Double Poisson bracket for covariant field theory

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Motivation for double Poisson algebras

- **Van den Bergh 2008:** for A a unital associative algebra on a field $\mathbb{K}$, when is its representations space $\mathrm{Rep}(A,N) = \mathrm{Hom}_{\mathrm{Alg}}(A, M_N(\mathbb{K}))$ a Poisson variety?

  If A has a double Poisson bracket $\boxed{\{\!\{\ ,\ \}\!\} : A \otimes A \to A \otimes A}$, then the coordinate ring $\mathcal{O}(\mathrm{Rep}(A,N))$ is a Poisson algebra with $\{a_{ij}, b_{uv}\} = \{\!\{a,b\}\!\}'_{uj} \cdot \{\!\{a,b\}\!\}''_{iv}$.

  Moreover $\boxed{\{\ ,\ \} : A \otimes A \xrightarrow{\{\!\{\ ,\ \}\!\}} A \otimes A \xrightarrow{m} A}$ makes $A/[A,A]$ a Poisson algebra.

- **Shedler 2009, Odesskii-Roubtsov-Sokolov 2013:** if $A = T(V)$ is free associative, then the restriction $\boxed{r = \{\!\{\ ,\ \}\!\}|_V : V \otimes V \to V \otimes V}$ to generators satisfies the associative Yang-Baxter equation [Aguiar 2001] $\boxed{r^{12} \circ r^{31} + r^{23} \circ r^{12} + r^{31} \circ r^{23} = 0}$

  and therefore also the classical YBE $\boxed{[r^{12}, r^{31}] + [r^{23}, r^{12}] + [r^{31}, r^{23}] = 0}$.

- **Developpments:** many examples (e.g. on quiver varieties, free and polynomial alg.), many extensions (e.g. vertex alg., Courant-Dorfmant alg., pre-Calabi-Yau alg.) and generalizations (e.g. trace bracket algebras, double quasi-Poisson).
  See https://mfairon.perso.math.cnrs.fr/DoubleBrackets.html

Plan of the talk

- **Aim:** infinite-dimensional example of a double Poisson algebra and of an r-matrix for multilocal observables in covariant field theory.

- **Plan:**

  1 - Double Poisson algebras

  2 - Poisson 2-algebras in symmetric 2-monoidal categories

  3 - Double Poisson bracket for observables on fields

- **Double Poisson algebra:** unital associative algebra $(A,\cdot)$ with double bracket

$$\{\{\ ,\ \}\} : A\otimes A \to A\otimes A,\ \ a\otimes b \mapsto \{\{a,b\}\} = \{\{a,b\}\}'\otimes\{\{a,b\}\}''$$

satisfying

skew-symmetry: $\{\{b,a\}\} = -\{\{a,b\}\}^{op} = -\{\{a,b\}\}''\otimes\{\{a,b\}\}'$

Jacobi identity: $\mathrm{Jac}(\ ,\ ,\ ) = \sum_{\tau\in C_3} \tau\circ\{\{\ ,\{\{\ ,\ \}\}\}\}_L\circ\tau^{-1} \equiv 0$

with $C_3 = \{1,(123),(132)\}$ and $\{\{a,b\otimes c\}\}_L = \{\{a,b\}\}\otimes c$

i.e. $$\mathrm{Jac}(a,b,c) = \{\{a,\{\{b,c\}\}'\}\}\otimes\{\{b,c\}\}'' + (123)\circ\{\{b,\{\{c,a\}\}'\}\}\otimes\{\{c,a\}\}'' + (132)\circ\{\{c,\{\{a,b\}\}'\}\}\otimes\{\{a,b\}\}''$$

Leibniz identity: $\{\{a,b\cdot c\}\} = \{\{a,b\}\}\cdot(1\otimes c) + (b\otimes 1)\cdot\{\{a,c\}\}$

- **Proposition:** if A is a double Poisson algebra with multiplication m, then
  1. $\{\{\ ,\ \}\}$ is a derivation of m
  2. $\{\ ,\ \} = m\circ\{\{\ ,\ \}\}$ induces a Poisson bracket on $A/[A,A]$
  3. $\{\{a,b,c\}\} = \mathrm{Jac}(a,b,c)$ is a triple bracket (cyclic skew-symmetric derivation in c).

