda}(|;q)\varphi^{0\lambda\lambda'}(|\vee t_2;q)\psi^0_{\lambda'\nu}(|;q).$$

We can rewrite this last equation as

$$\Pi_{\mu\nu}(q) = \sum_{|t|>0} e_0^{2|t|}\psi^0_{\mu\nu}(t;q) = \sum_t e_0^{2|t|+2}\psi^0_{\mu\nu}(|\vee t;q),$$

with $\psi^{0\mu\nu}(t_1\vee t_2;q) = 0$ if $t_1\neq |$ and

$$\psi^0_{\mu\nu}(|\vee t;q) = -\psi^0_{\mu\lambda}(|;q)\varphi^{0\lambda\lambda'}(|\vee t;q)\psi^0_{\lambda'\nu}(|;q). \quad (83)$$

Finally, we use (37) with $n=0$ and $t_1 = |$

$$\begin{aligned}\varphi^0_{\mu\nu}(|\vee t_2;q) = -iD^0_{\mu\lambda}(q)\int\frac{\mathrm{d}^4p}{(2\pi)^4}&\mathrm{tr}\big[\gamma^\lambda\varphi^1(t_2;p;\lambda',-q)\big]\\ &\times D^0_{\lambda'\nu}(q).\end{aligned}$$

We multiply this equation by ${D^0}^{-1}(q)$ on the left and on the right, and we use the fact that $\varphi^0_{\mu\nu}(|\vee t_2;q)$ is transverse to get, from (83), the relation

$$\psi^{0\mu\nu}(|\vee t;q) = i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\mu\varphi^1(t;p;\nu,-q)\big].$$

If we sum over trees t, the last equation becomes

$$\Pi^{\lambda\mu}(q) = ie_0^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\frac{\delta S(p)}{e_0\delta A^0_\mu(-q)}\big].$$

13.3 Proof of (50)

From the definition of self-energy ψ (Eq.(44)), we use the definition of the convolution (Eq.(39)) and of the pruning operator (Eq.(40)) to write, for a tree $t = t_l \vee t_r$

$$\begin{aligned}\psi^0(t) &= \psi^0(|)\varphi^0(t)\psi^0(|) + \psi^0(|)(\varphi^0 \star \psi^0)(t)\\ &= \psi^0(|)\varphi^0(t)\psi^0(|) + \psi^0(|)\varphi^0(t_l \vee |)\psi^0(t_r)\\ &\quad + \sum_{i=1}^{n(t_r)} \psi^0(|)\varphi^0(t_l \vee u_i)\psi^0(v_i),\end{aligned}$$

where u_i and v_i are the trees obtained by pruning t_r (i.e. $P(t_r) = \sum_i u_i \otimes v_i$).

The last equation will be transformed by using the recursive definition of φ^0 for the trees t, $t_l \vee |$ and $t_l \vee u_i$. For $n = 0$, Eq.(35) becomes

$$\varphi^0(t;q) = i\int \frac{\mathrm{d}^4p}{(2\pi)^4} S^0(q)\gamma^\lambda \varphi^0_{\lambda\lambda'}(t_l;p)\varphi^1(t_r;q-p;\lambda',p).$$

Therefore, we obtain

$$\begin{aligned}\psi^0(t;q) &= -i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda \varphi^0_{\lambda\lambda'}(t_l;p)\\ &\Big[\varphi^1(t_r;q-p;\lambda',p)\psi^0(|;q) + \varphi^1(|;q-p;\lambda',p)\psi^0(t_r;q)\\ &\quad + \sum_{i=1}^{n(t_r)} \varphi^1(u_i;q-p;\lambda',p)\psi^0(v_i;q)\Big].\end{aligned}$$

This can also be written

$$\psi^0(t;q) = i\int \frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda \varphi^0_{\lambda\lambda'}(t_l;p) g(t_r;q-p;\lambda',p),$$

where

$$\begin{aligned}g(t_r;q-p;\lambda',p) &= -\varphi^1(t_r;q-p;\lambda',p)\psi^0(|;q)\\ &\quad -(\varphi^1(q-p;\lambda',p)\star\psi^0(q))(t_r)\\ &\quad -\varphi^1(|;q-p;\lambda',p)\psi^0(t_r;q).\end{aligned}$$

