On the Leibniz cohomology of vector fields

Alessandra Frabetti[*], Friedrich Wagemann[†]

May 15, 2000

Abstract

I. M. Gelfand and D. B. Fuks have studied the cohomology of the Lie algebra of vector fields on a manifold. In this article, we generalize their main tools to compute the Leibniz cohomology, by extending the two spectral sequences associated to the diagonal and the order filtration. In particular, we determine some new generators for the diagonal Leibniz cohomology of the Lie algebra of vector fields on the circle.

1 Introduction

Let g be a Lie algebra. The Leibniz cohomology $HL^*(g)$ of g with trivial coefficients is the cohomology of the complex with cochain modules $CL^n(g) = \mathrm{Hom}(g^{\otimes n}, \mathbb{R})$ and coboundary operators $d : CL^n(g) \to CL^{n+1}(g)$, defined by

$$df(x_1, \ldots, x_{n+1}) = \sum_{i<j} (-1)^{j+1} f(x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x}_j, \ldots, x_{n+1}),$$

where $(x_1, \ldots, x_k)$ denotes the element $x_1 \otimes \ldots \otimes x_k$ in the tensor product $g^{\otimes k}$. The Leibniz cohomology is a generalization of the Chevalley-Eilenberg cohomology of Lie algebras, in the sense that it constitutes the natural cohomology theory for Leibniz algebras, which are non-antisymmetric generalizations of Lie algebras. It was introduced by J.-L. Loday and T. Pirashvili in [6]. When computed on Lie algebras, the Leibniz cohomology may detect new invariants in dimensions higher then 1, since $HL^0(g) = H^0(g) \cong \mathbb{R}$ and $HL^1(g) = H^1(g)$. The method to compare the Lie and the Leibniz cohomology of a Lie algebra in higher dimensions is given by the Pirashvili long exact sequence

$$\begin{aligned} 0 \longrightarrow H^2(g) \longrightarrow HL^2(g) \longrightarrow H^0_{rel}(g) \longrightarrow H^3(g) \longrightarrow \ldots \\ \ldots \longrightarrow H^n(g) \longrightarrow HL^n(g) \longrightarrow H^{n-2}_{rel}(g) \longrightarrow H^{n+1}(g) \longrightarrow \ldots \end{aligned}$$

induced by the projections $g^{\otimes n} \longrightarrow \Lambda^n g$, where $H^*_{rel}(g)$ is a relative cohomology described in [10].

Loday and J.-M. Oudom showed in [5], [9] that the Leibniz cohomology $HL^*(g)$ is endowed with the structure of a graded dual Leibniz algebra (in

[*]I.R.M.A. - Université L. Pasteur, 7 rue René Descartes, F-67084 Strasbourg Cedex. frabetti@math.u-strasbg.fr

[†]I.G.D. - Université Lyon-I, 43, bd du 11 Novembre 1918 F-69622 Villeurbanne Cedex. wagemann@desargues.univ-lyon1.fr

the sense of Koszul duality for operads). In fact, there is a cup product $\cup : HL^p(g) \otimes HL^q(g) \longrightarrow HL^{p+q}(g)$ defined on the cohomology classes of two cocycles $\alpha \in CL^p(g)$, $\beta \in CL^q(g)$ as the cohomology class of the cocycle

$$\alpha \cup \beta(x_1, \ldots, x_{p+q}) = \sum_{\sigma \in Sh_{p-1,q}} \operatorname{sgn}(\sigma) \alpha(x_1, x_{\sigma(2)}, \ldots, x_{\sigma(p)}) \beta(x_{\sigma(p+1)}, \ldots, x_{\sigma(p+q)}),$$

where $Sh_{p-1,q}$ denotes the set of the $(p-1,q)$-shuffles among the permutations on $\{2, 3, ..., p+q\}$. The cup product satisfies the relation of dual Leibniz algebras

$$(\alpha \cup \beta) \cup \gamma - \alpha \cup (\beta \cup \gamma) = (-1)^{|\beta||\gamma|} \alpha \cup (\gamma \cup \beta).$$

Therefore, in order to describe the Leibniz cohomology, it suffices to give its generators as a dual Leibniz algebra.

When g is a topological Lie algebra, we can consider the submodules of continuous cochains $CL^n_{cont}(g) = \mathrm{Hom}_{cont}(g^{\otimes n}, \mathbb{R}) \subset \mathrm{Hom}(g^{\otimes n}, \mathbb{R}) = CL^n(g)$. Since the Leibniz differential df of any continuous cochain f is still continuous, the sequence of continuous cochains $d : \mathrm{Hom}_{cont}(g^{\otimes n}, \mathbb{R}) \to \mathrm{Hom}_{cont}(g^{\otimes n+1}, \mathbb{R})$ forms a subcomplex of the Leibniz complex, whose cohomology is called continuous Leibniz cohomology of g. It is still denoted by $HL^*(g)$ when no confusion can arise.

If g is the topological Lie algebra $\mathrm{Vect}\, M$ of vector fields on a differentiable manifold M, the Leibniz cohomology $HL^*(\mathrm{Vect}\, M)$ is a generalization of the Lie cohomology $H^*(\mathrm{Vect}\, M)$ studied by I. M. Gelfand, D. B. Fuks, R. Bott, G. Segal and A. Haefliger in [3], [2], [1], [4] and references therein.

