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GÉOMÉTRIE PRESQUE-CRITIQUE ET DYNAMIQUES EN PHYSIQUE STATISTIQUE PLANAIRE

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devant le jury composé de

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à Laure et Victor

Contents

Chapter 1

Introduction

The idea of this document is to review the research that I have accomplished so far. It is presented for the *Habilitation à Diriger des Recherches (HDR)* degree. As it is customary, I will focus essentially on the results that I and coauthors have obtained after my PhD thesis (which included the works [G1, G2, G3] as well as an early sketch of [G4]). I will thus describe the works [G4] to [G15] and will discuss as well two related works in progress [G16, G17].

As it is suggested by the title of this memoir, a large fraction of my research since [G4] has been devoted to the study of the so called **near-critical geometry** of planar statistical physics models such as percolation, Ising model or the random cluster model. The point is to study the large-scale geometry of these models "near" their phase transition, where "near" is chosen in a suitable manner in order to obtain an interesting non-trivial geometry. These near-critical regimes have less symmetries than their critical analogs: for example conformal invariance is replaced by the notion of conformal covariance and the SLE processes (Schramm Loewner Evolution) are replaced by massive versions of these. My main result in this line of research is perhaps the proof in [G4, G5] with G. Pete and O. Schramm that **near-critical percolation** on the triangular lattice has a (unique) massive scaling limit. The works [G6, G9, G10, G11, G12] are also in one way or another connected to this subject. Still suggested by the title of this document, another important aspect of my research focused on dynamics of such statistical physics models at their critical point. This includes the proof that dynamical percolation on the triangular grid has a scaling limit (still [G5]), the study of critical percolation under conservative dynamics in [G8] and in some ways the work [G9] even though the latter one focuses more on the near-critical phenomena. Finally, I have done some excursions away from the above unified theme into the world of coalescing Brownian motions with the work [G15] and into the world of Liouville quantum gravity with the works [G13, G14]. The above picture illustrates well I believe the near-critical aspect of my research. It represents the fractal geometry of a snowflake at different temperatures around the "critical" temperature 0^oC.

Instead of providing detailed proofs, I wish to describe in a concise and hopefully appealing manner the main results of the works [G4] to [G15]. I will also highlight the main difficulties encountered along the way together with the mathematical ideas designed in each case to overcome these difficulties. In this respect, the style will be intentionally rather informal.

The works [G4] to [G15] are naturally divided into the following six groups:

(i) The first group consists of the works [G4, G5, G6]. It is divided into two chapters: Chapter 2 on **near-critical percolation** and Chapter 3 on its application to the model of **Minimal Spanning Tree**. As mentioned above, I consider this body of works to be the main scientific contribution to this *HDR*. All these works were initiated a long time ago in summer 2008 together with Gábor Pete and Oded Schramm who tragically passed away on September 1, 2008. These projects were at a very early stage at that time but we were all confident one would eventually come up with a proof. It then took Gábor and I nearly five years to complete this program which lead to the proof that near-critical and dynamical percolation have a scaling limit. See [G4] and [G5]. In the recent [G6], we apply these results to the scaling limit of the Minimal spanning tree in the plane.

(ii) The second group consists of the single article [G8] and corresponds to the content of Chapter 4 entitled **Critical percolation under conservative dynamics**. This is a natural extension of the main article in my thesis [G3] which gave optimal results on the **noise sensitivity** of critical percolation subjected to an "i.i.d. noising". In [G8], together with Erik Broman and Jeffrey Steif, we consider a variant of the classical model of dynamical percolation where "particles" undergo an **exclusion process** instead of independent exponential updates. As we will see in more details, the difficulty raised by such conservative dynamics lies in the fact that they are less suitable to the classical Fourier analysis approach.

(iii) The third group (Chapter 5) consists of the two papers [G10] and [G12] (joint with Federico Camia and Chuck Newman) which build the scaling limit of the **magnetization field** of the critical Ising model in the plane and study some of its properties.

(iv) The fourth group (Chapter 6) is composed of the papers [G9, G11] as well as part of [G12]. In [G9], together with Hugo Duminil-Copin and Gábor Pete, we study the near-critical behavior of the Ising model by changing the temperature, while in [G11, G12], together with Federico Camia and Chuck Newman, we study a different perturbation of the Ising model near its critical point by adding some small external magnetic field.

The last two groups are of a very different flavour:

(v) The fifth group (Chaper 7) which consists of the paper [G15] (joint with N. Berestycki and A. Sen) introduces a new approach for coalescing flows. This new approach is inspired from the Schramm-Smirnov space for critical percolation and is thus related to Chapter 2 in many ways. It has two main advantages: first, it simplifies and strengthens previous known hypothesis on the convergence of coalescing random walks to the **Brownian web**. And second, our approach is sufficiently simple that we can handle substantially more complicated coalescing flows with little extra work such as coalescing Brownian motions on the **Sierpinski gasket**. In the work in progress [G17], we show using a new technique (approximate randomized algorithms) that these flows lead to new examples of **blacknoises** in the sense of Tsirelson.

(vi) Finally, our last group (Chapter 8) consists of the papers [**G13**, **G14**] (joint with R. Rhodes and V. Vargas) which build a natural Feller diffusion in the framework of two-dimensional Liouville quantum gravity: the **Liouville Brownian motion**. This diffusion preserves the so-called **Liouville measure** which was introduced in [DS11] in order to prove a form of the celebrated **KPZ formula** from [KPZ88]. In particular, the Liouville Brownian motion is expected to be the scaling limit of simple random walks on planar maps suitably "uniformized" in the plane.

In the rest of this introduction, I will give a more detailed description of each Chapter. In each case, I will start by introducing the relevant objects and will explain what the main results are. More precise statements will be given in the corresponding chapters.

1. — Near-critical percolation and Minimal spanning tree in the plane (Chapters 2 and 3)

*The presentation of this Chapter is largely borrowed from our introduction in [**G5**]*

1.1. — The model of percolation

Percolation is a central model of statistical physics. It combines a very simple definition with an exceptionally rich behaviour. We will be concerned only with planar percolation in this text. On the triangular lattice $\mathbb{T}$, **site-percolation** is defined as follows: each site $x \in \mathbb{T}$ is kept (or declared *open* or colored black) with probability some fixed parameter $p \in [0,1]$ and is removed (or declared *closed* or colored white) with probability $1-p$ independently of the other sites. This way, one obtains a random configuration $\omega_p \sim \mathbb{P}_p$ in $\{0,1\}^{\mathbb{T}}$. This model undergoes a well known **phase transition** at the critical point $p_c = p_c(\mathbb{T}) = \frac{1}{2}$: if $p \le p_c$, then all open connected components or **clusters** are finite a.s. while if $p > p_c$, there is a.s. a unique infinite cluster. On the lattice $\mathbb{Z}^2$, the model of **edge-percolation** is defined in the same fashion: each edge is kept (or declared open) with probability p and is removed with probability $1-p$ independently of the other edges. This model undergoes a similar phase transition at the critical point $p_c(\mathbb{Z}^2) = 1/2$. The identification of these critical points to be $p_c(\mathbb{T}) = p_c(\mathbb{Z}^2) = 1/2$ goes back to Kesten's Theorem [Ke80]. For a complete account of percolation and historical references, see the book [Gri99] and for a thorough study of the two-dimensional case, see the lecture notes [We07].

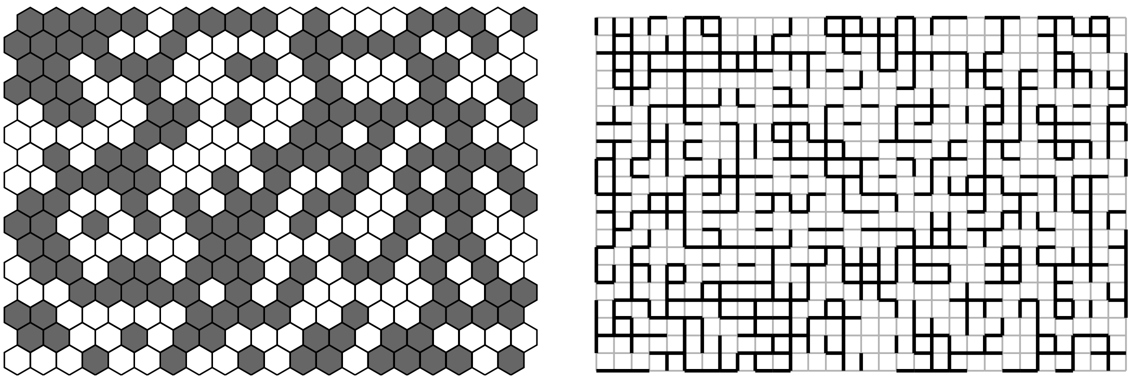


Figure 1.1: Pictures (by Oded Schramm) representing two percolation configurations respectively on $\mathbb{T}$ and on $\mathbb{Z}^2$ (both at $p = p_c = 1/2$). The sites of the triangular grid are represented by hexagons.

The following Theorem due to Russo, Seymour and Welsh (see [Gri99, We07]) has important consequences for percolation in dimension two.

Theorem 1.1 (Russo-Seymour-Welsh (RSW)). *For any $a > 1$, there exists a constant $c_a \in (0,1)$ such that uniformly in $n \geq 1$, the probability that a* ***critical*** *percolation on $\mathbb{T}$ or $\mathbb{Z}^2$ crosses a long rectangle $an \times n$ (in the sense that one can find an open path of ω_{p_c} which remains inside the rectangle and connects its left and right boundaries) is bounded above by $1 - c_a$ and bounded below by c_a.*

Let us highlight two significant applications of this theorem:

1. A first consequence of the RSW Theorem is the fact that the above phase transition is **continuous** on $\mathbb{T}$ and $\mathbb{Z}^2$ (one also says that the corresponding phase transition falls into the class of second-order phase transitions). It corresponds to the fact that the so-called **density functions**

$$\begin{cases} \theta_{\mathbb{T}}(p) := \mathbb{P}_p[0 \text{ is connected to infinity}] \\ \theta_{\mathbb{Z}^2}(p) := \mathbb{P}_p[0 \text{ is connected to infinity}] \end{cases}$$

are continuous on $[0,1]$. Note that such a continuity property remains a big open problem in the case of percolation on the three-dimensional lattice $\mathbb{Z}^3$.

2. A second striking consequence of RSW is the fact that large clusters in critical planar percolation have a rich *fractal geometry*. (For example the boundary of large clusters is made of multiple *fjords* at all scales and so on). As we will see below, this fractal geometry is now very well understood in the case of critical percolation on $\mathbb{T}$ due to Smirnov's Theorem 1.2.

1.2. — Near-critical percolation

When one deals with a statistical physics model which undergoes such a continuous phase transition, it is natural to understand the nature of its phase transition by studying the behaviour of the system near its critical point, at $p = p_c + \Delta p$.

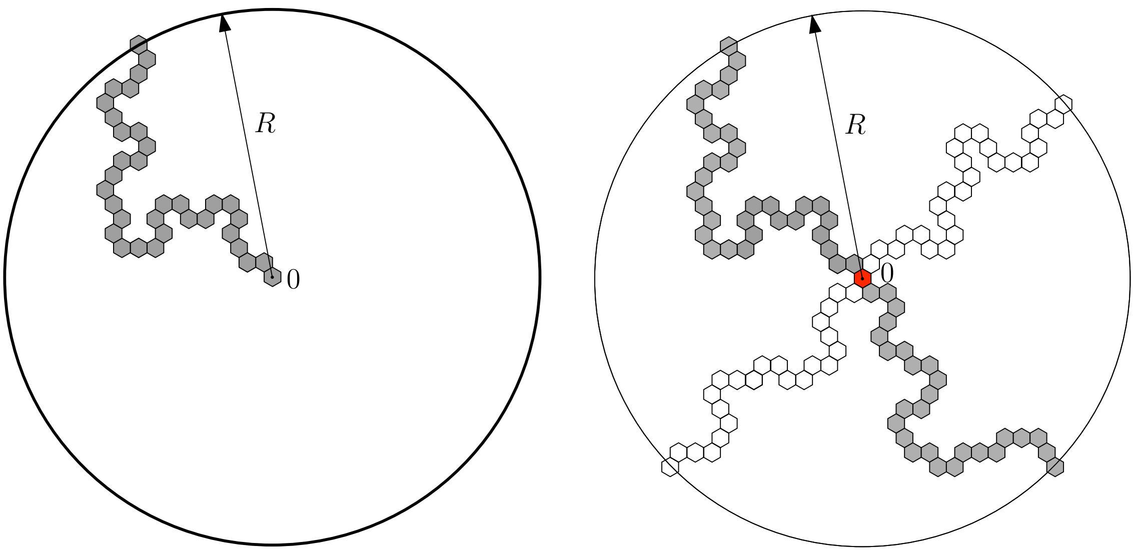
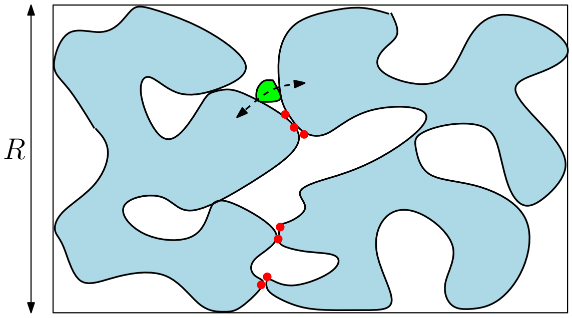


Figure 1.2: A *one-arm event* is realized on the left. Its probability is denoted by $\alpha_1(R)$. A *four-arm event* is realized on the right whose probability is denoted by $\alpha_4(R)$.

In order to study such systems near their critical point, it is very useful to introduce the concept of **correlation length** $L(p)$ for $p \approx p_c$. Roughly speaking, $p \mapsto L(p)$ is defined in such a way that, for $p \neq p_c$, the system "looks critical" on scales smaller than $L(p)$, while the non-critical behaviour becomes "striking" above this scale $L(p)$. See for example [We07, N08a, Ke87] for a precise definition and discussion of $L(p)$ in the case of percolation.