It will be useful to transform $g(t_r;q-p;\lambda',p)$. To do that, we rewrite the equation for $\psi^1(t;q;\lambda,p)$ given at the end of paragraph 6.4 in Ref.[10], so that it gives

$$\begin{aligned}\varphi^0(|;q)\psi^1(t;q;\lambda,p) &= \varphi^0(|;q)\gamma^\lambda\varphi^0(t;q+p)\psi^0(|;q+p)\\ &-\varphi^0(t;q)\gamma^\lambda - \varphi^1(t;q;\lambda,p)\psi^0(|;q+p)\\ &+\sum_{i=1}^{n(t)}\Big[\varphi^0(|;q)\gamma^\lambda\varphi^0(u_i;q+p)\psi^0(v_i;q+p)\\ &-\varphi^0(u_i;q)\psi^1(v_i;q;\lambda,p) - \varphi^1(u_i;q;\lambda,p)\psi^0(v_i;q+p)\Big].\end{aligned}$$

Now, we replace $\varphi^0(t;q+p)$ by its value given from Eq.(45), we add $g(t;q;\lambda,p)$ on both sides and we reorder a bit.

$$\begin{aligned}g(t;q;\lambda,p) &= \varphi^0(|;q)\psi^1(t;q;\lambda,p) + \varphi^0(t;q)\gamma^\lambda\\ &\quad +\varphi^0(|;q)\gamma^\lambda\varphi^0(t;q+p)\psi^0(|;q+p)\\ &\quad -\varphi^1(|;q;\lambda,p)\psi^0(|;q+p)\\ &\quad +\sum_{i=1}^{n(t)}\varphi^0(u_i;q+p)\psi^1(v_i;q;\lambda,p).\end{aligned}$$

Finally, we note that $\varphi^1(|;q;\lambda,p) = \varphi^0(|;q)\gamma^\lambda\varphi^0(t;q+p)$ and we obtain

$$\begin{aligned}g(t;q;\lambda,p) &= \varphi^0(|;q)\psi^1(t;q;\lambda,p) + \varphi^0(t;q)\psi^1(|;q;\lambda,p)\\ &\quad +\sum_{i=1}^{n(t)}\varphi^0(u_i;q)\psi^1(v_i;q;\lambda,p).\end{aligned}$$

14 Appendix2

In this appendix, we collect the relation between bare and renormalized photon and electron φ up to three loops.

14.1 Electron Green function for massless QED

14.1.1 One loop

$$\bar\varphi(\mathsf{Y}) = \varphi(\mathsf{Y}) - \zeta_2(\mathsf{Y})\varphi(|).$$

14.1.2 Two loops

$$\begin{aligned}\bar\varphi(\mathsf{Y}\vee|) &= \varphi(\mathsf{Y}\vee|) - \zeta_3(\mathsf{Y})\varphi(\mathsf{Y}) - \zeta_2(\mathsf{Y}\vee|)\varphi(|),\\ \bar\varphi(|\vee\mathsf{Y}) &= \varphi(|\vee\mathsf{Y}) - \zeta_2(\mathsf{Y})\varphi(\mathsf{Y}) - \zeta_2(|\vee\mathsf{Y})\varphi(|) + \zeta_2(\mathsf{Y})^2\varphi(|).\end{aligned}$$