In order to study the Leibniz cohomology of $\mathrm{Vect}\, \mathbb{R}^n$, J. Lodder starts with the Lie algebra of formal vector fields $W_n = \mathbb{R}[[x_1, ..., x_n]] \otimes \{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$, where $\mathbb{R}[[x_1, ..., x_n]]$ denotes the algebra of formal power series in n variables, and $\frac{\partial}{\partial x_i}$ the partial derivation in respect of the variable x_i. He showed in [8] that the first non-zero class in $HL^*(W_n)$ is the Godbillon-Vey class in dimension $2n+1$. Using Pirashvili's sequence, he describes $HL^*(W_n)$ as a dual Leibniz algebra with generators and relations given in [8]. For $n = 1$, this dual Leibniz algebra has one generator in degree 3 (the Godbillon-Vey class θ) and a new generator in degree 4 (called α) with relations $\theta^2 = 0$ and $\alpha \cup \theta = 0$.

The case of $HL^*(\mathrm{Vect}\, M)$, for general smooth manifolds, is much more complicated. Even in the simplest case $M = S^1$ there are too many non-zero terms appearing in the sequence, which prevent us to determine the dimension of the Leibniz cohomology groups. Hence Pirashvili's method cannot be reproduced in general.

In this paper, instead, we generalize the classical method of Gelfand and Fuks, regarding continuous Leibniz cochains on $\mathrm{Vect}\, M$ as generalized sections of the tensor powers of the tangent bundle TM, over the manifold $M \times ... \times M$. This guarantees for Leibniz cohomology the same general results previously determined by Gelfand and Fuks for Lie cohomology, and some explicit computations for manifolds with few non-trivial de Rham cohomology classes.

In particular, in section 2 we describe the spectral sequence associated to the diagonal filtration (filtration of a complex of distributions by their *support*). The first term of the sequence is given by some quotient complexes.

In section 3 we describe a spectral sequence converging to the cohomology of these quotient complexes. It is associated to the order filtration, i.e. the filtration of a complex of distributions which are supported on a given submanifold by their *order*.

In the last section, we show that the order spectral sequence for the Leibniz cohomology of $\mathrm{Vect}\, S^1$ collapses. The resulting diagonal cohomology is 4-periodic, with dimension respectively $0, 1, 2, 1$. Finally, we describe the diagonal cocycles which determine the new generators.

For the collapsing of the diagonal spectral sequence, and thus for an explicit desciption of all continuous Leibniz cohomology of $\mathrm{Vect}\, S^1$, we would need the dual Leibniz algebra structure which is still a mystery for us. The knowledge of this structure leads to study the relationship with Leibniz minimal models (as introduced by M. Livernet in [7]) and 'real' homotopy types in the spirit of Bott, Haefliger and Segal [4] [1].

2 Generalized sections and diagonal filtration

Let M be a smooth oriented manifold of dimension n. Let $\mathrm{Vect}\, M$ be the Lie algebra of vector fields on M, equipped with the C^∞ topology which makes it a topological Fréchet nuclear Lie algebra. Let $CL^*_{cont}(\mathrm{Vect}\, M)$ be the complex of continuous Leibniz cochains of $\mathrm{Vect}\, M$. Then, for any $m \geq 0$, the module of Leibniz m-cochains is the space of generalized sections of the bundle of tensor powers $TM^{\otimes m}$ over the cartesian product manifold M^m, meaning $TM^{\otimes m} = \bigotimes_{i=1}^m \pi_i^*(TM)$ where $\pi_i : M^m \to M$ is the projection on the ith factor. In other words, the Leibniz m-cochains of $\mathrm{Vect}\, M$ are the distributions on M^m with values in the bundle $TM^{\otimes m}$. Remark that the Chevalley-Eilenberg m-cochains of $\mathrm{Vect}\, M$, that is, the elements of $C^m_{cont}(\mathrm{Vect}\, M)$, can be seen as conveniently anti-symmetrized generalized sections of the same bundle, cf [3], [2]. Throughout this section we follow the notations of [2].

For any $k \geq 1$, let M^m_k be the subset of M^m of m-uples $(x_1, \ldots, x_m)$ which do not have more then k different entries x_i, that is,

$$M^m_k = \{(x_1, \ldots, x_m) \in M^m \mid \forall (i_1, \ldots, i_{k+1}) \subset (1, \ldots, m)\ \exists (i_r, i_s)\ :\ x_{i_r} = x_{i_s}\}.$$

Then, the submanifold $M^m_1 = \{(x, \ldots, x) \in M^m\} = \triangle$ coincides with the diagonal of M^m, all the subsets M^m_k for $k \geq m$ coincide with the whole M^m, and there is a sequence of inclusions

$$0 = M^m_0 \subset M^m_1 = \triangle \subset M^m_2 \subset \ldots \subset M^m_{m-1} \subset M^m_m = M^m.$$

Denote by $CL^m_k(M)$ the subspace of $CL^m_{cont}(\mathrm{Vect}\, M)$ of generalized sections of the bundle $TM^{\otimes m}$ with support on the subset M^m_k. Then there is an induced sequence of inclusions of cochains modules

$$\{0\} = CL^m_0(M) \subset CL^m_1(M) \subset \ldots \subset CL^m_{m-1}(M) \subset CL^m_m(M) = CL^m_{cont}(\mathrm{Vect}\, M).$$

In other words, as a distribution, a Leibniz m-cochain is concentrated on some subspace M^m_k (perhaps on the whole M^m).