Example: free associative double Poisson algebra [Odesskii-Roubtsov-Sokolov 2013]

- **Remark:** A double Poisson bracket $\{\!\{\ ,\ \}\!\}$ on $A = T(V)$ is uniquely determined by its restriction $\{\!\{\ ,\ \}\!\} : V \otimes V \to T(V) \otimes T(V)$ to the generators $V = \mathrm{Span}_{\mathbb{K}}\{v_i\}$.

- **Theorem:** Classification of first polynomial double Poisson brackets (DPB):

Constant: $\{\!\{v_i, v_j\}\!\} = c_{ij}\, 1 \otimes 1$ defines a DPB iff $c_{ji} = -c_{ij}$ (skew-symmetric)

Linear: $\{\!\{v_i, v_j\}\!\} = b_{ij}^k\, v_k \otimes 1 - b_{ji}^k\, 1 \otimes v_k$ defines a DPB iff $b_{ij}^k b_{kh}^l = b_{ik}^l b_{jh}^k$

Quadratic: $\{\!\{v_i, v_j\}\!\} = r_{ij}^{kl}\, v_k \otimes v_l + a_{ij}^{lk}\, v_k v_l \otimes 1 - a_{ji}^{kl}\, 1 \otimes v_l v_k$ defines a DPB iff

the operators $r, a : V \otimes V \to V \otimes V$ given by $\begin{cases} r(v_i \otimes v_j) = r_{ij}^{kl}\, v_k \otimes v_l \\ a(v_i \otimes v_j) = a_{ij}^{kl}\, v_k \otimes v_l \end{cases}$

satisfy

$$r^{12} = -r^{21} \quad \text{and} \quad r^{12} r^{31} + r^{23} r^{12} + r^{31} r^{23} = 0$$

$$a^{12} a^{31} = a^{31} a^{12}$$

$$r^{13} a^{12} - a^{32} r^{13} = a^{32} a^{12} \quad \text{and} \quad a^{12} r^{23} - r^{23} a^{12} = (23) a^{13} a^{12}$$

where $r^{\alpha\beta}$ and $a^{\alpha\beta}$ act at positions α and β.

Hence $\boxed{r = \{\!\{\ ,\ \}\!\}|_{V \otimes V}}$ is a skew-symmetric r-matrix satisfying the associative YBE.

2 - Symmetric 2-monoidal categories [Aguiar-Mahajan 2010]

- **Symmetric 2-monoidal category:** category $\mathcal{C}$ with
  - two tensor products $\otimes, \boxtimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with unit objects $I_\otimes$, $I_\boxtimes$
  - three morphisms $\mu : I_\otimes \boxtimes I_\otimes \to I_\otimes$, $\delta : I_\boxtimes \to I_\boxtimes \otimes I_\boxtimes$, $\nu : I_\boxtimes \to I_\otimes$ such that $(I_\otimes, \mu, \nu)$ is a monoid w.r.t. $\boxtimes$ and $(I_\boxtimes, \delta, \nu)$ is a comonoid w.r.t. $\otimes$
  - a shuffle map $\mathrm{sh} : (A \otimes B) \boxtimes (C \otimes D) \to (A \boxtimes C) \otimes (B \boxtimes D)$

  satisfying appropriate compatibility conditions.

- **Example:** graded vector spaces $V_\bullet = \bigoplus_{n \geqslant 0} V_n$ with

  Hadamard $\quad (V_\bullet \otimes W_\bullet)_n = V_n \otimes W_n \qquad (I_\otimes)_n = \mathbb{K}$

  Cauchy $\quad (V_\bullet \boxtimes W_\bullet)_n = \bigoplus_{p+q=n} V_p \otimes W_q \qquad (I_\boxtimes)_n = \begin{cases} \mathbb{K}, & n = 0 \\ 0, & n > 0 \end{cases}$

- **Example [Chase]:** oriented graphs $A = \left\{ x \xrightarrow{a \in A} y \right\} \overset{s}{\underset{t}{\rightrightarrows}} X$ with

  $A \otimes B = \left\{ (a, b) \in A \times B \mid x \xrightarrow{a} y \xrightarrow{b} z \right\} \qquad I_\otimes = X = \left\{ 1_x = \circlearrowleft_x \right\}$