Let us give a short heuristical derivation of the correlation length based on the side picture: fix $p = p_c + \delta p$ slightly above p_c. One is looking for a scale R above which the percolation configuration starts being very well connected. At the critical point, the number of **pivotal points** which lie between two large clusters of diameter R is typically of order $R^2\alpha_4(R)$, where $\alpha_4(R) = \alpha_{4,p_c}(R)$ stands for the probability of the alternating four-arm event up to radius R at the critical point of the planar percolation model considered. See Figure 1.2 for an illustration of this arm-event. This fact suggests that when the scale R is such that $R^2\alpha_4(R)\delta_p \gg 1$, then the percolation configuration should start being very well connected. On the other hand, it could be that the correlation length $L(p)$ is in fact much smaller than what this analysis suggests due to the effect of "microscopic" clusters such as the green one pictured on the above figure.

Kesten proved in his seminal paper [Ke87] that the above heuristical derivation indeed gives the right behavior for $L(p)$, namely he proved that

$$L(p) \asymp \inf \left\{ R \geq 1, \text{ s.t. } R^2\alpha_4(R) \geq \frac{1}{|p-p_c|} \right\} . \tag{1.1}$$

Notice in particular that this scale whose aim is to separate critical from non-critical effects at $p \approx p_c$ can be computed just by studying the **critical geometry** of the system (here, the quantity $\alpha_4(R)$). A detailed study of the near-critical system below its correlation length was given in [BC+01]. Furthermore, Kesten's notion of correlation length enabled him to prove in [Ke87] that, as $p > p_c$ tends to p_c, one has

$$\begin{aligned} \theta(p) &\asymp \mathbb{P}_p[0 \text{ is connected to } \partial B(0, L(p))] \\ &\asymp \mathbb{P}_{p_c}[0 \text{ is connected to } \partial B(0, L(p))] \\ &:= \alpha_{1,p_c}(L(p))\,. \end{aligned} \tag{1.2}$$

(See Figure 1.2 for an illustration of the probability of the one-arm event $\alpha_1(R)$). In particular, it is a striking fact that the density $\theta(p)$ of the infinite cluster near its critical point can be evaluated just using quantities which describe the critical system: $\alpha_1(R)$ and $\alpha_4(R)$.

Such critical quantities are not yet fully understood on $\mathbb{Z}^2$ at $p_c(\mathbb{Z}^2) = 1/2$, but there is one planar percolation model for which such quantities can be precisely estimated: the model of **site percolation on the triangular grid** $\mathbb{T}$ introduced above (where one also has $p_c(\mathbb{T}) = 1/2$). Indeed, one has in this case the following celebrated theorem by Smirnov:

Theorem 1.2 (Conformal invariance, Smirnov, [Sm01]). *If one considers critical site percolation on $\eta\mathbb{T}$, the triangular grid with small mesh $\eta > 0$, and lets $\eta \to 0$, then the limiting probabilities of crossing events are* ***conformally invariant****.*

This conformal invariance enables one to rely on the so-called Stochastic Loewner Evolution (or SLE) processes introduced by Schramm in [Sch00], which then can be used to obtain the following estimates:

(i) $\alpha_1(R) = R^{-5/48+o(1)}$ obtained in [LSW02],

(ii) $\alpha_4(R) = R^{-5/4+o(1)}$ obtained in [SW01],

(iii) $L(p) = \left|\frac{1}{p-p_c}\right|^{4/3+o(1)}$ obtained in [SW01],

(iv) $\theta_{\mathbb{T}}(p) = (p-p_c)^{5/36+o(1)} 1_{p>p_c}$ obtained in [SW01],

where the $o(1)$ are understood as $R \to \infty$ and $p \to p_c$, respectively. It is straightforward to check that items (iii) and (iv) follow from items (i), (ii) together with equations (1.1) and (1.2).

Items (iii) and (iv) are exactly the type of estimates which describe the so-called **near-critical** behaviour of a statistical physics model. To give another well-known example in this vein: for the Ising model on the lattice $\mathbb{Z}^2$, it is known since Onsager [On44] that $\theta(\beta) := \mathbb{P}^+_\beta[\sigma_0 = +] \asymp (\beta - \beta_c)^{1/8} 1_{\beta > \beta_c}$, which is a direct analog of Item (iv) if one interprets $\theta(\beta)$ in terms of its associated FK percolation ($q = 2$). Also the correlation length $\beta \mapsto L(\beta)$ defined in the spirit of Kesten's paper [Ke87] is known to be of order $\frac{1}{|\beta-\beta_c|}$. We will come back to this in the description of the work [**G9**] in Chapter 6.

The main question addressed in [**G4**, **G5**] is the following one: how does the system look below its correlation length $L(p)$? More precisely, let us redefine $L(p)$ to be exactly the above quantity $\inf\left\{R \geq 1, \text{ s.t. } R^2\alpha_4(R) \geq \frac{1}{|p-p_c|}\right\}$; of course, the exact choice of the constant factor in $1/|p-p_c|$ is arbitrary here. Then, for each $p \neq p_c$, one may consider the percolation configuration ω_p in the domain $[-L(p), L(p)]^2$ and rescale it to fit in the compact window $[-1,1]^2$ (one thus obtains a percolation configuration on the lattice $L(p)^{-1}\mathbb{T}$ with parameter $p \neq p_c$). A natural question is to prove that as $p \neq p_c$ tends to p_c, one obtains a nontrivial scaling limit: the **near-critical scaling limit**. Prior to the works [**G4**, **G5**], subsequential scaling limits were known to exist. As such, the status for near-critical percolation was the same as for critical percolation on $\mathbb{Z}^2$, where subsequential scaling limits (in the so-called **Schramm-Smirmov space** $\mathscr{H}$ yet to be defined in Definition 2.3) are also known to exist. The existence of such subsequential scaling limits is basically a consequence of the RSW theorem. Obtaining a (unique) scaling limit is in general a much harder task (for example, it follows from Smirnov's theorem [Sm01] for critical percolation on $\mathbb{T}$), and this is the main contribution of [**G4**, **G5**] where we prove the existence of the scaling limit (again in the space $\mathscr{H}$) for near-critical site percolation on the triangular grid $\mathbb{T}$ below its correlation length. See Corollary 1.5 where one obtains two different scaling limits as $p \to p_c$: ω^+_∞ and ω^-_∞ depending whether $p > p_c$ or not. One might think at this point that these near-critical scaling limits should be identical to the critical scaling limit ω_∞, since the correlation length $L(p)$ was defined in such a way that the system "looks" critical below $L(p)$. But, as it is shown in [NW09], although any subsequential scaling limit of near-critical percolation indeed "resembles" ω_∞ (the interfaces have the same Hausdorff dimension 7/4 for example), it is nevertheless singular w.r.t ω_∞.

1.3. — Near-critical coupling

The proof in [**G4**, **G5**] relies on a slightly tangential way of viewing near-critical percolation: via the so-called monotone couplings.

It is a classical fact that one can couple site-percolation configurations $\{\omega_p\}_{p\in[0,1]}$ on $\mathbb{T}$ in such a way that for any $p_1 < p_2$, one has $\omega_{p_1} \leq \omega_{p_2}$ with the obvious partial order on $\{0,1\}^{\mathbb{T}}$. One way to achieve such a coupling is to sample independently on each site $x \in \mathbb{T}$ a uniform random variable $u_x \sim \mathcal{U}([0,1])$, and then define $\omega_p(x) := 1_{u_x \leq p}$.

Remark 1.1. Note that defined this way, the process $p \in [0,1] \mapsto \omega_p$ is a.s. a càdlàg path in $\{0,1\}^{\mathbb{T}}$ endowed with the product topology. This remark already hints why we will later consider the Skorohod space on the Schramm-Smirnov space $\mathscr{H}$.

One would like to rescale this monotone coupling on a grid $\eta\mathbb{T}$ with small mesh $\eta > 0$ in order to obtain an interesting limiting coupling. If one just rescales space without rescaling the parameter p around p_c, it is easy to see that the monotone coupling $\{\eta\omega_p\}_{p\in[0,1]}$ on $\eta\mathbb{T}$ converges as a coupling to a trivial limit except for the slice corresponding to $p = p_c$ where one obtains the Schramm-Smirnov scaling limit of critical percolation, denoted by ω_∞. Thus, one should look for a monotone coupling $\{\omega_\eta^{\mathsf{nc}}(\lambda)\}_{\lambda\in\mathbb{R}}$, where $\omega_\eta^{\mathsf{nc}}(\lambda) = \eta\omega_p$ with $p = p_c + \lambda r(\eta)$, and where the zooming factor $r(\eta)$ goes to zero with the mesh. On the other hand, if it tends to zero too quickly, it is easy to check that $\{\omega_\eta^{\mathsf{nc}}(\lambda)\}_\lambda$ will also converge to a trivial coupling where all the slices are identical to the $\lambda = 0$ slice, i.e., the Schramm-Smirnov limit ω_∞. From the above heuristical explanation and especially from Kesten's work on the correlation length [Ke87] (see also [NW09] and [**G4**, **G5**]), it is natural to fix once and for all the zooming factor to be:

$$r(\eta) := \eta^2 \alpha_4^\eta(\eta,1)^{-1} (= \eta^{3/4+o(1)})\,, \tag{1.3}$$

where $\alpha_4^\eta(r,R)$ stands for the probability of the **alternating four-arm** event for critical percolation on $\eta\mathbb{T}$ from radius r to R. One disadvantage of the present definition of $\omega_\eta^{\mathsf{nc}}(\lambda)$ is that $\lambda \in \mathbb{R} \mapsto \omega_\eta^{\mathsf{nc}}(\lambda)$ is a time-*in*homogenous Markov process. To overcome this, we change slightly the definition of $\omega_\eta^{\mathsf{nc}}(\lambda)$ as follows:

Definition 1.1. *Let us define the* **near-critical coupling** $(\omega_\eta^{\mathsf{nc}}(\lambda))_{\lambda\in\mathbb{R}}$ *to be the following process:*

(i) Sample $\omega_\eta^{\mathsf{nc}}(\lambda = 0)$ according to $\mathbb{P}_\eta$, the law of critical percolation on $\eta\mathbb{T}$. We will sometimes represent this as a black-and-white colouring of the faces of the dual hexagonal lattice.

(ii) As λ increases, closed (white) hexagons switch to open (black) at exponential rate $r(\eta)$, defined by (1.3).

(iii) As λ decreases, open (black) hexagons switch to closed (white) at rate $r(\eta)$.

As such, for any $\lambda \in \mathbb{R}$, the near-critical percolation $\omega_\eta^{\mathsf{nc}}(\lambda)$ corresponds exactly to a percolation configuration on $\eta\mathbb{T}$ with parameter

$$\begin{cases} p = p_c + 1 - e^{-\lambda\, r(\eta)} & \text{if } \lambda \geq 0 \\ p = p_c - (1 - e^{-|\lambda|\, r(\eta)}) & \text{if } \lambda < 0\,, \end{cases}$$

thus making the link with our initial definition of $\omega_\eta^{\mathsf{nc}}(\lambda)$.

In this setting of monotone couplings, the main contribution of [**G4**, **G5**] is to prove the convergence of the monotone family $\{\omega_\eta^{\mathsf{nc}}(\lambda)\}_{\lambda\in\mathbb{R}}$ as $\eta \to 0$ to a limiting coupling $\{\omega_\infty^{\mathsf{nc}}(\lambda)\}_{\lambda\in\mathbb{R}}$. See Theorems 1.3 and 1.4 for precise statements. In some sense, this limiting object captures the birth of the infinite cluster seen from the scaling limit.

1.4. — Rescaled dynamical percolation

In [HPS97], the authors introduced a natural reversible dynamics on percolation configurations called **dynamical percolation**. This dynamics is very simple: each site (or bond in the case of bond-percolation) is updated independently of the other sites at rate one, according to the Bernoulli law $p\delta_1 + (1-p)\delta_0$. As such, the law $\mathbb{P}_p$ on $\{0,1\}^{\mathbb{T}}$ is invariant under the dynamics. Several intriguing properties like existence of exceptional times at $p = p_c$ where infinite clusters suddenly arise have been proved lately; see [SchSt10, **G3**, HPS12]. It is a natural desire to define a similar dynamics for the Schramm-Smirnov scaling limit of critical percolation $\omega_\infty \sim \mathbb{P}_\infty$, i.e., a process $t \mapsto \omega_\infty(t)$ which would preserve the measure $\mathbb{P}_\infty$ introduced later in Chapter 2 (Theorem 2.2). Defining such a process is a much more difficult task and a natural approach is to build this process as the scaling limit of dynamical percolation on $\eta\mathbb{T}$ properly rescaled (in space as well as in time). Using similar arguments as for near-critical percolation (see the detailed discussion in [**G4**]), the right way of rescaling dynamical percolation is as follows:

Definition 1.2. *In the rest of this paper, for each $\eta > 0$, the rescaled* **dynamical percolation** $t \mapsto \omega_\eta(t)$ *will correspond to the following process:*

(i) Sample the initial configuration $\omega_\eta(t=0)$ according to $\mathbb{P}_\eta$, the law of critical site percolation on $\eta\mathbb{T}$.

(ii) As time t increases, each hexagon is updated independently of the other sites at exponential rate $r(\eta)$ (defined in equation (1.3)). When an exponential clock rings, the state of the corresponding hexagon becomes either white with probability 1/2 or black with probability 1/2. (Hence the measure $\mathbb{P}_\eta$ is invariant).

Note the similarity between the processes $\lambda \mapsto \omega_\eta^{\mathsf{nc}}(\lambda)$ and $t \mapsto \omega_\eta(t)$. In particular, the second main achievement of [**G4**, **G5**] is to prove that the rescaled dynamical percolation process $t \mapsto \omega_\eta(t)$, seen as a càdlàg process in the Schramm-Smirnov space $\mathscr{H}$ has a scaling limit as the mesh $\eta \to 0$. See Theorem 1.6. This answers in particular Question 5.3 in [Sch07].