The above inclusions of Leibniz cochain modules define an increasing multiplicative filtration of the complex $CL^*_{cont}(\mathrm{Vect}\, M)$ called *diagonal filtration*, that is,

$$d(CL^m_k(M)) \subset CL^{m+1}_k(M),$$

for all k and m, and

$$CL^m_k(M) \cup CL^l_h(M) \subset CL^{m+l}_{k+h}(M),$$

for all k,h and m,l. To see this, it suffices to understand that the subspace $CL^m_k(M)$ consists exactely of Leibniz cochains which vanish on any family of m vector fields having the property $(\triangle_k)$. (Recall that a family of m-vector fields $\Gamma \subset \mathrm{Vect}\, M$ has the property $(\triangle_k)$ if for any set of k points $S \subset M$ at least one vector field of Γ vanishes in a neighbourhood of S.)

Hence, there exists a spectral sequence abutting to the Leibniz cohomology of $\mathrm{Vect}\, M$. To avoid confusion, we denote this spectral sequence by $B_*(M)$. The 0-th term is the bicomplex of the quotients

$$B_0^{m,k}(M) := CL^m_k(M)/CL^m_{k-1}(M),$$

with differentials $d_v : B_0^{m,k}(M) \longrightarrow B_0^{m+1,k}(M)$ and $d_t : B_0^{m,k}(M) \longrightarrow B_0^{m+1,k-1}(M)$ (of bidegree $(1,0)$ and $(1,-1)$) induced by the Leibniz coboundary. For $k = 1$, the differential d_v coincides with the Leibniz differential, hence the family $B_0^{m,1}(M) = CL^m_\triangle(M)$ is a subcomplex called *diagonal complex*. For $k = 0$ we have $B_0^{m,0}(M) = 0$, except for $m = 0$ where $B_0^{0,0}(M) = \mathbb{R}$.

Then the first term of the spectral sequence is the cohomology

$$\begin{array}{ll} B_1^{m,k}(M) = H^m(CL^*_k(M)/CL^*_{k-1}(M), d_v), & \text{for } k > 1 \\ B_1^{m,1}(M) = H^m(CL^*_\triangle(M), d_v), & \text{for } k = 1 \\ B_1^{m,0}(M) = \begin{cases} \mathbb{R}, & m = 0 \\ 0, & m > 0 \end{cases}, & \text{for } k = 0. \end{array}$$

The cohomology of the complex $(CL^*_\triangle(M), d_v)$ is also called the diagonal Leibniz cohomology of $\mathrm{Vect}\, M$, and denoted by $HL^*_\triangle(\mathrm{Vect}\, M)$. By abuse of notation, we call k-diagonal cochains the elements of $B_0^{*,k}(M)$, and k-diagonal cohomology the term $B_1^{*,k}(M)$, also denoted $HL^*_{(k)}(\mathrm{Vect}\, M)$. Of course, if $\tilde{d}_t$ is the differential induced by d_t on the first term, the second term of the spectral sequence is the cohomology $B_2^{m,k}(M) = H^k(B_1^{m+*,*}(M), \tilde{d}_t)$. The Leibniz cohomology of $\mathrm{Vect}\, M$ is then

$$HL^m(\mathrm{Vect}\, M) = \bigoplus_{k \geq m} B_\infty^{m,k}(M).$$

In this generality, as for the Gelfand-Fuks computations, the spectral sequence is rather intractable. In order to compute its first term, we introduce following Gelfand and Fuks, for each quotient complex $B_0^{*,k}(M) = CL^*_k(M)/C^*_{k-1}(M)$ a spectral sequence abutting to its cohomology $B_1^{m,k}(M)$. These spectral sequences are defined by the order filtration for Leibniz cochains, which is discussed in the next section.

3 The order filtration for diagonal cohomology

The set $B_0^{m,k}(M)$ of k-diagonal Leibniz cochains contains the distributions concentrated on the subset M^m_k modulo those concentrated on the subset M^m_{k-1}. The set $M^m_{(k)} = M^m_k \setminus M^m_{k-1}$ is now a submanifold without singularities, and a k-diagonal Leibniz cochain can then be described as a distributions defined on the jets of sections of the bundle $TM^{\otimes m}$, where the jet expansion is taken in the normal direction to the submanifold $M^m_{(k)}$ in M^m.

Recall that a generalized section of a vector bundle E over M, concentrated on a subset $S \subset M$, is of order $\leq l$ on S if it vanishes on any section having trivial l-jet at every point of S.