  $A \boxtimes B = \left\{ (a, b) \in A \times B \mid x \underset{b}{\overset{a}{\rightrightarrows}} y \right\} \qquad I_\boxtimes = X \times X = \left\{ x \xrightarrow{(x,y)} y \right\}$

Example: vector bundles over configuration spaces [F-Kravchenko-Ryvkin 2024]

- **Configuration space of a manifold M:** $\mathrm{Conf}(M) = \bigsqcup_{n\geqslant 0} \mathrm{Conf}_n(M)$

  space of unordered configurations $\underline{x} = \{x_1, \ldots, x_n\}$ of distinct points of M

$$\mathrm{Conf}_0(M) = \{\text{vacuum } \varnothing\}$$

$$\mathrm{Conf}_n(M) = (M^n \backslash \Delta^{(n)})/S_n = \{\{x_1, \ldots, x_n\} \mid x_1, \ldots, x_n \in M\}$$

- **Category of $\mathbb{K}$-vector bundles over $\mathrm{Conf}(M)$:** $VB(\mathrm{Conf}(M))$

  vector bundle: $\mathbf{V} = \bigsqcup_{\underline{x}\in\mathrm{Conf}(M)} \mathbf{V}_{\underline{x}}$ with usual local trivializations

  $\mathbf{V} = \bigsqcup_{n\geqslant 0} \mathbf{V}_n$ with $\mathbf{V}_n \to \mathrm{Conf}_n(M)$ usual vector bundle of rank r_n

  bundle map: $\mathbf{V} \to \mathbf{W}$ = collection of usual bundle maps $\mathbf{V}_k \to \mathbf{W}_k$

- **Theorem:** $VB(\mathrm{Conf}(M))$ is a symmetric 2-monoidal category with

  Hadamard $(\mathbf{V} \otimes \mathbf{W})_{\underline{x}} = V_{\underline{x}} \otimes W_{\underline{x}}$ $\qquad (I_\otimes)_{\underline{x}} = \mathbb{K}$

  Cauchy $(\mathbf{V} \boxtimes \mathbf{W})_{\underline{x}} = \bigoplus_{\underline{x}=\underline{x}'\sqcup\underline{x}''} V_{\underline{x}'} \otimes W_{\underline{x}''}$ $\qquad (I_\boxtimes)_n = \begin{cases} \mathbb{K}, & \underline{x} = \varnothing \\ 0, & \text{else} \end{cases}$

Example: $(\mathbf{V} \boxtimes \mathbf{W})_{\{x,y\}} = \mathbf{V}_{\{x,y\}} \otimes \mathbf{W}_\varnothing \oplus \mathbf{V}_x \otimes \mathbf{W}_y \oplus \mathbf{V}_y \otimes \mathbf{W}_x \oplus \mathbf{V}_\varnothing \otimes \mathbf{W}_{\{x,y\}}$

2-algebras and 2-coalgebras [Aguiar-Mahajan 2010, F-Kravchenko-Ryvkin 2024]

Fix a symmetric 2-monoidal category $(\mathcal{C}, \otimes, \boxtimes)$ enriched on vector spaces (need $\oplus$).

- 2-**algebra:** A with

$$m_\otimes : A \otimes A \to A \qquad u_\otimes : \mathbf{I}_\otimes \to A$$
$$m_\boxtimes : A \boxtimes A \to A \qquad u_\boxtimes : \mathbf{I}_\boxtimes \to A$$

plus compatibility using μ, δ, ν, sh.

2-**coalgebra:** C with

dual maps $\Delta_\otimes$, $\varepsilon_\otimes$, $\Delta_\boxtimes$, $\varepsilon_\boxtimes$
and dual relations

$\Longrightarrow C^* = \mathrm{Hom}(C, I_\otimes)$ is a 2-algebra.