1.5. — Main results

The first result we wish to state is that if $\lambda \in \mathbb{R}$ is fixed, then the near-critical percolation $\omega_\eta(\lambda)$ has a scaling limit as $\eta \to 0$. In order to state a proper theorem, one has to specify what the setup and the topology are. As it is discussed at the beginning of Section 1 in Chapter 2, there are several very different manners to represent or "encode" what a percolation configuration is (see also the very good discussion on this in [SchSm11]). In [**G4**, **G5**], we followed the approach by Schramm and Smirnov, which will be explained in details in Section 1 of Chapter 2. In this approach, each percolation configuration $\omega_\eta \in \{0,1\}^{\eta\mathbb{T}}$ corresponds to a point in the Schramm-Smirnov topological space $(\mathscr{H}, \mathcal{T})$ which has the advantage to be compact (see Theorem 2.1) and Polish. From [SchSm11] and [CN06], it follows that $\omega_\eta \sim \mathbb{P}_\eta$ (critical percolation on $\eta\mathbb{T}$) has a scaling limit in $(\mathscr{H}, \mathcal{T})$: i.e., it converges in law as $\eta \to 0$ under the topology $\mathcal{T}$ to a "continuum" percolation $\omega_\infty \sim \mathbb{P}_\infty$, where $\mathbb{P}_\infty$ is a Borel probability measure on $(\mathscr{H}, \mathcal{T})$. We may now state our first main result.

Theorem 1.3. *Let $\lambda \in \mathbb{R}$ be fixed. Then as $\eta \to 0$, the near-critical percolation $\omega_\eta^{\mathsf{nc}}(\lambda)$ converges in law (in the topological space $(\mathscr{H}, \mathcal{T})$) to a limiting random percolation configuration, which we will denote by $\omega_\infty^{\mathsf{nc}}(\lambda) \in \mathscr{H}$.*

As pointed out earlier, the process $\lambda \in \mathbb{R} \mapsto \omega_\eta^{\mathsf{nc}}(\lambda)$ is a càdlàg process in $(\mathscr{H}, \mathcal{T})$. One may thus wonder if it converges as $\eta \to 0$ to a limiting random càdlàg path. There is a well-known and very convenient functional setup for càdlàg paths with values in a Polish metric spaces (X, d): the **Skorohod space** introduced in Proposition 2.1. Fortunately, we know from Theorem 2.1 that the Schramm-Smirnov space $(\mathscr{H}, \mathcal{T})$ is metrizable. In particular, one can introduce a Skorohod space of càdlàg paths with values in $(\mathscr{H}, d_{\mathscr{H}})$ where $d_{\mathscr{H}}$ is some fixed distance compatible with the topology $\mathcal{T}$. This Skorohod space is defined in Lemma 2.2 and is denoted by $(\mathsf{Sk}, d_{\mathsf{Sk}})$. We have the following theorem:

Theorem 1.4. *As the mesh $\eta \to 0$, the càdlàg process $\lambda \mapsto \omega_\eta^{\mathsf{nc}}(\lambda)$ converges in law under the topology of d_{Sk} to a limiting random càdlàg process $\lambda \mapsto \omega_\infty^{\mathsf{nc}}(\lambda)$.*

Remark 1.2. Due to the topology given by d_{Sk} (see Lemma 2.2), it is not a priori obvious that the slice $\omega_\infty^{\mathsf{nc}}(\lambda)$ obtained from Theorem 1.4 is the same object as the scaling limit $\omega_\infty^{\mathsf{nc}}(\lambda)$ obtained in Theorem 1.3. Nonetheless, it is proved in Theorem 9.5 in [G5] that these two objects indeed coincide.

From the above theorem, it is easy to extract the following corollary which answers our initial motivation by describing how percolation looks below its correlation length.

Corollary 1.5. *For any $p \neq p_c$, let*

$$L(p) := \inf \left\{ R \geq 1, \ s.t. \ R^2 \alpha_4(R) \geq \frac{1}{|p - p_c|} \right\}.$$

Recall that for any $p \in [0,1]$, ω_p stands for percolation on $\mathbb{T}$ with intensity p. Then as $p - p_c > 0$ tends to zero, $L(p)^{-1}\omega_p$ converges in law in $(\mathscr{H}, d_{\mathscr{H}})$ to $\omega_\infty^+ := \omega_\infty^{\mathsf{nc}}(\lambda = 1)$ while as $p - p_c < 0$ tends to 0, $L(p)^{-1}\omega_p$ converges in law in $(\mathscr{H}, d_{\mathscr{H}})$ to $\omega_\infty^- := \omega_\infty^{\mathsf{nc}}(\lambda = -1)$.

We defined another càdlàg process of interest in Definition 1.2: the rescaled dynamical percolation process $t \mapsto \omega_\eta(t)$. This process also leaves in the Skorohod space Sk and we have the following scaling limit result:

Theorem 1.6. *As the mesh $\eta \to 0$, rescaled dyamical percolation converges in law (in $(\mathsf{Sk}, d_{\mathsf{Sk}})$) to a limiting stochastic process in $\mathscr{H}$ denoted by $t \mapsto \omega_\infty(t)$.*

By construction, $t \mapsto \omega_\eta(t)$ and $\lambda \mapsto \omega_\eta^{\mathsf{nc}}(\lambda)$ are Markov processes in $\mathscr{H}$. Yet there is absolutely no reason that the Markov property survives at the scaling limit. In fact, we wish to point out that the last author of [G4, G5], Oded Schramm, initially believed the opposite. Our strategy of proof for Theorems 1.4 and 1.6 (see below) in fact enables us to prove the following result.

Theorem 1.7.

- *The process $t \mapsto \omega_\infty(t)$ is a **Markov** process which is reversible w.r.t the measure $\mathbb{P}_\infty$, the scaling limit of critical percolation.*

- *The process $\lambda \mapsto \omega_\infty^{\mathsf{nc}}(\lambda)$ is a* ***time-homogeneous (but non-reversible) Markov*** *process in $(\mathscr{H}, d_{\mathscr{H}})$.*

Remark 1.3. Thus we obtain a natural **diffusion** on the Schramm-Smirnov space $\mathscr{H}$. Interestingly, it can be seen that this diffusion is **non-Feller**! See Remark 11.9 in [**G5**]. We do not know whether the strong Markov property is satisfied or not.

Furthermore the processes $\lambda \mapsto \omega_\infty^{\mathsf{nc}}(\lambda)$ and $t \mapsto \omega_\infty(t)$ turn out to be conformally covariant under the action of conformal maps. Roughly speaking, if $\tilde{\omega}_\infty(t) = \phi \cdot \omega_\infty(t)$ is the conformal mapping of a continuum dynamical percolation from a domain D to a domain $\tilde{D}$, then the process $t \mapsto \tilde{\omega}_\infty(t)$ evolves very quickly (in a precise quantitative manner) in regions of D' where $|\phi'|$ is large and very slowly in regions of D' where $|\phi'|$ is small. This type of invariance was conjectured in [Sch07], it was even coined a "relativistic" invariance due to the space-time dependency. When the conformal map is a scaling $z \in \mathbb{C} \mapsto \alpha \cdot z \in \mathbb{C}$, the conformal covariance reads as follows:

Theorem 1.8. *For any scaling parameter $\alpha > 0$ and any $\omega \in \mathscr{H}$, we will denote by $\alpha \cdot \omega$ the image by $z \mapsto \alpha\, z$ of the configuration ω. With these notations, we have the following identities in law:*

1.
$$\Big(\lambda \mapsto \alpha \cdot \omega_\infty^{\mathsf{nc}}(\lambda)\Big) \overset{(d)}{=} \Big(\lambda \mapsto \omega_\infty^{\mathsf{nc}}(\alpha^{-3/4}\lambda)\Big)$$

2.
$$\Big(t \geq 0 \mapsto \alpha \cdot \omega_\infty(t)\Big) \overset{(d)}{=} \Big(t \mapsto \omega_\infty(\alpha^{-3/4}t)\Big)$$

Note that this theorem is very interesting from a **renormalization group** perspective. Indeed, the mapping $F : \mathscr{H} \to \mathscr{H}$ which associates to a configuration $\omega \in \mathscr{H}$ the "renormalized" configuration $\frac{1}{2} \cdot \omega \in \mathscr{H}$ is a very natural renormalization map on the space $\mathscr{H}$. It is easy to check that the law $\mathbb{P}_\infty$ is a fixed point for this transformation. The above theorem shows that the one-dimensional line given by $\{\mathbb{P}_{\lambda,\infty}\}_{\lambda\in\mathbb{R}}$, where $\mathbb{P}_{\lambda,\infty}$ denotes the law of $\omega_\infty^{\mathsf{nc}}(\lambda)$, provides an **unstable variety** for the transformation $\omega \in \mathscr{H} \mapsto \frac{1}{2} \cdot \omega \in \mathscr{H}$.

In the last section of Chapter 2, we will list some further properties (such as an extension to the model of **gradient percolation**, massive SLE_6, correlation lengths for $\omega_\infty^{\mathsf{nc}}(\lambda)$ etc..)

1.6. — Global strategy

In order to prove Theorem 1.3 and Theorem 1.4, our strategy in [**G4**, **G5**] is to start by building the processes $\lambda \mapsto \omega_\infty^{\mathsf{nc}}(\lambda)$ and $t \mapsto \omega_\infty(t)$ and then show that they are the scaling limits of their discrete η-analogs. We will focus on the near-critical case, the dynamical case being handled similarly. Our strategy will be to start with the critical slice, i.e., the Schramm-Smirnov limit $\omega_\infty = \omega_\infty(\lambda = 0) \sim \mathbb{P}_\infty$ and then as λ will increase, we will randomly add in an appropriate manner some "infinitesimal" mass to $\omega_\infty(0)$. In the other direction, as λ will decrease below 0, we will randomly remove some "infinitesimal" mass to $\omega_\infty(0)$. Before passing to the limit, when one still has discrete configurations ω_η on a lattice $\eta\mathbb{T}$, this procedure of adding or removing mass is straightforward and is given by the Poisson point process induced by Definition 1.1. At the scaling limit, there are no sites or hexagons any more, hence one has to find a proper way to perturb the slice $\omega_\infty(0)$. Even though there are no black or white hexagons anymore, there are some specific points in $\omega_\infty(0)$ that should play a significant role and are measurable w.r.t. ω_∞: namely, the set of all **pivotal points** of ω_∞. We shall denote this set by $\bar{\mathcal{P}} = \bar{\mathcal{P}}(\omega_\infty)$, which can be proved to be measurable w.r.t. ω_∞ using the methods of [**G4**, Section 2] The "infinitesimal" mass we will add to the configuration $\omega_\infty(0)$ will be a certain random subset of $\bar{\mathcal{P}}$. Roughly speaking, one would like to define a mass measure $\bar{\mu}$ on $\bar{\mathcal{P}}$ and the infinitesimal mass should be given by a Poisson point process PPP on $(x, \lambda) \in \mathbb{C} \times \mathbb{R}$ with intensity measure $d\bar{\mu} \times d\lambda$. We would then build our limiting process $\lambda \mapsto \omega_\infty^{\mathsf{nc}}(\lambda)$ by "updating" the initial slice $\omega_\infty(0)$ according to the changes induced by the point process PPP. So far, the strategy we just outlined corresponds more-or-less to the conceptual framework from [CFN06].

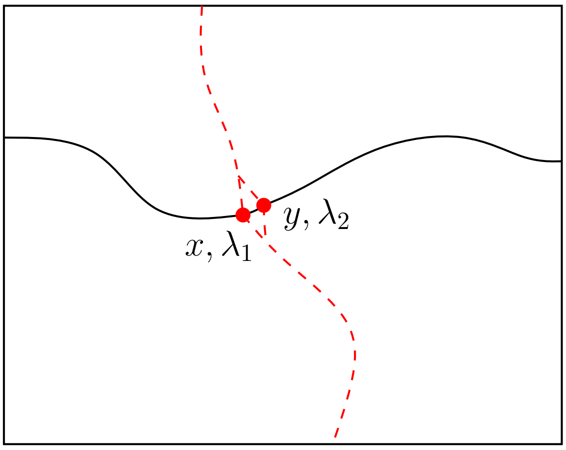
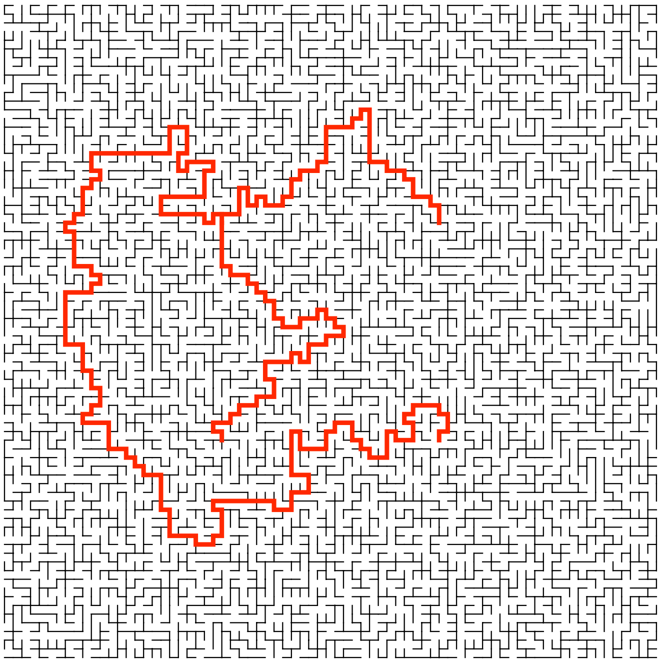


Figure 1.3: An example of a "cascade" configuration: at $\lambda = 0$ there is no left-right crossing and both points x and y have low importance, but at the level $\lambda_2 > \lambda_1 > 0$ there is a left-right crossing that we could not predict if we are not looking at low important points.

The main difficulty with this strategy is the fact that the set of pivotal points $\bar{\mathcal{P}}(\omega_\infty)$ is a.s. a dense subset of the plane of Hausdorff dimension $3/4$ and that the appropriate mass measure $\bar{\mu}$ on $\bar{\mathcal{P}}$ would be of infinite mass everywhere. This makes the above strategy too degenerate to work with. To overcome this, one introduces a small spatial cut-off $\epsilon > 0$ which will ultimately tend to zero. Instead of considering the set of all pivotal points, the idea is to focus only on the set of pivotal points which are initially pivotal up to scale ϵ. Let us denote by $\bar{\mathcal{P}}^\epsilon = \bar{\mathcal{P}}^\epsilon(\omega_\infty(\lambda = 0))$ this set of ϵ-pivotal points. The purpose of the first paper [**G4**] is to introduce a measure $\bar{\mu}^\epsilon = \bar{\mu}^\epsilon(\omega_\infty)$ on this set of ϵ-pivotal points. This limit corresponds to the weak limit of renormalized (by $r(\eta)$) counting measures on the set $\bar{\mathcal{P}}^\epsilon(\omega_\eta)$, and it can be seen as a "local time" measure on the pivotal points of percolation and is called the **pivotal measure**. See Theorem 2.4 where for technical reasons a slightly different set $\mathcal{P}^\epsilon$ with its corresponding measure μ^ϵ is used. Once such a spatial cut-off ϵ is introduced, the idea is to "perturb" $\omega_\infty(\lambda = 0)$ using a Poisson point process $\mathsf{PPP} = \mathsf{PPP}(\mu^\epsilon)$ of intensity measure $d\mu^\epsilon \times d\lambda$ (we now switch to the actual measure μ^ϵ used throughout and which is introduced in Definition 2.12). This will enable us to define a cut-off trajectory $\lambda \mapsto \omega_\infty^{\mathsf{nc},\epsilon}(\lambda)$. (In fact the construction of this process requires a lot of work, most of [**G5**], see Section 3 in Chapter 2). The main problem that remains is to show that this procedure in some sense stabilizes as the cut-off $\epsilon \to 0$. This is far from being obvious since there could exist "cascades" from the microscopic world which would have macroscopic effects as is illustrated in Figure 1.3.