For any integer p, denote by $F^p B_0^{m,k}(M)$ the subspace of Leibniz k-diagonal m-cochains of $\operatorname{Vect} M$ which are of order $\leq m-p$. The family of such spaces defines a decreasing filtration of the complex $CL_k^*(M)/CL_{k-1}^*(M)$ of k-diagonal Leibniz cochains. In particular, for $k=1$, the family $F^p B_0^{m,1}(M) = F^p CL_\triangle^m(M)$ gives a filtration of the diagonal complex.

Theorem 3.1 *For each $k>0$ there is a spectral sequence abutting to the k-diagonal Leibniz cohomology of* $\operatorname{Vect} M$, *with the following E_2-term:*

$$^{(k)}E_2^{p,q} = H_{-p}(M^k, M_{k-1}^k) \otimes \bigoplus_{\substack{q_1+\ldots+q_k=q \\ q_1,\ldots,q_k>0}} HL^{q_1}(W_n) \otimes \ldots \otimes HL^{q_k}(W_n),$$

where $H_{-p}(M^k, M_{k-1}^k)$ denotes the relative cohomology of the manifold M^k with respect to the subspace M_{k-1}^k.

In particular, for $k=1$, there is a spectral sequence

$$E_2^{p,q} = H_{-p}(M) \otimes HL^q(W_n) \Rightarrow HL_\triangle^{p+q}(\operatorname{Vect} M)$$

abutting to the diagonal cohomology of $\operatorname{Vect} M$.

Proof: Since the proof for Leibniz cohomology does not substantially differs from that for Lie cohomology, we only stress the main differences in the simple case of the diagonal complex.

By definition, the 0-th term of the spectral sequence defined by the order filtration on the diagonal complex $CL_\triangle^*(M)$ is the bicomplex

$$E_0^{p,q}(M) = F^p CL_\triangle^{p+q}(M)/F^{p+1}CL_\triangle^{p+q}(M)$$

that is, $E_0^{p,q}$ is the quotient space of cochains from $CL_\triangle^{p+q}(\operatorname{Vect} M)$ which are of order $\leq q$ with respect to the diagonal $\triangle$, modulo those of order $<q$. In other words, an element of the space $E_0^{p,q}$ is a generalized section of the bundle over M

$$\mathcal{E}_0^{p,q} = \operatorname{Hom}(S^q \operatorname{norm}_{M^{p+q}} \triangle, TM^{\otimes p+q}|_\triangle).$$

Here, $\operatorname{norm}_{M^{p+q}} \triangle$ is the normal bundle of the diagonal (closed) submanifold $\triangle \subset M^{p+q}$, and is related to the fact that the jet expansion is in the normal direction with respect to the diagonal. The qth symmetric power S^q arises from the fact that the generalized sections are non-zero exactly on those elements with non-trivial q-jet on the diagonal modulo those with trivial $q+1$-jet. The fiber of the bundle $\mathcal{E}_0^{p,q}$ in a point $x \in M$ is

$$\operatorname{Hom}(S^q(\bigoplus^{p+q} V/V_\triangle), V^{\otimes p+q}),$$

where $V = T_x M$ is the fiber of TM and $V_\triangle$ denotes the image of the diagonal inclusion $V \hookrightarrow V \oplus \ldots \oplus V$. The vector space $S^q(\bigoplus^{p+q} V/V_\triangle)$ admits the standard Koszul resolution

$$0 \leftarrow S^q((\bigoplus^{p+q} V)/V_\triangle) \stackrel{pr}{\leftarrow} S^q(\bigoplus^{p+q} V) \leftarrow S^{q-1}(\bigoplus^{p+q} V) \otimes V \leftarrow \dots$$
$$\dots \leftarrow S^{q-i+1}(\bigoplus^{p+q} V) \otimes \Lambda^{i-1} V \leftarrow \dots$$

Using the isomorphisms

$$\begin{aligned}\mathrm{Hom}(S^{q-i}(\bigoplus^{p+q} V) \otimes \Lambda^i V, V^{\otimes p+q}) &\cong \Lambda^i V' \otimes \mathrm{Hom}(S^{q-i}(\bigoplus^{p+q} V), V^{\otimes p+q}) \\ &\cong \Lambda^i V' \otimes \mathrm{Hom}(S^{q-i}(\bigoplus^{p+q} V) \otimes (V')^{\otimes p+q}, \mathbb{R}),\end{aligned}$$

and

$$S^*(\bigoplus^{p+q} V) \otimes (V')^{\otimes p+q} \cong (S^*(V) \otimes (V'))^{\otimes p+q} \cong W_n^{\otimes p+q},$$

where W_n is the Lie algebra of formal vector fields on $\mathbb{R}^n$, we finally have an isomorphism

$$\mathrm{Hom}(S^{q-i}(\bigoplus^{p+q} V) \otimes \Lambda^i V, V^{\otimes p+q}) \cong \Lambda^i V' \otimes CL^{p+q}_{(-p-i)}(W_n)$$

where the index l in $CL^{p+q}_{(l)}(W_n)$ means the weight with respect to the adjoint action of the Euler field $e_0 = \sum_{i=1}^n x_i \frac{d}{dx_i}$. Then the Koszul resolution gives rise to an exact sequence of fibres. ¿From this, we get the corresponding exact sequence of bundles, and of sections, because the global sections functor is exact on $\mathcal{C}^\infty$ fibre bundles. Finally, we pass to the exact sequence of generalized sections by a lemma on duals of Fréchet nuclear spaces.