14.1.3 Three loops

$$\begin{aligned}\bar\varphi((\mathsf{Y}\vee|)\vee|) &= \varphi((\mathsf{Y}\vee|)\vee|) - 2\zeta_3(\mathsf{Y})\varphi(\mathsf{Y}\vee|) + \zeta_3(\mathsf{Y})^2\varphi(\mathsf{Y}) - \zeta_2((\mathsf{Y}\vee|)\vee|)\varphi(|),\\ \bar\varphi((|\vee\mathsf{Y})\vee|) &= \varphi((|\vee\mathsf{Y})\vee|) - \zeta_3(|\vee\mathsf{Y})\varphi(\mathsf{Y}) - \zeta_2((|\vee\mathsf{Y})\vee|)\varphi(|),\\ \bar\varphi(\mathsf{Y}\vee\mathsf{Y}) &= \varphi(\mathsf{Y}\vee\mathsf{Y}) - \zeta_3(\mathsf{Y})\varphi(|\vee\mathsf{Y}) - \zeta_2(\mathsf{Y})\varphi(\mathsf{Y}\vee|)\\ &\quad +\zeta_2(\mathsf{Y})\zeta_3(\mathsf{Y})\varphi(\mathsf{Y}) - \zeta_2(\mathsf{Y}\vee\mathsf{Y})\varphi(|) + \zeta_2(\mathsf{Y}\vee|)\zeta_2(\mathsf{Y})\varphi(|),\\ \bar\varphi(|\vee(\mathsf{Y}\vee|)) &= \varphi(|\vee(\mathsf{Y}\vee|)) - \zeta_3(\mathsf{Y})\varphi(|\vee\mathsf{Y}) - \zeta_2(\mathsf{Y}\vee|)\varphi(\mathsf{Y})\\ &\quad -\zeta_2(|\vee(\mathsf{Y}\vee|))\varphi(|) + \zeta_2(\mathsf{Y}\vee|)\zeta_2(\mathsf{Y})\varphi(|),\\ \bar\varphi(|\vee(|\vee\mathsf{Y})) &= \varphi(|\vee(|\vee\mathsf{Y})) - \zeta_2(\mathsf{Y})\varphi(|\vee\mathsf{Y}) - \zeta_2(|\vee\mathsf{Y})\varphi(\mathsf{Y})\\ &\quad +\zeta_2(\mathsf{Y})^2\varphi(\mathsf{Y}) + [-\zeta_2(|\vee(|\vee\mathsf{Y})) + 2\zeta_2(|\vee\mathsf{Y})\zeta_2(\mathsf{Y}) - \zeta_2(\mathsf{Y})^3]\varphi(|).\end{aligned}$$

14.2 Photon Green function for massless QED

14.2.1 One loop

$$\bar\varphi_{\lambda\mu}(\mathsf{Y}) = \varphi_{\lambda\mu}(\mathsf{Y}) - \zeta_3(\mathsf{Y})\varphi_{\lambda\mu}(|).$$

14.2.2 Two loops

$$\begin{aligned}\bar\varphi_{\lambda\mu}(\mathsf{Y}\vee|) &= \varphi_{\lambda\mu}(\mathsf{Y}\vee|) - 2\zeta_3(\mathsf{Y})\varphi_{\lambda\mu}(\mathsf{Y}) + \zeta_3(\mathsf{Y})^2\varphi_{\lambda\mu}(|),\\ \bar\varphi_{\lambda\mu}(|\vee\mathsf{Y}) &= \varphi_{\lambda\mu}(|\vee\mathsf{Y}) - \zeta_3(|\vee\mathsf{Y})\varphi_{\lambda\mu}(|).\end{aligned}$$

14.2.3 Three loops

$$\begin{aligned}
\bar\varphi_{\lambda\mu}((\mathrm{Y}\vee|)\vee|) &= \varphi_{\lambda\mu}((\mathrm{Y}\vee|)\vee|) - 3\zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(\mathrm{Y}\vee|) \\
&\quad +3\zeta_3(\mathrm{Y})^2\varphi_{\lambda\mu}(\mathrm{Y}) - \zeta_3(\mathrm{Y})^3\varphi_{\lambda\mu}(|), \\
\bar\varphi_{\lambda\mu}((|\vee\mathrm{Y})\vee|) &= \varphi_{\lambda\mu}((|\vee\mathrm{Y})\vee|) - \zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(|\vee\mathrm{Y}) - \zeta_3(|\vee\mathrm{Y})\varphi_{\lambda\mu}(\mathrm{Y}) \\
&\quad +\zeta_3(|\vee\mathrm{Y})\zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(|), \\
\bar\varphi_{\lambda\mu}(\mathrm{Y}\vee\mathrm{Y}) &= \varphi_{\lambda\mu}(\mathrm{Y}\vee\mathrm{Y}) - \zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(|\vee\mathrm{Y}) - \zeta_3(|\vee\mathrm{Y})\varphi_{\lambda\mu}(\mathrm{Y}) \\
&\quad +\zeta_3(|\vee\mathrm{Y})\zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(|), \\
\bar\varphi_{\lambda\mu}(|\vee(\mathrm{Y}\vee|)) &= \varphi_{\lambda\mu}(|\vee(\mathrm{Y}\vee|)) - \zeta_3(\mathrm{Y})\varphi_{\lambda\mu}(|\vee\mathrm{Y}) - \zeta_3(|\vee(\mathrm{Y}\vee|))\varphi_{\lambda\mu}(|), \\
\bar\varphi_{\lambda\mu}(|\vee(|\vee\mathrm{Y})) &= \varphi_{\lambda\mu}(|\vee(|\vee\mathrm{Y})) - \zeta_3(|\vee(|\vee\mathrm{Y}))\varphi_{\lambda\mu}(|).
\end{aligned}$$