1.7. — Scaling limit of the Minimal Spanning Tree (Chapter 3)



In our recent work [G6], we prove a scaling limit result for a (version of) the model of **Minimal Spanning Tree** in the plane. This work as we shall see below is based on [G4, G5] which are described in Chapter 2.

The Minimal Spanning Tree of is a classical combinatorial object. On a finite graph $G = (V, E)$, it may be defined as follows: For each edge $e \in E(G)$, let $U(e)$ be an independent Unif$[0, 1]$ label. The **Minimal Spanning Tree** on G, denoted by MST, is the spanning tree T for which $\sum_{e \in T} U(e)$ is minimal. As opposed to the celebrated **travelling salesman problem (TSP)**, there exist fast algorithms which compute the MST given the labels $\{U(e)\}_{e \in E}$:

1. **Prim's** algorithm: start from any vertex $x \in V$ and run an **invasion percolation** until the whole graph is covered. I.e., let $V_0 := \{x\}$ and build $V_1, \ldots, V_i, \ldots$ until $V_n = V$ as follows: for any $i \geq 0$, among all edges which leave the set V_i, choose the edge with minimal label (there is a.s a unique one if labels are independent Unif$[0, 1]$ variables) and add its exiting endpoint to V_i in order to obtain V_{i+1}.

2. **Kruskal's** algorithm: order edges in such a way that $U(e_1) < U(e_2) < \ldots U(e_{|E|})$ and let $M_0 = \emptyset$. Define $M_1, \ldots, M_{|E|}$ inductively as follows until one obtains a spanning tree of G: at step i, add the edge e_i to M_{i-1} if it does not create any cycle; otherwise "delete" e_i and go to step $i + 1$.

3. **Reversed Kruskal:** delete from each cycle of edges $(e_1, \ldots, e_n = e_1)$ the edge with maximal weight. The set of remaining edges gives MST.

These three algorithms obviously have a similar flavor. They also shows that MST depends only on the ordering of the labels, not on the values themselves. Moreover, the third algorithm also makes sense on any infinite graph, and produces what in general is called the Free Minimal Spanning Forest (FMSF) of the infinite graph. The Wired Minimal Spanning Forest (WMSF) is the one when we also remove the edge with the highest label from cycles that "go through infinity", i.e., which are the union of two disjoint infinite simple paths starting from a vertex. For the case of Euclidean planar lattices, these two measures on spanning forests are known to be the same, again denoted by MST, and it almost surely consists of a single tree [AM94]. This measure can also be obtained as a **thermodynamical limit**: take any exhaustion by finite subgraphs $G_n(V_n, E_n)$, introduce a boundary condition by identifying some of the vertices on the boundary of G_n (i.e., elements of V_n that have neighbors in G outside of V_n), and then take the weak limit. Studying these measures has a rich history on $\mathbb{Z}^d$, on point processes in $\mathbb{R}^d$, and on general transitive graphs; see for example [LyP13] and the references therein. We will be concerned only with the planar case here.

Note that the second algorithm in particular is intimately related to the **standard coupling** for Bernoulli percolation discussed above (see Remark 1.1). Indeed if $(\omega_p)_{p\in[0,1]}$ denotes a standard coupling of Bernoulli percolations on $G = (V, E)$, then the minimal spanning tree can be obtained by increasing the level p from 0 to 1 and adding edges one at a time with the condition that they should not create cycles.

With this algorithm in mind, imagine one wishes to understand the large scale geometry of a planar Minimal Spanning Tree, for example on a large N by N box on $\mathbb{Z}^2$ (see the above figure). Let ω_p^N be a standard coupling on this large finite graph. If one raises the level p, then from the above discussion on **near-critical percolation**, we will not create **macroscopic** branches of the minimal spanning Tree MST before getting very near to $p \approx p_c$. In fact it is not hard to convince oneself that all the macroscopic geometry of MST arises from what happens in the **near-critical window**.

Figure 1.4: The MST connects the percolation p-clusters without creating cycles, yielding the cluster-tree MST^p.

Following this informal discussion, it should then be possible to extract a "continuum" minimal spanning tree MST_∞ out of the near-critical coupling $(\omega_\infty^{\mathsf{nc}}(\lambda))_{\lambda\in\mathbb{R}}$ introduced in [**G5**] (see our main Theorem 1.4 above). Furthermore, this limiting tree MST_∞ should be the scaling limit (under some appropriate topology) of a discrete Minimal Spanning Tree MST_η. Of course, we will not be able to obtain a scaling limit result for the MST on the square lattice $\mathbb{Z}^2$ (as pictured above) since in that case we don't have a scaling limit even for $\omega_\eta(\lambda = 0)$. Yet, it turns out that there is a natural notion of MST on the triangular lattice $\mathbb{T}$ which is associated to **site**-percolation on $\mathbb{T}$. (See Section 1 in Chapter 3). For this planar MST on the triangular grid $\mathbb{T}$, we prove the following theorem:

Theorem 1.9 (Limit of MST_η in $\mathbb{C}$, [G6]). *As $\eta \to 0$, the spanning tree MST_η on $\eta\mathbb{T}$ converges in distribution (under the setup introduced in [AB+99]) to a unique scaling limit MST_∞ that is invariant under translations, scalings, and rotations.*

Remark 1.4. Note that subsequential scaling limits were known to exist since the work [AB+99] which introduced a certain Polish space which we rely on in [**G6**].

Remark 1.5. We obtain a similar scaling limit result in [**G6**] for the related model of **invasion percolation**.

Remark 1.6. The recent works [ABG12, ABGM13] follow a strategy similar to ours, but in a very different setting: namely, in the **mean-field** case. It is well-known that there is a phase transition at $p = 1/n$ for the **Erdös-Rényi random graphs** $G(n,p)$. Similarly to the above case of planar percolation, it is a natural problem to study the geometry of these random graphs near the transition $p_c = 1/n$. It turns out in this case that the meaningful rescaling is to work with $p = 1/n + \lambda/n^{4/3}$, $\lambda \in \mathbb{R}$. If $R_n(\lambda) = (C_n^1(\lambda), C_n^2(\lambda), \ldots)$ denotes the sequence of clusters at $p = 1/n + \lambda/n^{4/3}$, ordered in decreasing order of size, say, then it is proved in [ABG12] that as $n \to \infty$, the normalized sequence $n^{-1/3}\, R_n(\lambda)$ converges in law to a limiting object $R_\infty(\lambda)$ for a certain topology on sequences of compact spaces which relies on the Gromov-Hausdorff distance. This near-critical coupling $\{R_\infty(\lambda)\}_{\lambda\in\mathbb{R}}$ has then been used in [ABGM13] to obtain a scaling limit as $n \to \infty$ (in the Gromov-Hausdorff sense) of the MST on the complete graph with n vertices. One could say that [**G5**] is the Euclidean ($d = 2$) analogue of the mean-field case [ABG12], and that [**G6**] is the analogue of [ABGM13]. However, an important difference is that in the mean-field case one is interested in the intrinsic metric properties (and hence works with the Gromov-Hausdorff distance between metric spaces), while in the Euclidean case one is first of all interested in how the graph is embedded in the plane.

We will explain in Chapter 3 the main steps which lead in [**G6**] to the proof of this theorem. We will also discuss some almost sure properties statisfied by MST_∞ (estimates on the Hausdorff dimension of branches, maximal degree of points etc...)

2. — Critical percolation under conservative dynamics (Chapter 4)

In the standard model of **dynamical percolation** introduced in 1996 by Häggström, Peres and Steif in [HPS97], sites (or edges) evolve independently of each other according to Poisson Point Processes in such a way that the product measure $\mathbb{P}_p$ is preserved by the dynamics. For example, if $\omega(0) \sim \mathbb{P}_p$ for some intensity $p \in [0,1]$, then open sites switch to closed ones at rate $1-p$ while closed sites switch to open ones at rate p. This defines a natural dynamics $t \mapsto \omega(t)$ in the space of percolation configurations which is such that for any time $t \geq 0$, $\omega(t) \sim \mathbb{P}_p$. The general question studied in dynamical percolation is whether, when we start with the stationary distribution $\mathbb{P}_p$, there exist atypical times at which the percolation structure looks markedly different than that at a fixed time. In almost all cases, the term "markedly different" refers to the existence or nonexistence of an infinite connected component. Let us briefly review the main results in this area:

1. In [HPS97], the authors show (among other things) that for dynamical percolation on $\mathbb{Z}^d, d \geq 19$ at the critical point $p_c(\mathbb{Z}^d)$, there are a.s. no **exceptional times** along the dynamics where an infinite cluster suddenly appears. Their proof relies crucially

on the fact that $p \mapsto \theta_{\mathbb{Z}^d}(p)$ is **linear** near $p_c(\mathbb{Z}^d)$ by a famous result of Hara and Slade ([HS94]). Since the density function for planar percolation is no longer linear near p_c, the authors in [HPS97] left open the natural question of the existence (or not) of exceptional times in critical planar dynamical percolation.

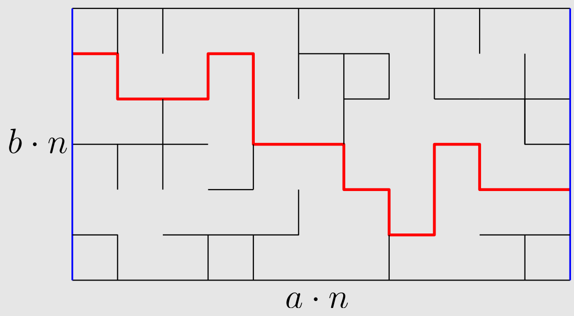
2. Mostly motivated by this question of exceptional times in planar dynamical percolation, Benjamini, Kalai and Schramm introduced in [BKS99] the very fruitful concept of **noise sensitivity of Boolean functions**. (See also the survey [**G18**] and the lecture notes [**G20**]). Let us point out that the authors of [BKS99] were also motivated at the time by a strategy to prove conformal invariance in critical planar percolation. This strategy did not work out and the proof by Smirnov for site percolation in [Sm01] uses a completely different approach. In the particular case of percolation, the idea in [BKS99] is to study how macroscopic (large) clusters are affected by small i.i.d perturbations (where only a small fraction, say ϵ of the edges is resampled). One of their main theorem states that the large clusters of $\omega(t = 0)$ are "independent" of the large clusters in $\omega(t = \epsilon)$. One says in this case that critical percolation is **noise sensitive**. They proved their result by studying the **"spectrum"** of critical percolation and by showing that this spectrum essentially consists of high frequencies. We will come back to this below.

3. The spectral approach initiated in [BKS99] was promising but not quantitative enough yet to yield the existence of exceptional times for dynamical percolation in $2d$. In [SchSt10], Schramm and Steif provide more quantitative results on the noise sensitivity of planar critical percolation (Theorem 1.11 below). This enabled them to prove for the first time the existence of exceptional times for dynamical site percolation on the triangular lattice $\mathbb{T}$ at $p_c(\mathbb{T}) = 1/2$. Furthermore, if $\mathcal{E}$ denotes the random set of these exceptional times, they prove that the Hausdorff dimension of $\mathcal{E}$ a.s. lies in the interval $[1/6, 31/36]$.

4. Finally, in [**G3**], we obtained optimal results on the spectrum of critical percolation (using yet a different approach). This enabled us to strengthen the known results from [SchSt10] on the triangular lattice: for example we proved that the Hausdorff dimension of the set of exceptional times $\mathcal{E}$ is a.s. equal to $31/36$ and we established the existence of exceptional times where an infinite (primal) cluster coexists with an infinite (dual) cluster, which is a rather counter-intuitive fact in percolation theory. Furthermore, we also obtained the existence of exceptional times for dynamical percolation on $\mathbb{Z}^2$, even though conformal invariance and critical exponents are still lacking in this case.

Let us very briefly explain what the **spectral approach** is since [BKS99] (we refer to [**G18**, **G20**] for more details). First one notices that the large scale geometry of critical percolation is encoded by macroscopic crossing events such as the event f_n introduced below. (Note that this is also the point of view which led Schramm and Smirnov to introduce their topological space $\mathscr{H}$ which encodes a percolation configuration ω by the set of quads which are traversed by ω).

Definition 1.3 (Percolation crossings).



Let $a, b > 0$. Consider the rectangle $[0, a \cdot n] \times [0, b \cdot n]$. The left to right crossing event corresponds to the Boolean function $f_n : \{-1,1\}^{O(1)n^2} \to \{-1,1\}$ defined as follows:

$$f_n(\omega) := \begin{cases} 1 & \text{if there is a left-right crossing} \\ -1 & \text{otherwise} \end{cases}$$

Proving the existence of exceptional times essentially boils down to showing that these crossing events decorrelate rapidly. In [BKS99], the authors study the behaviour as $n \to \infty$ of

$$\mathrm{Cov}\big[f_n(\omega(0)), f_n(\omega(\epsilon))\big] = \mathbb{E}\big[f_n(\omega(0))\, f_n(\omega(\epsilon))\big] - \mathbb{E}\big[f_n\big]^2 . \tag{1.4}$$

For this, they decompose the Boolean functions $f_n \in L^2(\{-1,1\}^{O(n^2)})$ into **Fourier-Walsh series**:

$$f_n = \sum_{S \subset [0,an]\times[0,bn]} \hat{f}_n(S)\, \chi_S ,$$

where the so-called **characters** $\{\chi_S\}$ are simply defined by

$$\chi_S(x_1, \ldots, x_m) := \prod_{i \in S} x_i .$$

See Section 3 in Chapter 4 for a short account on this. This spectral representation enables them to rewrite the above covariance in terms of the Fourier coefficients as follows:

$$\mathrm{Cov}\big[f_n(\omega(0)), f_n(\omega(\epsilon))\big] = \sum_{\emptyset \neq S \subset [0,an]\times[0,bn]} \hat{f}_n(S)^2\, e^{-|S|\epsilon} . \tag{1.5}$$

In particular, we see that **noise sensitivity** corresponds to a Fourier spectrum supported on large frequencies $|S| \gg 1$. From then on, the study of the **Fourier spectrum** of critical percolation has received a lot of attention. Let us now briefly review the main quantitative results in this direction.