In conclusion, we have an exact sequence of complexes (see [2] p.146 for details)

$$0 \leftarrow E_0^{p,*} \leftarrow Sec' \xi_0^{p,*} \leftarrow Sec' \xi_1^{p,*} \leftarrow \dots$$

where $\xi_i^{p,q}$ the bundle associated with TM with typical fiber $\Lambda^i(TM)' \otimes \gamma_i^{p,q}$, and $\gamma_i^{p,q}$ is the bundle associated with TM with typical fiber $CL^{p+q}_{(-p-i)}(W_n)'$. We use lemma (A.1) to get the acyclicity of all complexes in the above sequence except $E_0^{p,*}$ and $Sec'\xi_{-p}^{p,*}$. The identification of the differential $d_1^{p,q}$ is clearly the same as in [3], so we get the stated result. □

This proof can easily be adapted to obtain the higher diagonal E_2-terms.

4 Leibniz cohomology of the Lie algebra of vector fields on S^1

Now we consider the circle S^1. We choose coordinates $z = \exp(i\varphi)$, for $\varphi \in [0, 2\pi]$. As a topological Lie algebra, $\mathrm{Vect}\, S^1$ is spanned by the vector fields $e_k := i \exp(ik\varphi) \frac{d}{d\varphi}$, for all integers k, with Lie bracket $[e_k, e_l] = (k-l)e_{k+l}$.

The Lie cohomology of $\mathrm{Vect}\, S^1$ was computed by Gelfand and Fuks. It is a free graded-commutative algebra with one even generator in degree 2, represented by the Gelfand-Fuks cocycle ω, and one odd generator in degree 3, represented by the Godbillon-Vey cocycle θ. Hence

$$H^*(\mathrm{Vect}\, S^1) = \bigoplus_{k \geq 0} \mathbb{C}[\omega^k] \oplus \bigoplus_{k \geq 0} \mathbb{C}[\theta \cup \omega^k].$$

For each $\varphi_0 \in S^1$, the Taylor expansion of vector fields defines a map

$$\begin{array}{rcl} \pi_{\varphi_0} : \mathrm{Vect}\, S^1 & \longrightarrow & W_1 \\ f(\varphi)\frac{d}{d\varphi} & \longmapsto & \sum_{n\geq 0} \frac{d^n f(\varphi_0)}{d\varphi^n} x^n \frac{d}{dx}, \end{array}$$

where $W_1 = \mathbb{C}[[x]] \otimes \{\frac{d}{dx}\}$ is the complex Lie algebra of formal vector fields in the origin of $\mathbb{R}$ and $x = \varphi - \varphi_0$. The pull back π^* gives a map from the Lie cocycles of W_1 to the Lie cocycles of $\mathrm{Vect}\, S^1$, and similarly a map between the Leibniz cocycles. A cocycle $\gamma_{\varphi_0} = \pi^*_{\varphi_0}\gamma$ on $\mathrm{Vect}\, S^1$ which is pulled back from a cocycle γ on W_1 is *local* on S^1, in the sense that as a distribution it has support on the single point φ_0. Changing the point φ_0 of evaluation produces cocycles which are cohomologous, hence a local cocycle admits an S^1-invariant form given by the integration

$$\gamma := \int_0^{2\pi} \gamma_\varphi \, d\varphi,$$

where S^1 is meant to act on $\mathrm{Vect}\, S^1$ by rotations.

The Godbillon-Vey cocycle θ on $\mathrm{Vect}\, S^1$ is precisely pulled back from a cocycle on W_1, hence it is a local. Instead, the Gelfand-Fuks cocycle ω is a *diagonal* cocycle, that is, it is a distribution with support on the diagonal of $S^1 \times S^1$.

The Leibniz cohomology of W_1, computed by Lodder, is the dual Leibniz algebra generated by the Godbillon-Vey cocycle θ and a new generator α in degree 4. The cup products θ^2 and $\alpha \cup \theta$ are zero, because of the definition of the cup product. The cup product $\alpha^2 \cup \alpha$ is equal to $2\alpha \cup \alpha^2$, because of the relation of dual Leibniz algebra and the fact that α has even degree. Hence the only higher order Leibniz cocycles for W_1 are $\alpha^k := \alpha \cup \alpha^{k-1}$ and $\theta \cup \alpha^k$.

The pull back on $\mathrm{Vect}\, S^1$ of these cocycles gives all the local Leibniz cocycles which appear in $HL^*(\mathrm{Vect}\, S^1)$. We now determine the diagonal ones.

The diagonal Leibniz cohomology

The diagonal cohomology $HL^*_\Delta(\mathrm{Vect}\, S^1)$ is given by the cohomology classes which are represented by diagonal cocycles. In particular, it also contains local classes. The subset of local classes forms a dual Leibniz algebra with the cup product. In fact, the cup product of two local cocycles is still a local cocycle. Instead, the cup product of a local cocycle by a diagonal one, or the product of two diagonal cocycles, is not diagonal anymore, but 2-diagonal. In general, the cup product of k diagonal cocycle (with at least a non-local one) is k-diagonal. Therefore, the diagonal cohomology $HL^*_\Delta(\mathrm{Vect}\, S^1)$ is not a dual Leibniz algebra with the cup product, and it can only be described as a vector space.