14.3 Electron self-energy for massless QED

14.3.1 One loop

$$\bar\psi(\mathrm{Y}) = \psi(\mathrm{Y}) + \zeta_2(\mathrm{Y})\psi(|).$$

14.3.2 Two loops

$$\begin{aligned}
\bar\psi(\mathrm{Y}\vee|) &= \psi(\mathrm{Y}\vee|) - \zeta_3(\mathrm{Y})\psi(\mathrm{Y}) + \zeta_2(\mathrm{Y}\vee|)\psi(|), \\
\bar\psi(|\vee\mathrm{Y}) &= \psi(|\vee\mathrm{Y}) + \zeta_2(\mathrm{Y})\psi(\mathrm{Y}) + \zeta_2(|\vee\mathrm{Y})\psi(|).
\end{aligned}$$

14.3.3 Three loops

$$\begin{aligned}
\bar\psi((\mathrm{Y}\vee|)\vee|) &= \psi((\mathrm{Y}\vee|)\vee|) - 2\zeta_3(\mathrm{Y})\psi(\mathrm{Y}\vee|) + \zeta_3(\mathrm{Y})^2\psi(\mathrm{Y}) + \zeta_2((\mathrm{Y}\vee|)\vee|)\psi(|), \\
\bar\psi((|\vee\mathrm{Y})\vee|) &= \psi((|\vee\mathrm{Y})\vee|) - \zeta_3(|\vee\mathrm{Y})\psi(\mathrm{Y}) + \zeta_2((|\vee\mathrm{Y})\vee|)\psi(|), \\
\bar\psi(\mathrm{Y}\vee\mathrm{Y}) &= \psi(\mathrm{Y}\vee\mathrm{Y}) - \zeta_3(\mathrm{Y})\psi(|\vee\mathrm{Y}) + \zeta_2(\mathrm{Y}\vee|)\psi(\mathrm{Y}) + \zeta_2(\mathrm{Y}\vee\mathrm{Y})\psi(|), \\
\bar\psi(|\vee(\mathrm{Y}\vee|)) &= \psi(|\vee(\mathrm{Y}\vee|)) - \zeta_3(\mathrm{Y})\psi(|\vee\mathrm{Y}) + \zeta_2(|\vee(\mathrm{Y}\vee|))\psi(|) \\
&\quad +\zeta_2(\mathrm{Y})\psi(\mathrm{Y}\vee|) - \zeta_2(\mathrm{Y})\zeta_3(\mathrm{Y})\psi(\mathrm{Y}), \\
\bar\psi(|\vee(|\vee\mathrm{Y})) &= \psi(|\vee(|\vee\mathrm{Y})) + \zeta_2(\mathrm{Y})\psi(|\vee\mathrm{Y}) + \zeta_2(|\vee\mathrm{Y})\psi(\mathrm{Y}) + \zeta_2(|\vee(|\vee\mathrm{Y}))\psi(|).
\end{aligned}$$

14.4 Vacuum polarization for massless QED

14.4.1 One loop

$$\bar\psi_{\lambda\mu}(\mathrm{Y}) = \psi_{\lambda\mu}(\mathrm{Y}) - \zeta_3(\mathrm{Y})\psi_{\lambda\mu}(|).$$

14.4.2 Two loops

$$\begin{aligned}
\bar\psi_{\lambda\mu}(\mathrm{Y}\vee|) &= 0, \\
\bar\psi_{\lambda\mu}(|\vee\mathrm{Y}) &= \psi_{\lambda\mu}(|\vee\mathrm{Y}) - \zeta_3(|\vee\mathrm{Y})\psi_{\lambda\mu}(|).
\end{aligned}$$