The first result is due to Benjamini, Kalai and Schramm:

Theorem 1.10 ([BKS99]). *There exists a constant $c > 0$ such that*

$$\sum_{0<|S|<c \log n} \hat{f}_n(S)^2 \to 0 , \tag{1.6}$$

as the size of the system (n) go to infinity. This implies in particular the following **quantitative noise sensitivity result***: let the amount of noise ϵ_n depend on the size of the system in such a way that $\epsilon_n \gg \frac{1}{\log n}$. Then,*

$$\mathrm{Cov}\big[f_n(\omega(0)), f_n(\omega(\epsilon_n))\big] \to 0 .$$

To prove this theorem, Benjamini, Kalai and Schramm relied on the **hypercontractive inequality** which had already been used in the context of Boolean functions by Kahn, Kalai and Linial in their seminal work [KKL88] (the $\log n$ factor in (1.6) is reminiscent of this hypercontractive technique).

As mentioned above, this noise sensitivity result was not strong enough to imply the existence of exceptional times. Since the technique based on the hypercontractive inequality cannot go beyond the logarithmic scale in (1.6), Schramm and Steif relied on a completely different approach (based on **randomized algorithm**) in [SchSt10] to prove the following more quantitative result (we give here a simplified version):

Theorem 1.11 ([SchSt10]). *Consider critical site percolation on $\mathbb{T}$. For any $\epsilon > 0$, one has*

$$\sum_{0<|S|<n^{1/8-\epsilon}} \hat{f}_n(S)^2 \to 0\,,$$

as $n \to \infty$, where f_n is the analog on the triangular grid $\mathbb{T}$ of the crossing event defined in Definition 1.3. This result improves greatly on (1.6) and yields a polynomial noise sensitivity result for the crossing events $\{f_n\}$.

Finally, using yet a different approach (a "geometric" study of the spectral sets), the following result is proved in [G3] (we also give here a simplified version):

Theorem 1.12 ([G3]). *For any $1 \le r \le n$,*

$$\sum_{0<|S|<r^2\alpha_4(r)} \hat{f}_n(S)^2 \asymp \left(\frac{n}{r}\right)^2 \alpha_4(r,n)^2\,,$$

where the constants involved in $\asymp$ are uniform. This exact tail-behaviour (up to constants) of the spectral sample holds on the triangular grid $\mathbb{T}$ as well as on $\mathbb{Z}^2$. On the triangular grid, it implies in particular the following bound valid for any $\epsilon > 0$:

$$\sum_{0<|S|<n^{3/4-\epsilon}} \hat{f}_n(S)^2 \to 0\,.$$

All these results gave a better and better understanding of the noise sensitivity of critical planar percolation under "i.i.d noising", which then implied a richer understanding of (standard) dynamical percolation.

In the work [G8], we consider a different type of dynamics on percolation configurations which is **conservative** and still preserves the product measure $\mathbb{P}_p$: This dynamics is very natural and well-known: sites (or edges) now evolve according to a symmetric exclusion process with some **symmetric transition kernel** $\{P(x,y)\}$, $(x,y) \in \mathbb{E}^2 \times \mathbb{E}^2$ or $(x,y) \in \mathbb{T} \times \mathbb{T}$. See Section 1 in Chapter 4 as well as Figure 1.5 which illustrates a nearest-neighbour simple exclusion dynamics in the case of site-percolation on $\mathbb{Z}^2$.
Let $t \mapsto \omega^P(t)$ denote the trajectory of such a conservative dynamics with kernel P. Since for all $t \ge 0$, one has $\omega^P(t) \sim \mathbb{P}_p$ (assuming one starts at equilibrium), it is natural to wonder, similarly as in the i.i.d case, wether the macroscopic geometry is still **noise sensitive** or not under such dynamics (we will stick to the planar case here). It is useful to point out at

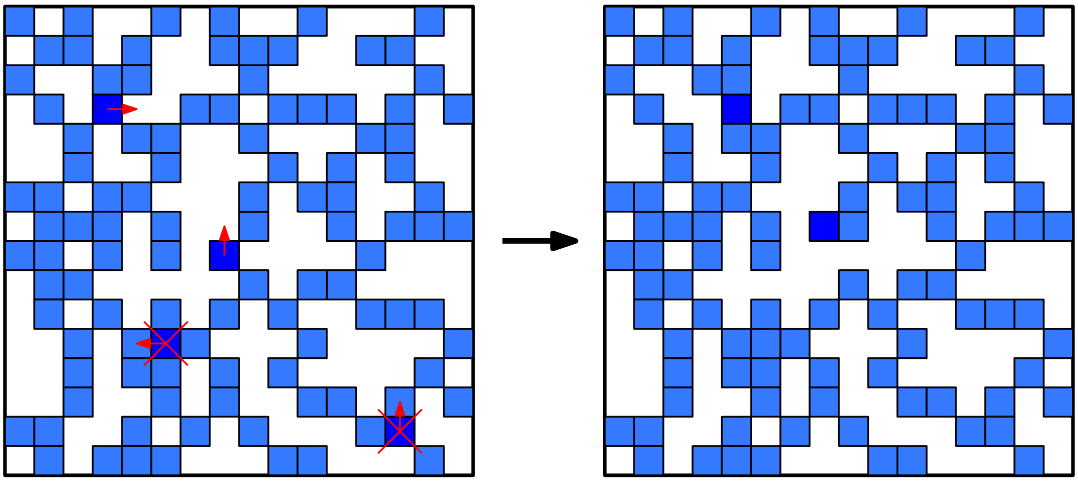


Figure 1.5:

this point that there are plenty of Boolean functions which are highly noise-sensitive under an i.i.d noising but remain stable under conservative dynamics. The most extreme example is given by the following Boolean functions on $\{-1,1\}^m$ (called **parity functions**):

$$g_m(x_1,\ldots,x_m) := \prod_{i=1}^m x_i \,.$$

These functions are very unstable if one rerandomizes a small fraction of the bits, but are completely stable under any kind of conservative dynamics. This simple example illustrates why noise sensitivity in the i.i.d regime does not necessarily transfer to noise sensitivity under an exclusion process (which we call in [**G8**] **exclusion sensitivity**). This is why a careful analysis is needed in [**G8**] in the case of percolation.

Here is another reason why **exclusion sensitivity** is in general much harder to study than (standard) i.i.d noise sensitivity: even though noise sensitivity is intimately related to the typical structure of **pivotal points**, there is no existing proof of noise sensitivity of percolation which is based on the properties of the **pivotal set**. All the proofs so far go through the study of the Fourier spectrum of percolation. Therefore, there is no hope to obtain a proof of noise sensitivity of percolation under conservative dynamics using simple considerations on the pivotal points. One thus needs to understand decorrelations such as

$$\mathrm{Cov}\big[f_n(\omega^P(0)), f_n(\omega^P(\epsilon))\big] \,, \tag{1.7}$$

as $n \to \infty$. The Fourier-Walsh decomposition we encountered earlier was very natural in the i.i.d case since the characters χ_S are **eigenfunctions** of the dynamics in the sense that

$$\mathbb{E}\big[\chi_S(\omega(t)) \;\big|\; \omega(0)\big] = e^{-t|S|}\chi_S(\omega(0)) \,.$$

These characters obviously no longer diagonalize the conservative dynamics $t \mapsto \omega^P(t)$. This makes the study of decorrelations such as (1.7) harder since ideally one would prefer to project $\omega^P(\cdot)$ on some orthonormal basis which diagonalizes the symmetric P-exclusion process. The eigenfunctions of the latter one are in general much more complicated than the characters χ_S and make the techniques from [BKS99, SchSt10, **G3**] obsolete. We will see in Chapter 4 how to overcome this difficulty.

The work [**G8**] studies the notion of **exclusion sensitivity** for general Boolean functions and then focuses on the particular case of critical percolation. Our main result on the **exclusion sensitivity of percolation** can be stated as follows:

Theorem 1.13 ([G8]). *Consider critical percolation on the triangular lattice $\mathbb{T}$ under an exclusion dynamics with symmetric Kernel*

$$P(x,y) \asymp \frac{1}{\|x-y\|^{2+\alpha}}\,,$$

for some exponent $\alpha > 0$ (the larger α is, the more localized the dynamics is). Then,

$$\mathrm{Cov}\big[f_n(\omega^P(0)), f_n(\omega^P(t))\big] \to 0\,,$$

as $n \to \infty$. Furthermore, this remains true is $t = t_n \geq n^{-\beta(\alpha)}$ for some exponent $\beta(\alpha) > 0$.

In particular, we obtain a **polynomial noise sensitivity result** similar as the one obtained by Schramm and Steif in Theorem 1.11. Unfortunately, our control is not good enough to imply the existence of exceptional times for $\omega^P(\cdot)$. Also, the higher α is, the worse our control gets and the limiting **nearest-neighbour exclusion dynamics** remains open due to the difficulty of its spectral approach.

3. — Magnetization field of the critical Ising model (Chapter 5)

Consider the Ising model (we recall its definition below) on a finite domain $\Lambda_N := [-N,N]^2$ with, say $+$ boundary conditions around $\partial\Lambda_N$. This corresponds to a random configuration of spins $\{\sigma_x\}_{x\in\Lambda_N}$ whose distribution depends on the inverse temperature β. The (total) magnetization $M_N := \sum_{x\in\Lambda_N}\sigma_x$ is a quantity that has received considerable attention. We will be interested here in the convergence in law of quantities such as the total magnetization (once properly renormalized) as $N\to\infty$. It is well-known that if $\beta \neq \beta_c$ (the critical inverse temperature, see below), then there exists a constant $a_\beta = \langle \sigma_0\rangle_{\beta,+} \geq 0$ such that as $N \to \infty$,

$$\frac{\sum_{x\in\Lambda_N}\sigma_x - a_\beta N^2}{N} \longrightarrow \mathcal{N}(0,\sigma_\beta^2)\,. \tag{1.8}$$

In other words the fluctuations of the total magnetization are Gaussian away from the critical point (β_c). Since the variance $\sigma_\beta^2 \nearrow \infty$ as $\beta \to \beta_c$, it is natural to wonder what is the law which governs the fluctuations of the total magnetization in the **critical regime**. The purpose of [G10, G12] is precisely to answer this type of question: more generally, the works [G10, G12] focus on the (renormalized) **magnetization field** which will be defined below.

— *The Ising model* —

Let us start by briefly recalling the definition of the Ising model in a finite domain of $\mathbb{Z}^2$:

Definition 1.4. *The* **Ising model** *on a finite domain $\Lambda\subset\mathbb{Z}^2$ with $+$ boundary condition and with external field $h\geq 0$ is a probability measure on $\{-1,1\}^\Lambda$, $\mathbb{P}^{\beta,h,+}$, defined as follows. For any spin configuration $\sigma\in\{-1,1\}^\Lambda$, let*

$$E(\sigma) := -\sum_{x\sim y}\sigma_x\sigma_y - \sum_{x\in\partial\Lambda}\sigma_x \tag{1.9}$$

be the interaction energy, where the first sum is over nearest neighbor pairs in Λ and the second is over sites in $\partial\Lambda$, the (interior) boundary of Λ. Let also

$$M(\sigma) := \sum_{x\in\Lambda} \sigma_x \tag{1.10}$$

be the total magnetization in Λ. The probability measure $\mathbb{P}^{\beta,h,+}$ on $\{-1,1\}^\Lambda$ is defined by

$$\mathbb{P}^{\beta,h,+}\big[\sigma\big] := \frac{1}{Z_{\beta,h}} e^{-\beta\, E(\sigma)+h\, M(\sigma)}\,, \tag{1.11}$$

where the **partition function** *$Z_{\beta,h}$ is simply defined as $\sum_\sigma e^{-\beta\, E(\sigma)+h\, M(\sigma)}$.*

Remark 1.7. All the results in [**G10**, **G12**] hold without any exterior magnetic field ($h = 0$). Yet we will need this exterior magnetic field h in the proofs and it will also be an important object in Chapter 6.

Remark 1.8. It is a classical fact that there are infinite volume limits (on $\mathbb{Z}^2$) for the above measures. We will consider in particular, the critical Ising model on the full plane $\mathbb{Z}^2$. Also, one may consider the Ising model on a finite domain $\Lambda \subset \mathbb{Z}^2$ with *free* boundary conditions (instead of $+$). This amounts to removing the second term in (1.9)

As is well-known, this model undergoes a phase transition at the **critical inverse temperature** $\beta_c(\mathbb{Z}^2) = \frac{1}{2}\log(1+\sqrt{2})$ (Onsager [On44], see also [BD12a] for a recent beautiful proof). This phase transition can be described for example as follows: the constant $a_\beta := \mathbb{E}^{\beta,+}_{\mathbb{Z}^2}\big[\sigma_0\big]$ introduced above is > 0 if $\beta > \beta_c$ and is equal to zero otherwise. See [Gri06] and references therein for more on this model.

— ***The renormalized magnetization field*** —

In order to obtain a limiting law describing the fluctuations of the total magnetization, we will rescale the Ising model on the lattice $a\,\mathbb{Z}^2$ with vanishing mesh $a \searrow 0$. This will enable us to rely on the recent breakthrough results [Sm10, CS12] by Smirnov and Chelkak-Smirnov on the conformal invariance of FK-percolation ($q = 2$) and site Ising model as well as on the scaling limit of the n-point spin correlation functions obtained by Chelkak, Hongler and Izyurov [CHI12].