Theorem 4.1 *The diagonal cohomology of* $\mathrm{Vect}\, S^1$ *is the graded vector space spanned by the classes of the local cocycles*

$$\begin{array}{cl} \theta \cup \alpha^r & \text{in degree } 3+4r, \text{ for } r \geq 0 \\ \alpha^s & \text{in degree } 4s, \text{ for } s \geq 1 \end{array}$$

and by the classes of the diagonal cocycles

$$\begin{array}{cl} \omega_r & \text{in degree } 2+4r, \text{ for } r \geq 0 \\ \beta_s & \text{in degree } 4s-1, \text{ for } s \geq 1 \end{array}$$

where $\omega_0 = \omega$ is the Gelfand-Fuks cocycle in degree 2 and ω_r, β_s determine new invariants for $r, s \geq 1$.

Proof: The spectral sequence of theorem (3.1), which abuts to $HL^*_\triangle(\mathrm{Vect}\, S^1)$, degenerates at the second term for $M = S^1$, because

$$E_2^{p,q}(S^1) = H_{-p}(S^1) \otimes HL^q(W_1)$$

differs from 0 only for $-p = 0, 1$, hence all the induced differentials in the higher terms are zero. So, we have

$$HL^m_\triangle(\mathrm{Vect}\, S^1) = \bigoplus_{\substack{p+q=m \\ q>0}} H_{-p}(S^1) \otimes HL^q(W_1),$$

for all $m > 0$, where the Leibniz cohomology of the formal vector fields W_1 was computed by Lodder, in [8], as

$$HL^q(W_1) = \bigoplus_{r+s=q} \Lambda^r[\theta] \otimes T^s[\alpha].$$

Thus, we obtain

$$HL^m_\triangle(\mathrm{Vect}\, S^1) = \bigoplus_{\substack{p+r+s=m \\ r+s>0}} \Lambda^p[\eta] \otimes \Lambda^r[\theta] \otimes T^s[\alpha],$$

where η is a generator in degree -1, θ is the Godbillon-Vey generator in degree 3, and α is the Lodder generator in degree 4. Since $\Lambda^p[\eta]$ differs from 0 only for $p = 0, -1$, and $\Lambda^r[\theta]$ differs from 0 only for $r = 0, 3$, we have

$$\begin{aligned} HL^m_\triangle(\mathrm{Vect}\, S^1) &= \bigoplus_{r+s=m+1} \mathbb{R}[\eta] \otimes \Lambda^r[\theta] \otimes T^s[\alpha] \oplus \bigoplus_{r+s=m} \Lambda^r[\theta] \otimes T^s[\alpha] \\ &= \mathbb{R}[\eta \otimes \theta] \otimes T^{m-2}[\alpha] \oplus \mathbb{R}[\theta] \otimes T^{m-3}[\alpha] \oplus \mathbb{R}[\eta] \otimes T^{m+1}[\alpha] \oplus T^m[\alpha]. \end{aligned}$$

If we call the new diagonal cocycles

$$\begin{aligned} \omega_r &:= \eta \otimes \theta \otimes \alpha^r, && \text{for } r \geq 0 \\ \beta_s &:= \eta \otimes \alpha^s, && \text{for } s \geq 1, \end{aligned}$$

and we remark that the higher-degree local cocycles are represented by the cup products, we get the final result

$$HL^*_\triangle(\mathrm{Vect}\, S^1) = \Lambda^*[\theta] \otimes T^*[\alpha] \oplus \bigoplus_{r\geq 0} \mathbb{C}[\omega_r] \oplus \bigoplus_{s \geq 1} \mathbb{C}[\beta_s].$$

□

Corollary 4.2 *The diagonal Leibniz cohomology $HL^*_\triangle(\mathrm{Vect}\, S^1)$ is periodic of period 4. The dimensions of $HL^{n+4k}_\triangle(\mathrm{Vect}\, S^1)$ for $n = 1, 2, 3, 4$ are respectively $0, 1, 2, 1$.*

In particular, the second cohomology group $HL^2(\mathrm{Vect}\, S^1)$ has dimension 1, generated by the Gelfand-Fuks class. This result was obtained by Loday and Pirashvili in [6].

Representative cocycles for the cohomology classes

Recall that the Godbillon-Vey class in $H^3(W_1)$ has a representative cocycle

$$\theta(F,G,H)=\left|\begin{array}{ccc} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{array}\right|_{x=0},$$

where $F=f(x)\frac{d}{dx}$, $G=g(x)\frac{d}{dx}$ and $H=h(x)\frac{d}{dx}$ denote some formal vector fields, and all the functions are evaluated at $x=0$. Now, there are two ways to derive from θ the Gelfand-Fuks cocycle.