14.4.3 Three loops

$$\begin{aligned}
\bar\psi_{\lambda\mu}((\mathrm{Y}\vee|)\vee|) &= 0, \\
\bar\psi_{\lambda\mu}((|\vee\mathrm{Y})\vee|) &= 0, \\
\bar\psi_{\lambda\mu}(\mathrm{Y}\vee\mathrm{Y}) &= 0, \\
\bar\psi_{\lambda\mu}(|\vee(\mathrm{Y}\vee|)) &= \psi_{\lambda\mu}(|\vee(\mathrm{Y}\vee|)) - \zeta_3(\mathrm{Y})\psi_{\lambda\mu}(|\vee\mathrm{Y}) - \zeta_3(|\vee(\mathrm{Y}\vee|))\psi_{\lambda\mu}(|), \\
\bar\psi_{\lambda\mu}(|\vee(|\vee\mathrm{Y})) &= \psi_{\lambda\mu}(|\vee(|\vee\mathrm{Y})) - \zeta_3(|\vee(|\vee\mathrm{Y}))\psi_{\lambda\mu}(|).
\end{aligned}$$

15 Appendix 3: The first trees

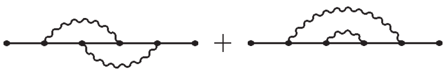
This appendix gives the Feynman diagrams corresponding to the first trees for the electron and photon Green functions.

15.1 Electron Green function

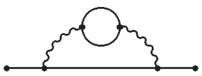
For the electron Green functions, all electron loops are oriented anticlockwise and the propagator is oriented from right to left, as indicated in $\varphi^0(|)$. Once the Feynman diagrams for $\varphi^0(t)$ are known, those for $\varphi^1(t)$ are obtained by summing all possible insertions of a photon dangling bond at each electron propagator , and so on for $\varphi^n(t)$. Notice that the last two diagrams of $\varphi^0(|\vee(\mathrm{Y}\vee|))$ are zero by Furry's theorem. However, they are useful to generate Feynman diagrams for higher order trees.