Definition 1.5. *For any $a > 0$, define the renormalized* **magnetization field** *to be the following random distribution on the plane:*

$$\Phi^a := a^{15/8} \sum_{x\in a\mathbb{Z}^2} \sigma_x\,,$$

where $\{\sigma_x\}_{x\in a\mathbb{Z}^2}$ is distributed according to a critical Ising model in the plane.

This definition easily extends to the magnetization field in a bounded domain Ω equipped with free *or $+$ boundary conditions. Namely,*

$$\Phi^a_\Omega := a^{15/8} \sum_{x\in \Omega_a} \sigma_x\,,$$

where Ω_a is an approximation of Ω by the grid $a\mathbb{Z}^2$ (for example, the largest connected component of $\Omega \cap a\mathbb{Z}^2$).

Remark 1.9. Notice that the renormalization here is different from the case $\beta \neq \beta_c$. We will explain in Section 1 where the term $a^{15/8}$ comes from.

— ***Scaling limit result*** —

Recall our main goal was to prove a limit in law of the above random distribution Φ^a as $a \searrow 0$. Similarly as in Chapter 2, one needs to specify here a convenient space in which the family $\{\Phi^a\}$ will be tight. We are looking here for a functional space of distributions. As we will briefly sketch in section 2 of Chapter 5, in the case of a bounded smooth domain Ω, the Sobolev space of negative index $\mathcal{H}^{-3} = \mathcal{H}^{-3}(\Omega)$ defined as the dual space of the Sobolev space $\mathcal{H}_0^3(\Omega)$ will be our choice.

Our main theorem can be stated as follows.

Theorem 1.14 (Scaling limit). *Let Ω be a bounded smooth domain of the plane. Consider the critical Ising model in Ω_a with + of* free *boundary conditions. Then the magnetization field $\Phi^a_\Omega = \Phi^a$ converges in law as the mesh size $a \searrow 0$ to a limiting random distribution $\Phi^\infty_\Omega = \Phi^\infty$. The convergence in law holds in the Sobolev space $\mathcal{H}^{-3} = \mathcal{H}^{-3}(\Omega)$ under the topology given by $\|\cdot\|_{\mathcal{H}^{-3}}$.*

Remark 1.10. In the full plane, the magnetization field Φ^a also converges in law as the mesh size $a \searrow 0$ to a limiting random distribution Φ^∞. In this case, the convergence holds under a product topology on the product of Sobolev spaces $\mathcal{H}^{-3}(\Omega_k)$ for some increasing sequence of domains Ω_k. See [**G10**].

Remark 1.11. Notice that if $\Omega := [-1,1]^2$ is equipped with + boundary conditions, then the random variable $\langle \Phi^\infty, 1_{[-1,1]^2}\rangle$ answers our earlier motivation i.e. the limit in law of the rescaled total magnetization $N^{-15/8} M_N$ in Λ_N at $\beta = \beta_c = \frac{1}{2}\log(1+\sqrt{2})$.

Brief sketch of proof(s):

The proof of Theorem 1.14 starts by showing that the sequence $\{\Phi^a_\Omega\}_a$ is indeed **tight** in the space $\mathcal{H}^{-3}(\Omega)$. We will highlight how to do this in section 2.

Then, the main part of the proof consists as usual in showing the uniqueness of possible subsequential scaling limits. For this, we provide two different proofs in [**G10**]:

1. The first proof relies on the FK representation of the Ising model (see Definition 1.6) which allows us to decompose the distribution Φ^a as a sum over the FK clusters, where each cluster C carries an independent random sign $\sigma_C \in \{-1,1\}$. The idea of the proof is to construct **area measures** on the FK clusters, similarly as the **pivotal measures** constructed in Chapter 2 (Definition 2.11). Then, one shows that the limiting object (Φ^∞) is well approximated by the signed measures given by the sum of the area measures (signed according to their spin σ_C) of the "macroscopic" FK clusters (say of diameters larger than ϵ). Two important ingredients in this proof are the RSW theorem for FK-Ising percolation from [DHN11] as well as the convergence of exploration paths of FK percolation to $\mathrm{SLE}_{16/3}$ from [CD+13]. The drawback of this approach is that we need to rely on the uniqueness of the full scaling limit of FK percolation (see Assumption 5.1 which is the analog of Theorem 2.2 for critical percolation).

2. Our second proof, as opposed to the first one, does not rely on any assumption. For any bounded domain Ω, the idea is to characterize the limit of Φ^a by showing that

the quantities

$$\phi_{\Phi^a}(f) = \mathbb{E}\left[e^{i\langle \Phi^a, f\rangle}\right],$$

converge as $a \searrow 0$ for any test function $f \in \mathcal{H}^3$. The main ingredients are the breakthrough results by Chelkak, Hongler and Izyurov in [CHI12] on the convergence of the k-point correlation functions together with a local control of these k-point functions and a control of the **exponential moments** of the total magnetization in Ω, both given in [**G10**].

See Section 3 in Chapter 5 where these two approaches are explained in more details.

— ***Properties of the field*** Φ^∞ —

Once we obtain such a limiting field Φ^∞, it is natural to study its properties. The first natural guess which comes to mind is that, as in the Central Limit Theorem, Φ^∞ might be a *Gaussian field*. This is indeed the case when $\beta \neq \beta_c$ as suggested by the Gaussian limit in equation (1.8). More precisely, when $\beta \neq \beta_c$, the magnetization field (properly rescaled) converges to a **two-dimensional Gaussian white noise**. Nevertheless, as we will see below, the above random field Φ^∞ is non-Gaussian! This is why one is motivated in studying how it behaves. The following properties of Φ^∞ are established in [**G10**, **G12**]:

1. A first natural direction is to study the **tail behavior** of Φ^∞. More precisely, if one considers the total magnetization $m^\infty = m^\infty_\Omega := \langle \Phi^\infty, 1_\Omega\rangle$, it is proved in [**G12**] that the tail probabilities of m^∞ behave like $\exp(-c\,x^{16})$. See Theorem 5.3 for a precise statement. In particular, one sees here that Φ^∞ cannot be Gaussian. There is also another way to see why Φ^∞ is non-Gaussian: the k-point correlation function from [CHI12] do not satisfy Wick's formula.

2. In [**G10**], we establish that Φ^∞ is **conformally covariant** under the action of conformal maps. See Theorem 5.1.

3. Finally, one may wish to find explicit density functions for the random variables $m^\infty = \langle \Phi^\infty, 1_\Omega\rangle$. We did not succeed in finding such explicit formulas. In fact, even the fact that the m^∞ should be absolutely continuous w.r.t the Lebesgue measure is not easy. In [**G12**], we prove not only that m^∞ is absolutely continuous but that its density function is very regular (an entire function on $\mathbb{C}$). This is done by studying the Fourier transform of m^∞. See Section 7 and Theorem 5.5.

Sections 5, 4 and 7 in Chapter 5 will give more details on these three properties satisfied by the field Φ^∞.

4. — Near-critical Ising model (Chapter 6)

In Chapter 6, we will be interested in the **near-critical behavior** of the Ising model when perturbed away from its critical point along two distinct directions: first by changing slightly the temperature in [**G9**] and then by adding some small exterior magnetic field in [**G11**, **G12**]. We will also mention a related work in preparation [**G16**] about critical dynamics of FK percolation.

4.1. — Model of FK percolation

Let us start by introducing the celebrated FK percolation model which generalizes the (standard) model of percolation by adding some dependency structure between edges as follows:

Definition 1.6 (FK percolation). *Let $G = (V, E)$ be a finite graph. The* **FK percolation** *or* **random-cluster model** *on G with parameters $p \in [0,1]$ and $q \geq 1$ is a probability measure on the subgraphs of $G = (V, E)$, defined for all $\omega \subset E$ by*

$$\mathbb{P}_{p,q}[\omega] := \frac{p^{\#\text{ open edges}}(1-p)^{\#\text{ closed edges}} q^{\#\text{ clusters}}}{Z_{p,q}}, \tag{1.12}$$

where $Z_{p,q}$ is the normalization constant such that $\mathbb{P}_{p,q}$ is a probability measure.

Notice that the case $q = 1$ corresponds to (standard) bond percolation. This model would also make sense for $q \in (0,1)$, but the very useful FKG inequality would no longer hold in this case. See [Gri06]. As we shall see below, the case $q = 2$ corresponds to the Ising model (introduced in Definition 1.4).

Even though the random-cluster model can be defined on any graph, we will restrict ourselves to the case of the square lattice $\mathbb{Z}^2$. Infinite volume measures in this case can be constructed using limits of the above measures along exhaustions by finite subsets Λ with different possible boundary conditions on $\partial\Lambda$: free, wired, etc. See [Gri06]. Random-cluster models exhibit a phase transition at some critical parameter $p_c = p_c(q)$. On $\mathbb{Z}^2$, this value does not depend on which infinite volume limit we are using, and, as in standard percolation, below this threshold, clusters are almost surely finite, while above this threshold, there exists (a.s.) a unique infinite cluster.

As claimed above, FK percolation with $q = 2$ corresponds to the Ising model (as such it is also called the **FK-Ising** percolation). Let us illustrate this by briefly explaining how to sample an Ising model in a finite domain $\Lambda \subset \mathbb{Z}^2$ with free boundary conditions out of an FK-Ising percolation ($q = 2$) in Λ with free b.c. (See [Gri06] for the case of an Ising model with $+$ or $-$ b.c. which is then related to an FK percolation with wired b.c.).

1. Sample an FK-Ising configuration ω in Λ with free b.c.

2. Independently for each cluster (connected component) C of ω, sample an unbiaised spin $\sigma_C \in \{\pm\}$ and for each vertex $x \in C$, declare $\sigma_x := \sigma_C$. The resulting configuration $\{\sigma_x\}_{x\in\Lambda}$ has the desired distribution.

There exists also an inverse procedure (i.e. from Ising to FK). Both procedures are part of the so-called **Edwards-Sokal coupling** where the parameter p of the FK-Ising percolation is related to the inverse temperature β of the Ising model as follows:

$$p = 1 - e^{-2\beta}. \tag{1.13}$$

As we have already seen, the critical parameter is known to be equal to $1/2$ for bond percolation on the square lattice ($q = 1$). For the FK-Ising percolation ($q = 2$), $p_c(2) = \frac{\sqrt{2}}{1+\sqrt{2}}$ is known since Onsager [On44] via the Edwards-Sokal coupling. See also the recent [BD12a] for an alternative proof of this fact. More recently, the general equality $p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ was proved for every $q \geq 1$ in [BD12b].

The above close relationship between Ising and FK-Ising percolation will allow us to understand the near-critical behavior of the Ising model when $\beta = \beta_c + \delta\beta$ by studying instead the near-critical geometry of the FK-Ising percolation when $p = p_c(q = 2) + \delta p$.

Finally, let us point out that similarly as in the case $q = 1$ where Smirnov proved the conformal invariance of critical site-percolation on $\mathbb{T}$ (see Theorem 1.2 above from [Sm01]), Smirnov also proved some years later a conformal invariance Theorem for critical FK-Ising percolation ($q = 2$) on $\mathbb{Z}^2$ in [Sm10]. His approach in [Sm10] is very different from the approach he used in [Sm01] for the case $q = 1$ and relies on the so-called **fermionic** or **Smirnov** observable which we be introduced later (see Definition 6.1).

4.2. — Correlation length for FK percolation

We already encountered the notion of **correlation length** in the description of Chapter 2. We did not need a precise and quantitative definition at the time but we will need one here in order to state our main result. The intuitive idea is to define for each $p > p_c$ a scale $L = L(p)$ above which the infinite cluster starts being very visible. One standard way as in the definition below is to rely on crossing events of long rectangles and to detect a scale above which they start being easily traversed.

Definition 1.7 (Correlation length). *Fix $\rho > 0$. For any $n \geq 0$, let R_n be the rectangle $[0, \rho n] \times [0, n]$. If $p > p_c$, then define for all $\epsilon > 0$ and all "boundary conditions" ξ around R_n,*

$$L^{\xi}_{\rho,\epsilon}(p) := \inf_{n>0} \left\{ \mathbb{P}^{\xi}_{p}\big[\text{there is a left-right crossing in } R_n\big] > 1 - \epsilon \right\}$$

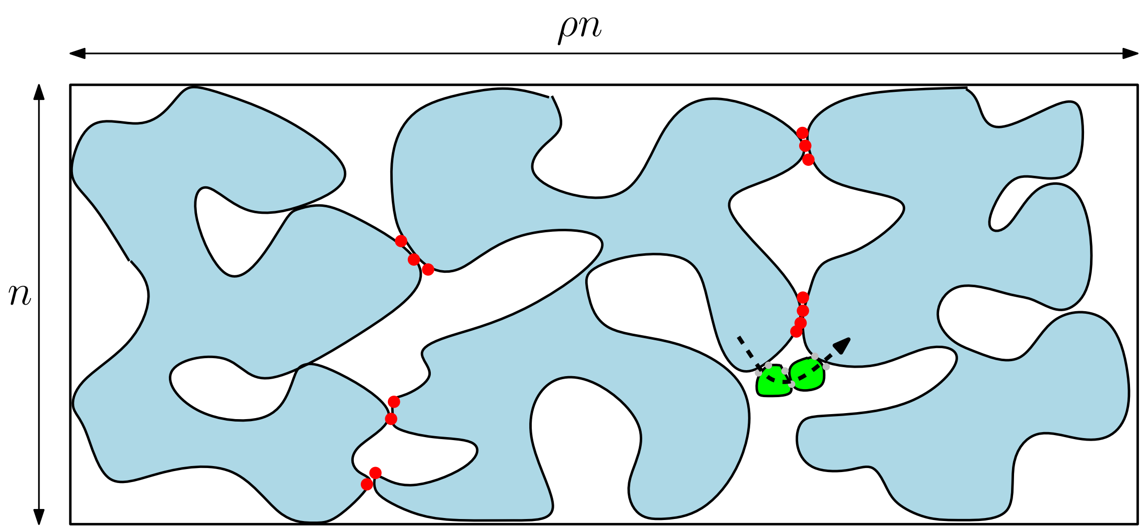


Figure 1.6:

Since the behavior of FK percolation in a finite domain highly depends on the chosen boundary conditions, the above correlation length depends on the choice of b.c. ξ around the rectangles R_n. By monotony (FKG), it is easy to check that for any $p > p_c(2)$ and any parameters $\rho, \epsilon > 0$:

$$L^{\text{free}}_{\rho,\epsilon}(p) \geq L^{\text{wired}}_{\rho,\epsilon}(p) \tag{1.14}$$

4.3. — An unexpected phenomenon

In order to understand the near-critical behavior of (standard $q = 1$) percolation, as we have already seen earlier, **pivotal points** play a very important role since the earlier work of Kesten (in particular [Ke87]). Looking at Figure 1.6, the intuition is that in the rectangle R_n, there are about $n^2\alpha_4(n)$ pivotal points between large (diameter $\geq n$ clusters). In particular, if $\delta p\, n^2\alpha_4(n) = (p-p_c)n^2\alpha_4(n) \gg 1$, then it seems plausible that large clusters should typically be well connected and the scale n should be above the correlation length. This is what Kesten proves in [Ke87] which leads to his very important "scaling relation":

$$L(p)^2\alpha_4(L(p)) \asymp \frac{1}{|p-p_c|}\,. \tag{1.15}$$

To our knowledge, it was widely believed so far that this scaling relation should apply in great generality. Namely that the correlation length of most $2d$ statistical physics models should be driven by the amount of pivotal points in the critical regime.