Under the pull back $\pi^*_{\varphi_0}: C^*(W_1)\longrightarrow C^*(\mathrm{Vect}\, S^1)$, θ determines a local cohomology class in $H^3(\mathrm{Vect}\, S^1)$. Its representative cocycle $\theta_{\varphi_0}(F,G,H)$ can be expressed by the same local formula, where $F=f(\varphi)\frac{d}{d\varphi}$, G and H are now vector fields on S^1 and the functions are evaluated at $\varphi=\varphi_0$. In fact, the integral form of local cocycles has an advantage, since it may allow to determine automatically diagonal cocycles of one degree less, by contracting any of the variables with the Euler field $e_0=i\frac{d}{d\varphi}$. We explain this in detail.

The only cochains which may contribute to the cohomology are those of zero weight, i.e. those γ for which $ad_{e_0}(\gamma)=0$, because the subcomplex of non-zero weight cochains is contractible, as shown in appendix A. Among these, those of the form $\gamma=\gamma'\wedge\epsilon^0$, where $\epsilon^0=e_0^*$ is the dual cochain of the Euler vector field and γ' has zero weight, are such that the contraction gives $i_{e_0}(\gamma)=\gamma'$. Since the Lie coboundary operator satisfies Cartan's formula, we have

$$d(\gamma)=d(\gamma')\wedge\epsilon^0.$$

Hence, γ is a cocycle if and only if γ' is.

However, if the local cocycle γ is not in its integral form, the dependence $\gamma=\gamma_{\varphi_0}$ from the evaluation point φ_0 is reflected onto $\gamma'=\gamma'_{\varphi_0}$, while the new cocycle γ' is surely not local. Indeed, the term which leads to locality is precisely the ϵ^0 that γ', being antisymmetric, cannot contain. Since the contraction and the differential commute with the integration, we can avoid the dependence on the point of evaluation, by applying the procedure to the integral mean of $\gamma\wedge\epsilon^0$.

The Godbillon-Vey cocycle is an example of a cocycle of the form $\gamma\wedge\epsilon^0$, with $\gamma=-i/2\,\epsilon^{-1}\wedge\epsilon^1$ and $\epsilon^p=e_p^*$. For the Gelfand-Fuks class in $H^2(\mathrm{Vect}\, S^1)$, the well known representative cocycle

$$\omega(F,G)=\int_{S^1}\left|\begin{array}{cc} f'(\varphi) & g'(\varphi) \\ f''(\varphi) & g''(\varphi) \end{array}\right| d\varphi$$

can be obtained from the Godbillon-Vey cocycle as follows,

$$\omega=i_{e_0}\int_{S^1}\theta_\varphi d\varphi.$$

We apply the same method to get diagonal k-cocycles on S^1 from local Leibniz $k+1$-cocycles.

The Godbillon-Vey class in $HL^3(W_1)$ has the same representative cocycle θ, and by pull back we obtain the local class θ_{φ_0} in $HL^3(\mathrm{Vect}\, S^1)$. By contracting its integral form we recover the Leibniz cocycle ω which represents the Gelfand-Fuks class.

In fact, as a Leibniz cocycle, θ is also cohomologous to the cocycle

$$\tilde{\theta}(F,G,H) = (fgh''' - fg'''h)|_{x=0},$$

which, by integration and contraction, produces again the Gelfand-Fuks cocycle. However, we prefer to use the antisymmetric form of θ since it is advantageous in the computation of the cup products.

The Lodder class in $HL^4(W_1)$ is the image of the non trivial Lie cohomology class in $H^3(W_1;W_1')$ with values in the adjoint representation, under the map $H^3(W_1;W_1') \longrightarrow HL^3(W_1;W_1') \cong H^4(W_1)$. It is represented by the cocycle

$$\alpha(L,F,G,H) = l'(0) \left| \begin{array}{ccc} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{array} \right|_{x=0},$$

where $L = l(x)\frac{d}{dx}$ denote another formal vector field. Then, the first new class for the Leibniz cohomology of Vect S^1 is represented by the cocycle

$$\beta_1(L,F,G) = \int_{S^1} l'(\varphi) \left| \begin{array}{cc} f'(\varphi) & g'(\varphi) \\ f''(\varphi) & g''(\varphi) \end{array} \right| d\varphi,$$

where, as before, L, F, G now denote vector fields on S^1.

The second method to obtain the Gelfand-Fuks cocycle from the Godbillon-Vey cocycle does not differ too much from the first one: one can show that

$$\frac{d}{d\phi}\theta_\phi = d\left(\left| \begin{array}{cc} \_' & \_' \\ \_'' & \_'' \end{array} \right| \right).$$

This permits to define ω as integral over S^1 of the expression in paranthesis on the right hand side, because it gives automatically a cocycle by the above equation. It is easily seen that β_1 can also be obtained via this method, i.e. that in the Leibniz setting

$$\frac{d}{d\phi}\alpha_\phi = d\left((-') \left| \begin{array}{cc} \_' & \_' \\ \_'' & \_'' \end{array} \right| \right).$$

The representative cocycles for the other new diagonal cohomology classes can be obtained in the same manner, starting from the cup products of the two local cocycles θ and α. The formulas are though complicated by the presence of the shuffles, and we omit them.