- **Example:** $\otimes$ and $\boxtimes$-tensors (co)algebras on V with dual $V^* = W$:

$$\begin{array}{ccc}
T^{a,\boxtimes}(V) = \bigoplus_n V^{\boxtimes n},\ \boxtimes & \xleftarrow{gr*} & T^{c,\boxtimes}(W) = \bigoplus_n W^{\boxtimes n},\ \Delta_\boxtimes \\
\text{algebra} \big\downarrow & & \big\uparrow \text{coalgebra} \\
S^\boxtimes(V) = \bigoplus_n V^{\boxtimes n}/S_n,\ \boxdot & \xrightarrow[Sym]{\cong} & \Sigma^\boxtimes(W) = \bigoplus_n (W^{\boxtimes n})^{S_n},\ \Delta_\boxdot
\end{array}$$

(same for $\otimes$)

Combine $\otimes$ and $\boxtimes$-tensors using sh and get tensor 2-(co)algebras:

$$\begin{array}{ccc}
T^{a,\boxtimes} T^{a,\otimes}(V) & \xrightarrow{sh} & T^{a,\otimes} T^{a,\boxtimes}(V) \\
\big\downarrow & & \big\downarrow \\
S^\boxtimes S^\otimes(V) & \xrightarrow{sh} & S^\otimes S^\boxtimes(V)
\end{array}
\qquad
\begin{array}{ccc}
T^{c,\boxtimes} T^{c,\otimes}(W) & \xrightarrow{sh} & T^{c,\otimes} T^{c,\boxtimes}(W) \\
\big\uparrow & & \big\uparrow \\
\Sigma^\boxtimes \Sigma^\otimes(W) & \xrightarrow{sh} & \Sigma^\otimes \Sigma^\boxtimes(W)
\end{array}$$

Poisson 2-algebras in $\mathcal{VB}(\mathrm{Conf}(M))$ [F-Kravchenko-Ryvkin 2024]

- **Poisson 2-algebra:** $(P, \bullet_{\otimes}, \bullet_{\boxtimes})$ commutative with $\boxed{\{\ ,\ \} : P \boxtimes P \to P}$ such that

  skew-symmetry: $\{a, b\} = -\{b, a\}$

  Jacobi identity: $\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0$

  $m_{\boxtimes}$-Leibniz rule: $\{a, b \bullet_{\boxtimes} c\} = \{a, b\} \bullet_{\boxtimes} c + b \bullet_{\boxtimes} \{a, c\}$

  $m_{\otimes}$-Leibniz rule: $\{a, b \bullet_{\otimes} c\} = \{a, b\} \bullet_{\otimes} (1_{\otimes} \bullet_{\boxtimes} c) + (1_{\otimes} \bullet_{\boxtimes} b) \bullet_{\otimes} \{a, c\}$

- **Abelian extension:** if P is a Poisson 2-algebra and A is a commutative $\boxtimes$-algebra, then $P \otimes A$ is a Poisson $\boxtimes$-algebra with $\boxed{\{p \otimes a, q \otimes b\} = \{p, q\} \otimes ab}$

- **Theorem:** In the category $\mathcal{VB}(\mathrm{Conf}(M))$, we have:
  1. A vector bundle $V \to M$ determines a 2-algebra bundle $\mathbf{S}^{\boxtimes} S^{\otimes}(V)$.
  2. Any skew-symmetric bundle map $k : V \boxtimes V \longrightarrow I_{\otimes} \boxdot I_{\otimes} \cong \mathbb{K}$ over $\mathrm{Conf}_2(M)$ determines a Poisson bracket on $\mathbf{S}^{\boxtimes} S^{\otimes}(V)$.
  3. Densities $\mathrm{Dens}_{\mathrm{Conf}(M)} \cong S^{\boxtimes}(\mathrm{Dens}_M)$ form a commutative $\boxtimes$-algebra bundle.

  Hence $\boxed{\mathbf{P}(V) = \mathbf{S}^{\boxtimes} S^{\otimes} V \otimes \mathrm{Dens}_{\mathrm{Conf}(M)}}$ is a Poisson $\boxtimes$-algebra bundle on $\mathrm{Conf}(M)$.

- **Analogy with $\{\{\ ,\ \}\}$ on $T(V)$:** the generators of the $\boxtimes$-algebra are $S^{\otimes}(V)$, not V!