At least, this is what we believed initially. I thus studied with H. Duminil-Copin in [DG] the *four-arms* event of critical FK percolation (with $q=2$) and established that

$$\alpha_4^{\mathrm{FK}}(R) = R^{-35/24+o(1)}\,. \tag{1.16}$$

Plugging this new **critical exponent** into the scaling relation (1.15) would in principle lead to

$$L^{\xi}_{\rho,\epsilon} \approx \left|\frac{1}{p-p_c}\right|^{24/13}\,,$$

which is in contradiction with the exact near-critical formula for the Ising model obtained already by Onsager in [On44]. See the discussion in [**G9**]. This contradiction was quite of a "shock" for us. Either our computation of α_4^{FK} in [DG] was wrong or the mechanism relating pivotal points and correlation length was very different in the case of FK-Ising percolation. As we will see in Chapter 6, we realized that it is the second option, i.e. that the scaling relation (1.15) is in fact very particular to the case of standard percolation ($q=1$) and is not true in general. By trying to understand what is the mechanism governing the correlation length for the FK-Ising percolation, we found out in [**G9**] an interesting **self-organized mechanism** which we will highlight in Chapter 6.

4.4. — Main result on the near-critical FK percolation ($q=2$)

As we have seen above, the mechanism which relates pivotal points and correlation length turns out to be extremely complicated in the case of FK percolation with $q>1$. In particular, it seems very difficult to prove any result about the correlation lengths $L^{\xi}_{\rho,\epsilon}(p)$ introduced in Definition 1.7 by using the fact that $\alpha_4^{\mathrm{FK}}(R) = R^{-35/24+o(1)}$. In [**G9**], we thus follow a completely different route than Kesten in [Ke87] in order to obtain the following analogous result for FK-Ising percolation:

Theorem 1.15 ([G9]). *Fix $q=2$. For every $\epsilon,\rho>0$, there is a constant $c=c(\epsilon,\rho)>0$ s.t.*

$$c\,\frac{1}{|p-p_c|} \leq L^{\xi}_{\rho,\epsilon}(p) \leq c^{-1}\frac{1}{|p-p_c|}\,\log\frac{1}{|p-p_c|}$$

for all $p\neq p_c$, whatever the choice of the boundary condition ξ is.

Remark 1.12. Note that this result is consistent with Onsager's celebrated formula

$$\langle \sigma_0 \rangle_{\beta,\mathbb{Z}^2,+} \asymp (\beta - \beta_c)_+^{1/8} ,$$

but cannot be derived from it.

4.5. — A more classical use of pivotal points: heat-bath FK dynamics

As we have seen at lengths in the presentation of Chapter 2, pivotal points play a similar role in both near-critical and dynamical percolation. When $q > 1$, we have just seen that understanding the near-critical behavior of FK percolation is not so related after all with pivotal points. Interestingly, we will see in Chapter 6 that for natural dynamics which preserve the critical FK measure, pivotal points still dictate somehow the behavior of the dynamics. This will be the content of our work in progress [**G16**]. In this work, we also discovered another unexpected phenomenon: there exists planar statistical physics models with pivotal points at all scales but which are NOT **dynamically sensitive**, i.e. they do not have exceptional times with an infinite cluster along a natural measure preserving dynamics as standard percolation does. As we will see in Chapter 6, we conjecture that this is the case in particular for critical FK percolation with $q > q^* := 4\cos^2(\frac{\pi}{4}\sqrt{14}) \approx 3.83$. These are the first natural models that are expected to be noise sensitive but not dynamically sensitive.

4.6. — Near-critical Ising model along the h-direction

Finally in the works [**G11**] and (part of) [**G12**], we are interested in the Ising model at the critical inverse temperature β_c with a small magnetic field $h > 0$. Our main result in [**G11**] is an analog of the following celebrated near-critical results:

1. For site percolation on the triangular lattice, one has ([Ke87, SW01]):

$$\theta(p) := \mathbb{P}\big[0 \leftrightarrow \infty\big] = |p - p_c|^{5/36+o(1)} \quad \text{as } p \searrow p_c$$

2. For the Ising model on $\mathbb{Z}^2$, one has ([On44]):

$$\langle \sigma_0 \rangle_\beta^+ \asymp |\beta - \beta_c|^{1/8} \quad \text{as } \beta \searrow \beta_c .$$

Our analogous result in [**G11**] may be stated as follows:

Theorem 1.16 ([G11]). *Consider the Ising model on $\mathbb{Z}^2$ at β_c with a positive external magnetic field $h > 0$, then*

$$\langle \sigma_0 \rangle_{\beta_c,h} \asymp h^{\frac{1}{15}} .$$

As we will see in Chapter 6, the proof of this result is surprisingly simpler than the above two near-critical results.

Our main result in [**G12**] is of a different spirit. Indeed, we focus in part of [**G12**] on **massive scaling limits** as the mesh and the exterior magnetic field tend to zero in the appropriate scaling.

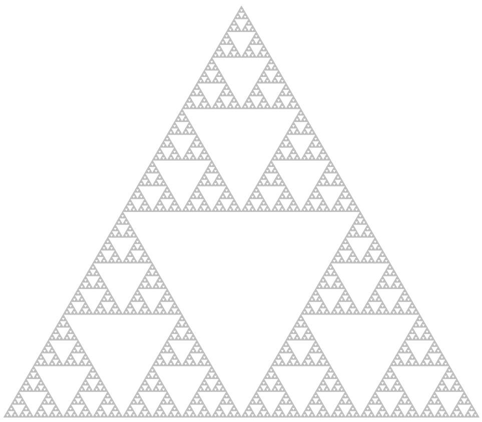
Theorem 1.17 ([G12]). *Fix some $h > 0$. Consider the Ising model on $a\mathbb{Z}^2$ at $\beta = \beta_c$ and with vanishingly small external magnetic field equal to $a^{15/8}h$. Let $\Phi^{a,h}$ be the near-critical magnetization field in the plane defined, as in [G10] (where $h = 0$), by*

$$\Phi^{a,h} := \sum_{x \in a\mathbb{Z}^2} \delta_x \, \sigma_x \, a^{15/8} \,,$$

where $\{\sigma_x\}_{x\in a\mathbb{Z}^2}$ is a realization of the above Ising model with external magnetic field equal to $h\,a^{15/8}$. Then, as the mesh $a \searrow 0$, the random distribution $\Phi^{a,h}$ converges in law to a near-critical field $\Phi^{\infty,h}$ under the topology of $\mathcal{H}^{-3}$ in the full plane defined in Section A.2 of [G10].

This proof of this massive scaling limit is straightforward in a smooth bounded domain by using the fact the magnetization field Φ^∞ has exponential moments. (See Proposition 5.6). More work is needed to extend it to the full plane. See [**G12**] where a coupling using the so-called **ghost vertex** is used.

5. — Coalescing flows of Brownian motions: a new perspective (Chapter 7)



The coalescing Brownian flow on $\mathbb{R}$ is a process which was introduced by Arratia [Arr79, Arr81] and Tóth and Werner [TW98], and which intuitively corresponds to starting coalescing Brownian motions from "every" space-time point. Already on the real-line $\mathbb{R}$, it is not a priori easy to turn this informal description into a rigorous mathematical object. This was the purpose of a series of papers by Fontes et al. [FINR04] where they construct the so-called **Brownian web** as a random variable in a certain metric space and they study the convergence of discrete coalescing flows towards it. Now imagine one wants to study a coalescing flow of Brownian motions on a more complex structure such as the **Sierpinski gasket** G pictured above. See for example [Bar98, BP88, Sa97] for the existence and uniqueness of Brownian motions on such fractal sets. Informally, such a coalescing process would consist of particles that perform independent coalescing Brownian motions starting from every space-time point $(x, t) \in G \times \mathbb{R}_+$, where G is the (infinite of compact) Sierpinski gasket. The situation becomes much more complicated since in that case, the so-called **non-crossing property** does not hold. This property says that paths cannot cross without coalescing, an obvious topological fact in dimension one which underlies Arratia's original approach and much of the work on the subject.

A new space of coalescing flow. In order to handle this lack of monotony, we introduce in [**G15**] a new topological space for coalescing flows which is inspired from the **Schramm-Smirnov space** $\mathscr{H}$ (Definition 2.3). As such, Chapter 7 is intimately related to Chapter 2. We call this new metric space the **Schramm-Smirnov space of coalescing flows** or simply the **space of coalescing flows** and we will denote it by $(\mathscr{C}, d_{\mathscr{C}})$ to make a clear distinction with $(\mathscr{H}, d_{\mathscr{H}})$. The idea is roughly to view a coalescing flow ξ as the set all topological **tubes** (see Figure 7.1) which are traversed by at least one particle. More precisely, as in the Schramm-Smirnov space $\mathscr{H}$, one considers the space of closed

hereditary subsets of $\mathcal{T}$, the space of all tubes. See Section 1 (in particular Definition 7.2) where we give a more precise description of the setup used in [**G15**].

— ***Main results*** —

The advantage of this new setup is that it makes the proof of convergence of discrete objects to their limiting "Brownian" flow surprisingly simple. In particular, even though our original motivation was to handle coalescing flows on more complex structures than the real line, it turns out that even in the classical setting of coalescing flows on the real line $\mathbb{R}$, our approach implies new results (see below). We start by stating our main results for coalescing flows on $\mathbb{R}$ and then for coalescing flows on the Sierpinski Gasket G.

Coalescing flows on $\mathbb{R}$. We prove an invariance principle (Theorem 1.18 below) for the convergence of scaled coalescing random walks on $\mathbb{Z}$ towards Arratia's flow under an optimal finite variance assumption on the random walk. Note that in previous works, the topological setup was different and required a non-trivial proof already for tightness. In particular, in [BM+06], it is shown that in order to obtain a tightness criterion under [FINR04]'s setup, a $3+\epsilon$ finite moment is sufficient, while a $3-\epsilon$ is needed. We prove in our present setting that a finite variance is necessary and sufficient for the convergence to hold.

Before stating a proper invariance principle, we need to define a limiting object in $\mathscr{C}$. As we will see in Section 2, we construct a natural probability measure $\xi_\infty \sim \mathbb{P}_\infty$ on the space of coalescing flows $\mathscr{C}$ (See Theorem 7.2). It can be shown a posteriori that this object is the measurable image of the Brownian web of Fontes at al. As such, it may be called **Brownian web** as well.

Consider now a system of independent coalescing random walks started from every space-time point (x,t) on $\mathbb{Z}\times\mathbb{Z}$. We assume that the step distribution μ satisfies

$$\mathbb{E}[\mu]=0, \quad \mathbb{E}[\mu^2]=\sigma^2<\infty \ \text{ and } \mu \ \textbf{is aperiodic}. \tag{1.17}$$

Under diffusive scaling, this gives rise to a coalescing flow $\xi_\eta \sim \mathbb{P}_\eta$ on the rescaled lattice $\mathbb{L}_\eta := \sigma^{-1}\eta\mathbb{Z}\times\eta^2\mathbb{Z}$ and ξ_η naturally belongs to the space $\mathscr{C}$. Note hat in ξ_η, two paths can cross over each other several times before they finally merge at some point in $\mathbb{L}_\eta$. Our main result about coalescing flows on $\mathbb{R}$ can be stated as follows:

Theorem 1.18 (Theorem 4.1 in [G15]). *Assume (1.17). Then as $\eta\to 0$,*

$$\mathbb{P}_\eta \to \mathbb{P}_\infty\,,$$

weakly in $\mathscr{C}$, where $\mathbb{P}_\infty$ is the law of the coalescing Brownian flow on $\mathscr{C}$, as defined in Theorem 7.2.

Coalescing flow on the Sierpinski gasket. Let us now turn to our initial motivation. In what follows and in Chapter 7, we will denote by G, either the usual Sierpinski gasket or the infinite one (i.e. $G := \bigcup 2^n G_0$). See Section 5.1 in [**G15**] for precise definitions. As in the case of Brownian flows on $\mathbb{R}$, the first step is to define a natural limiting flow on G. This will be the purpose of Theorem 7.4 where we build a natural coalescing flow of Brownian motions on G denoted by $\xi_\infty^G \sim \mathbb{P}_\infty^G$. Due to the flexibility provided by the space $(\mathscr{C}, d_{\mathscr{C}})$, this step is not much more complicated than in the case of the real-line (in particular the **non-crossing** property is not needed).

Then, we prove an invariance principle where the discrete model may be defined as follows. Let $\eta=2^{-n}$ and consider a **discrete** Sierpinski gasket G_n with mesh size 2^{-n}.

(See again Section 5 in [**G15**] or also Figure 7.5). We consider coalescing random walks on G_n defined as follows: initially there is a particle at every vertex of G_n. These perform independent simple random walks, jumping every 5^{-n} units of time, and coalesce when they are on the same vertex of G_n. (Note that these random walks are aperiodic). Consider the law $\mathbb{P}_\eta^G$ on $\mathscr{C}$ that this coalescing system induces. We prove the following invariance principle:

Theorem 1.19 (Invariance principle, [G15]). *As $\eta \to 0$,*

$$\mathbb{P}_\eta^G \to \mathbb{P}_\infty^G \,,$$

weakly in $(\mathscr{C}, d_{\mathscr{C}})$, where $\mathbb{P}_\infty^G$ is the law of the coalescing Brownian flow on G, as defined in Theorem 7.4.