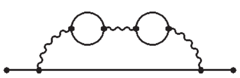
$\varphi^0(|) =$

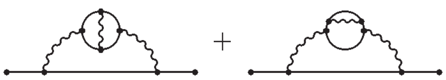
$\varphi^0(\mathrm{Y}) =$ 

$\varphi^0(|\vee\mathrm{Y}) =$ 

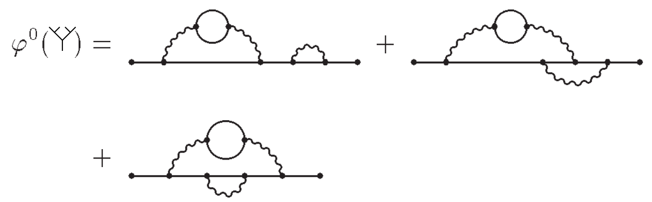
$+$ 

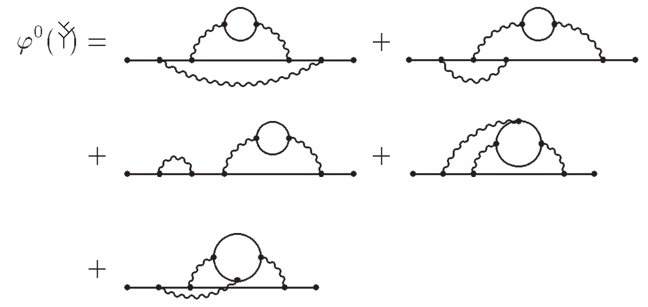
$\varphi^0(\mathrm{Y}\vee|) =$ 

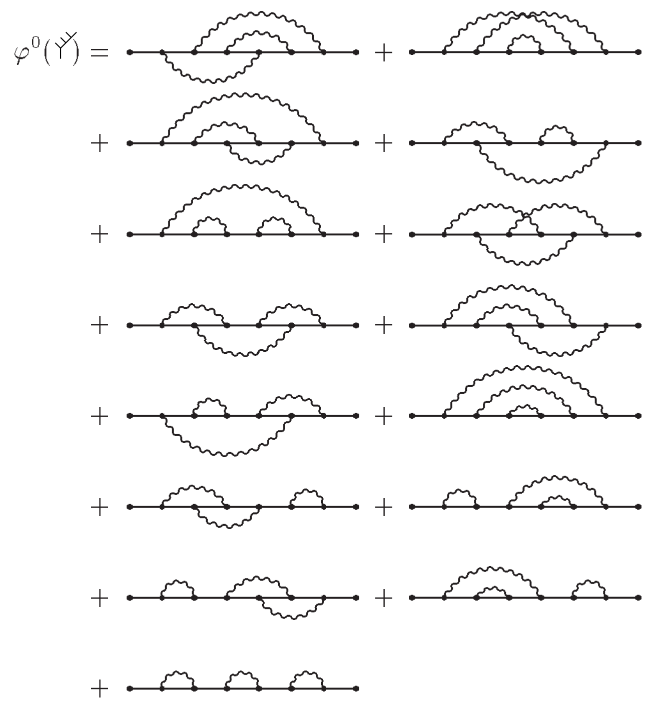
$\varphi^0((\mathrm{Y}\vee|)\vee|) =$ 

$\varphi^0((|\vee\mathrm{Y})\vee|) =$ 

$+$ 

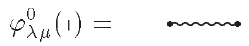


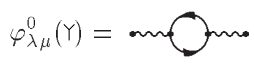


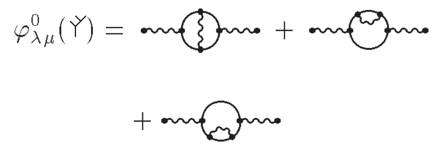


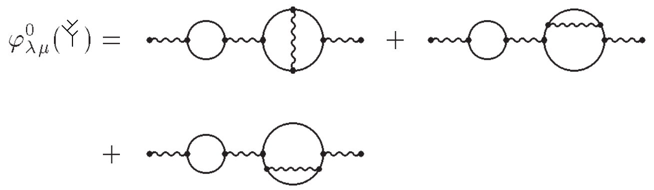
15.2 Photon Green function

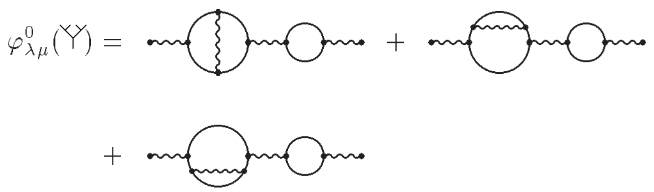
For the photon Green functions, all electron loops are oriented anticlockwise. This is only indicated explicitly for $\varphi^0_{\lambda\mu}(\curlyvee)$.

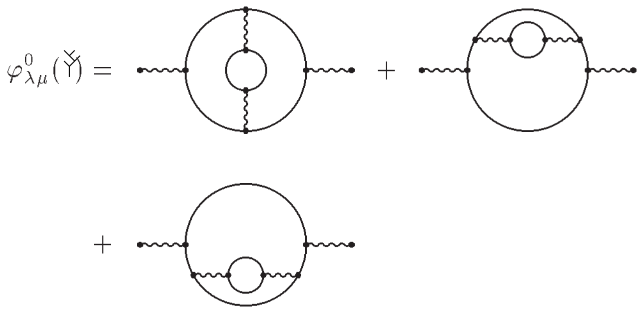


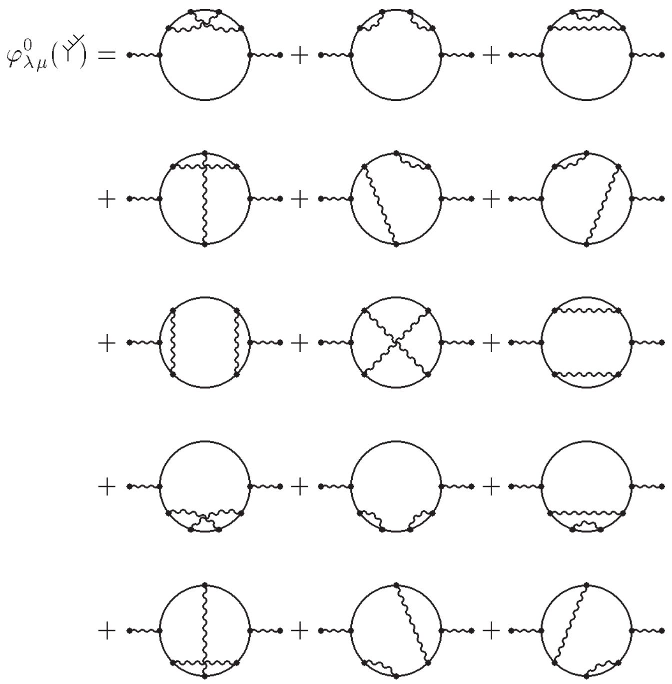












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