- **Covariant description of field theory:**
  Fields: $\mathcal{E}(M,E) = \{\text{smooth sections } \varphi : M \to E \text{ of a vector bundle over spacetime}\}$
  dynamics ruled by a Lagrangian $\mathcal{L} : JE \to \mathrm{Dens}_M$

  Off-shell observables: $C^\infty(\mathcal{E}(M,E)) = \{\text{smooth functions } F : \mathcal{E}(M,E) \to \mathbb{R}\}$
  multiplication $(F_1 \cdot F_2)(\varphi) = F_1(\varphi)F_2(\varphi)$
  Poisson bracket $\{F_1, F_2\}(\varphi) = F_1'(\varphi)\Delta_{\mathcal{L}} F_2'(\varphi)$ via the causal propagator of $\mathcal{L}$

  QFT: deformation quantization of a subalgebra $\mathcal{A}_0 \subset C^\infty(\mathcal{E}(M,E))$
  plus time-ordered product = different deformation using Feynman propagator.

- **Theorem:** The causal propagator is a distributional kernel k yielding a Poisson bracket on the bundle $\boxed{\mathbf{P}(E) = \mathbf{S}^{\boxtimes} S^{\otimes}(JE)^* \otimes \mathrm{Dens}_{\mathrm{Conf}(M)}}$ and there is a Poisson map

$$\boxed{F : \mathcal{D}'_{Pois}(\mathrm{Conf}(M), \mathbf{P}(E)) \longrightarrow C^\infty(\mathcal{E}(M,E))}$$

sending a distribution T to the function $F_T : \mathcal{E}(M,E) \to \mathbb{R}$ defined on $\varphi \in \mathcal{E}(M,E)$ by

$$F_T(\varphi) = \langle T, e(\varphi)\rangle = \int_{\mathrm{Conf}(M)} \langle T(\underline{x}), e(\varphi)(\underline{x})\rangle$$

where $\quad e(\varphi) = \sum \frac{1}{n_1! \cdots n_k!} (j\varphi)^{\otimes n_1} \boxtimes \cdots \boxtimes (j\varphi)^{\otimes n_1} \quad$ is a section of $\mathbf{\Sigma}^{\boxtimes}\mathbf{\Sigma}^{\otimes}(JE)$.

Let $V = (JE)^* \to M$ be a vector bundle of rank d (usually jets are of a given order).

- **Claim 1:** Any skew-symmetric bundle map $\tilde{k} : V \boxtimes V \to I_\otimes \boxtimes I_\otimes$ over $\mathrm{Conf}_2(M)$ determines a quadratic double Poisson bracket $\{\{\ ,\ \}\}$ on $\mathbf{S}^{\boxtimes} S^{\otimes}(V)$ over $\mathrm{Conf}(M)$ such that $\{\ ,\ \} = \boxdot \circ \{\{\ ,\ \}\}$ coincides with the Poisson bracket induced by $k = \boxdot \circ \tilde{k}$.

- **Explicit formulas:** If $\{e_i(x) \mid i = 1, \dots, d\}$ is a basis of V_x and $\{e_{n_1,\dots,n_d}(x) = e_1^{n_1}(x) \cdots e_d^{n_d}(x) \mid n_1, \dots, n_d \in \mathbb{N}\}$ is the PBW basis of $S^{\otimes}(V)_x$, then:

  on 2-generators V: $\{\{e_i(x), e_j(y)\}\} = \tilde{k}(e_i(x), e_j(y)) \in (I_\otimes \boxtimes I_\otimes)_{\{x,y\}}$

  on $\boxtimes$-generators $S^{\otimes}(V)$:

$$\{\{e_{n_1,\dots,n_d}(x), e_{m_1,\dots,m_d}(y)\}\} = \sum_{i,j} n_i m_j \tilde{k}(e_i(x), e_j(y))\, e_{n_1,\dots,n_i-1,\dots,n_d}(x) \boxtimes e_{m_1,\dots,m_j-1,\dots,m_d}(y)$$

  i.e. $\{\{e_N(x), e_M(y)\}\} = \sum_{JK} r^{JK}_{NM}(x,y)\, e_J(x) \boxtimes e_K(y)$ and $a^{JK}_{NM}(x,y) \equiv 0$.

- **Claim 2:** The causal propagator determines a distributional r-matrix $r^{JK}_{NM}(x,y)$ on local polynomial observables of fields, represented by sections of $S^{\otimes}(JE)^*$.

Thank you for the attention!