To end this section, let us just remark that the higher diagonal Leibniz cohomology spectral sequences also collapse at the second term. This is due to the fact that $H^*((S^1)^k, (S^1)^k_{k-1})$ is non-zero only in two dimensions, see [3].

Aknowledgements

The authors would like to thank Olivier Mathieu for many useful discussions and suggestions on the subject of the paper and Muriel Livernet for kindly remarking some oversights.

A Reduction to Euler-invariant cochains

Suppose a Lie algebra g possesses an element e_0, called the Euler field, and a basis of eigenvectors for the adjoint operator $ad_{e_0} : X \mapsto [X, e_0]$. For instance, let $\{e_k, k \in \mathbb{Z}, k \neq 0\}$ be the basis of elements such that $ad_{e_0}(e_k) = ke_k$.

The adjoint operator ad_{e_0} is a derivation of the Lie algebra g. It can be extended to the tensor powers of g as a graded derivation (with respect to the tensor product), and consequently to the Leibniz cochains $CL^*(g)$.

For any $\lambda \in \mathbb{Z}$, let $CL^*_{(\lambda)}(g)$ be the subset of Leibniz cochains γ such that $ad_{e_0}(c) = \lambda c$. Formula (iv) of proposition (3.1) in [6] means that the adjoint operator ad_{e_0} commutes with the Leibniz differential. Hence, $CL^*_{(\lambda)}(g)$ is a subcomplex for any $\lambda \in \mathbb{Z}$, and the Leibniz complex $CL^*(g) = \oplus_\lambda CL^*_{(\lambda)}(g)$ splits up into a direct sum of such subcomplexes. In particular, the elements c of $CL^*_{(0)}(g)$ are called *Euler-invariant cochains*, because $ad_{e_0}(c) = 0$.

The same splitting, as a completed direct sum, occurs if the Lie algebra has a topology and the basis $\{e_k, k \in \mathbb{Z}, k \neq 0\}$ is a topological one.

As for Lie cohomology, cf. theorem 1.5.2 [2], we then have:

Lemma A.1 *Suppose g is a Lie algebra. Under the previous assumptions the Euler-invariant cohomology determines the Leibniz cohomology of g, that is,*

$$HL^*_{cont}(g) \cong H^*(CL^*_{(0)}(g)).$$

Proof :We construct a contracting homotopy for the cochain complexes $CL^*_{(\lambda)}(g)$ with $\lambda \neq 0$. For $\lambda \neq 0$, let $D^p_{(\lambda)} : CL^{p+1}_{(\lambda)}(g) \longrightarrow CL^p_{(\lambda)}(g)$ be defined by

$$(D^p_{(\lambda)}c)(x_1, \ldots, x_{p-1}) = c(x_1, \ldots, x_{p-1}, e_0)$$

for any p-cochain c. Cartan's formula (formula (i) in proposition (3.1) [6]) shows that

$$(d\tilde{D}^p_{(\lambda)}c)(x_1, \ldots, x_p) = (\lambda - \tilde{D}^{p+1}_{(\lambda)}d)(c)(x_1, \ldots, x_p).$$

where we have set $\lambda = \sum_{i=1}^p \lambda_i$ with $ad_{e_0}(x_i) =: \lambda_i x_i$, and $\tilde{D}^p_{(\lambda)} := (-1)^p D^p_{(\lambda)}$. This means that $\tilde{D}^p_{(\lambda)}$ is a contracting homotopy for $CL^*_{(\lambda)}(g)$ for $\lambda \neq 0$. □

References

[1] Bott, R., Segal, G.: The cohomology of the vector fields on a manifold. Topology **16** (1977) 285–298

[2] Fuks, D. B.: Cohomology of Infinite Dimensional Lie Algebras. Consultant Bureau 1986

[3] Gelfand, I. M., Fuks, D. B.: Cohomologies of Lie Algebra of Tangential Vector Fields of a Smooth Manifold. Funct. Ana. and its Appl. **3** (1969) 194–210

[4] Haefliger, A.: Sur la cohomologie de l'algèbre de Lie des champs de vecteurs. Ann. Sci. ENS, $4^{ème}$ série, t. 9, 503-532 (1976)

[5] Loday, J.-L.: Cup-product for Leibniz cohomology and dual Leibniz algebras, Math. Scand. **77** n.2 (1995), 189-196.

[6] Loday, J.-L., Pirashvili, T.: Universal Enveloping Algebras of Leibniz Algebras and (Co)homology, Math. Annalen **296** (1993), 569-572.

[7] Livernet, M.: Rational homotopy of Leibniz algebras, Manuscripta Math. **96** (1998), 295-315.

[8] Lodder, J.: Leibniz cohomology for differentiable manifolds. Ann. Inst. Fourier, Grenoble **48**, 1 (1998) 73-95

[9] Oudom, J.-M.: Diagonale et cup-produit en cohomologie des algèbres de Leibniz, Comm. Algebra **24** n.11 (1996), 3623-3635.

[10] Pirashvili, T.: On Leibniz Homology, Ann. Inst. Fourier, Grenoble, **44,2** (1994), 401-411.