We will briefly explain how one proves such results along Chapter 7. One of the main tools besides the space $(\mathscr{C}, d_{\mathscr{C}})$ is to prove a **uniform coming down from infinity** result for the discrete coalescing systems. See Proposition 7.2 whose proof is inspired from [EMS09, ABL12].

— *Black noise* —

The Brownian web was the first example of a one-dimensional **black noise** in the sense of Tsirelson (see [Tsi04]). The first proof of this appeared in [LJR04b]. See also [Tsi04] for a different proof. In [**G17**], we prove that the Brownian flow on the Sierpinski gasket $\xi_\infty^G \sim \mathbb{P}_\infty^G$ gives a new example of a **black noise**. (Note that the scaling limit of critical percolation $\omega_\infty \sim \mathbb{P}_\infty$ defined in Theorem 2.2 is also a Black noise as shown in [SchSm11]; in fact it is the only known two-dimensional black noise). To prove that the noise generated by the flow ξ_∞^G is black, one needs to prove a **noise sensitivity** result. The proof we use is different from the proofs in the one-dimensional case and relies on an **approximate randomized algorithm** for the discrete coalescing system. See Section 5. This technique was invented in [SchSt10] and is used for the first time in [**G17**].

6. — Liouville Brownian motion in $2d$ quantum gravity (Chapter 8)

Let $\mathcal{T}_n$ be the set of all triangulations of the two-dimensional sphere $\mathbb{S}^2$ with n faces viewed up to orientation preserving homeomorphisms of $\mathbb{S}^2$ (i.e. up to "deformations"). As such $\mathcal{T}_n$ is a finite set and one can consider uniformly chosen triangulations $T = T_n$ in $\mathcal{T}_n$. See Figure 1.7 for an illustration. These random triangulations T_n are a particular instance of the so-called **planar maps**. See for example [LGM11]. Over the last years, planar maps have been studied intensively. For example, in the recent breakthrough works [LG13, Mie13], an invariance principle for the rescaled maps $n^{-1/4}T_n$ (viewed as random metric spaces in the Gromov-Hausdorff space) towards a limiting object called the **Brownian map** is proved [1].

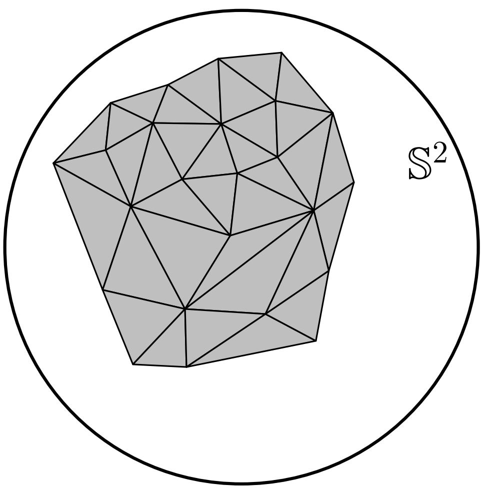
One of the main motivations behind this intense study of random planar maps goes back to the foundational work by Knizhnik, Polyakov and Zamolodchikov (KPZ) in [KPZ88]. In this paper, they came up with a novel and far-reaching approach in order to understand the critical behavior of many **two-dimensional statistical physics models**, including random walks, percolation as well as the Ising model. The main underlying idea of their approach is to study these models along the following two-step procedure:

[1] The proof in [Mie13] is restricted to random quadrangulations Q_n

(i) First, instead of considering the model on some regular lattice of the plane (such as $\mathbb{Z}^2$ for example), one defines it one some well-chosen "random planar lattice". Doing so corresponds to studying the model in its **quantum gravity** form. In the case of percolation, the appropriate choice of random lattice matches with the above uniform **planar maps**, see [LGM11].

(ii) Then it remains to get back to the actual **Euclidean** setup. This is done thanks to their celebrated **KPZ formula** ([KPZ88]) which gives a very precise correspondence between the geometric properties of models in their quantum gravity formulation and their analogs in the Euclidean case.

The nature and the origin of such a powerful correspondence remained rather mysterious for a long time and is still currently far from being fully understood. The main difficulty lies in the fact that the approach from [KPZ88] connects statistical physics models which leave on very different lattices: random ones on one side, and euclidean ones on the other. The recent work [DS11] sheds a lot of light on this puzzling **KPZ relation**: based on the idea that critical statistical physics models are in general "conformally invariant", they suggest to conformally map (using either Riemann's uniformization Theorem or a more combinatorial uniformization based on circle packings) the random planar lattices in the Euclidean plane. This way one obtains a random Riemannian structure on the plane. The analysis in [DS11] leads them to a very convincing conjecture on what this random Riemannian structure should look like as the size of the random lattices go to infinity. More precisely, for each **universality class** (parametrized by a one real parameter: either the **central charge** c in physics, or the κ parameter in SLE processes or the γ parameter in [DS11]), they identify a certain random measure on the plane, the so-called **Liouville measure** M_γ for which they prove a form of the KPZ formula: for each deterministic set X of the plane, they prove that the "size" of X measured according to the Lebesgue measure is a.s. related via the KPZ formula to the "size" of X measured according to the Liouville measure M_γ. Note that their KPZ relation in not "metric". (See also [BS09, RV11] for similar KPZ relations). In fact it remains a big open problem in the area to prove that one can extract a meaningful "quantum metric" out of the Liouville measures M_γ. Let us give two special cases:



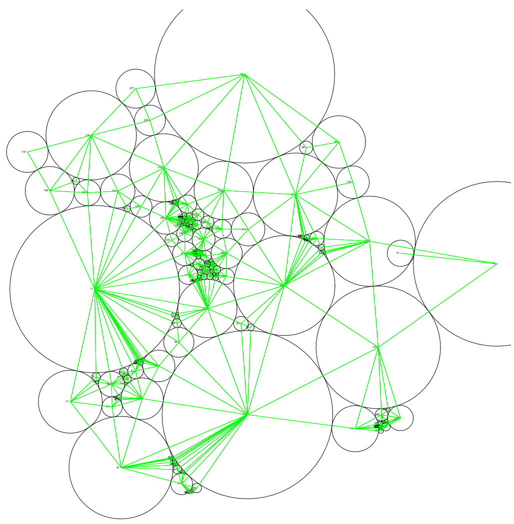


Figure 1.7: On the left, we represented a portion of a "uniform" triangulation of the sphere T_n. On the right, a simulation by M. Krikun of a circle packing of a uniform triangulation.

1. If one is interested in critical percolation or in the model of **self-avoiding walks**, the natural model of random lattices on the quantum gravity side is the model of uniform **planar maps** such as the above uniform triangulation T_n in $\mathcal{T}_n$. Following [DS11], if one "uniformizes" the random triangulations T_n in the sphere, for example using Köbe Theorem on circle packings as in Figure 1.7, then the pushforward of the measure which assigns a weight $1/n$ on each triangle of T_n should converge as $n \to \infty$ to the following **Liouville measure**

$$“M_{\sqrt{8/3}}(dz) = e^{\sqrt{8/3}X(z)}\, dz” ,$$

where X is a **Gaussian Free Field** on the sphere $\mathbb{S}^2$ with vanishing mean. See [Ben10, Sh10] for precise conjectures which take into account the 3 remaining degrees of freedom given by the Möbius transformations.

2. If one is interested in the critical Ising model, then the model of discrete random lattice is different (far from "uniform") and their "uniformized uniform measures" should converge this time ([DS11]) to

$$“M_{\sqrt{3}}(dz) = e^{\sqrt{3}X(z)}\, dz” .$$

See our survey [**G19**] for more on these topics.

From the above discussion, for each parameter $\gamma \leq \gamma_c = 2$ (corresponding for the more SLE inclined readers to $\kappa \equiv \gamma^2 \in [0,4]$), it is thus natural to try to equip the sphere $\mathbb{S}^2$ with the following natural Riemannian structures:

(A) a **volume form**: "$e^{\gamma X}\, dxdy$", where X is a Gaussian Free Field.

(B) a **metric tensor**: "$e^{\gamma X}(dx^2 + dy^2)$".

(C) a **Brownian motion**: "$dB_t = e^{-\frac{\gamma}{2}X} dW_t$".

Recall that the **Gaussian Free Field (GFF)** X with Dirichlet boundary conditions in a domain Ω informally corresponds to a Gaussian process indexed by the points $x \in \Omega$ such that "$\mathrm{Cov}\big[X(x), X(y)\big] := G_\Omega(x,y)$", the **Green function** in the domain Ω. This is of course very informal since it would imply an infinite variance at each point $x \in \Omega$ ($G_D(x,x) = \infty$). Yet, there are many ways to turn this informal description into a well-defined object, see [Sh07, **G19**]. On the sphere (where there is no boundary), the GFF may be defined as follows:

$$X(x) = \sum_{n\geq 1} a_n \frac{1}{\sqrt{\lambda_n}} \phi_n(x) , \tag{1.18}$$

where $\{\phi_n\}_n$ is an orthornormal basis of (non-constant) eigenfunctions of the Laplacian with eigenvalues $\lambda_n \neq 0$ and where $\{a_n\}_n$ is a family of independent Gaussian random variables $\mathcal{N}(0,1)$. Furthermore this series expansion makes sense only in a Sobolev space of negative index $\mathcal{H}^{-\epsilon}$, $\epsilon > 0$.

It thus appears that the Gaussian Free Field is far from being a smooth function: in particular all the structures listed above, (A) to (C) need to be defined carefully through some **regularization procedure**. The volume form in Item (A) has been constructed in [DS11] but follows also from the theory of the **multiplicative Chaos** developed by Kahane [Kah85]. See [RV10, RV13]. Item (B) has not been constructed so far and as mentioned above, this is one of the main conjectures in the field. The purpose of our work [**G13**] is to build a natural Liouville diffusion corresponding to Item (C) with the hope that it may shed some light on the conjectural Item (B). As we shall see below one of the difficulties lies in the fact that the SDE in Item (C) does not have strong solutions.

— *Main results and strategy of proof* —

Our main result in [G13] may be stated as follows:

Theorem 1.20 ([G13]). *For any parameter $\gamma < \gamma_c = 2$, given a* **Gaussian Free Field** *X with vanishing mean on $\mathbb{S}^2$, one can define a* **semi-group** *$(P_t^X)_{t\geq 0}$ on $\mathbb{S}^2$ such that a.s. in the realisation of X, one has*

(i) $(P_t^X)_{t\geq 0}$ *is a* **Feller** *process on $\mathbb{S}^2$.*

(ii) (P_t^X) *is reversible w.r.t the* **Liouville measure** *M_γ.*

Remark 1.13. Note that the above diffusion from [G13] should correspond to the scaling limit of simple random walks on uniformized (or circle-packed) triangulations as represented on the right of Figure 1.7.

Let us now briefly explain how one proves such a result. Due to the degeneracy of the GFF X, one needs to rely on a regularization/renormalization procedure. Two different natural regularizations are used in the literature:

1. Either one integrates the GFF against the uniform measures $\nu_\epsilon(x)$ on the circle of radius ϵ around points x. I.e, $X_\epsilon(x) := \langle X, \nu_\epsilon(x)\rangle$. This is the approach used in [DS11].

2. Or in certain special cases, one can decompose the Gaussian process X into a sum of independent non-degenerate Gaussian processes Y_k whose covariance matrices K_k are positive and positive-definite. Namely, $X = \sum_{k\geq 1} Y_k$, with $K(x,y) = \sum_{k\geq 1} K_k(x,y)$. This allows us to approximate the field X using $X_n(x) := \sum_{k=1}^n Y_k(x)$. This is the approach used by Kahane [Kah85]. See also [RV10, RV13].

We will use the latter approach since it is more suitable to the use of martingales. Indeed, for any subset $A \subset \mathbb{S}^2$, note that for any $\gamma \geq 0$:

$$M_n(A) := \int_A e^{\gamma X_n(x) - \frac{\gamma^2}{2}\mathbb{E}\left[X_n(x)^2\right]} dx\,, \tag{1.19}$$

is a martingale (while it would not be the case with the corresponding $M_\epsilon(A)$ in [DS11]'s setup). Kahane proved in [Kah85] that this martingale is **uniformly integrable** if and only if $\gamma^2 < 4$. This explains the above critical parameter $\gamma_c = 2$ (which also corresponds to the critical $\kappa = 4$ in SLE$_\kappa$ processes). When $\gamma < \gamma_c$, one obtains this way and as $n \to \infty$ the above **Liouville measure** M_γ.

Inspired by the above exact martingale, it is thus natural to regularize the SDE in item (C), i.e. "$dB_t = e^{-\frac{\gamma}{2}X} dW_t$" as follows.

Definition 1.8. *Let $(\bar{B}_t)_{t\geq 0}$ be a fixed* **driving Brownian motion**. *For each $n \geq 1$ and each starting point $x \in \mathbb{S}^2$, define the* **n-regularized Liouville Brownian motion** *as follows:*

$$\begin{cases} \mathcal{B}_{t=0}^{n,x} = x \\ d\mathcal{B}_t^{n,x} = e^{-\frac{\gamma}{2}X_n(\mathcal{B}_t^{n,x}) + \frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}_t^{n,x})^2]}\, d\bar{B}_t. \end{cases} \tag{1.20}$$

Briefly the proof of our main Theorem 1.20 is divided into the following steps:

1. First one analyzes what happens starting from a **fixed** point $x \in \mathbb{S}^2$ (in particular x may not depend on the realization of the GFF X). In this case, one proves that a.s. in the realization of the GFF X, the processes $\mathcal{B}^{n,x}$ weakly converge (under the uniform topology on compacts sets) to a process $(\mathcal{B}^x_t)_{t\geq 0}$ whose law is measurable w.r.t the GFF X. Note that this Liouville Brownian motion starting form one point was also introduced independently in [Ber13]. As we will explain in Section 1, constructing this process boils down to a certain **time-change** of a standard two-dimensional Brownian motion and earlier results from Kahane allow us to control this time-change. Interestingly, we show in [**G13**] that the limiting process $(\mathcal{B}^x_t)_{t\geq 0}$ is necessarily independent of the driving Brownian motion $\bar{B}_t$ thus explaining why there are no strong solutions for the SDE (C). See Theorem 8.1 (in particular item (iii)) for a precise statement. In a way, this can be interpreted as a creation of randomness by strongly pinching the Brownian curve $\bar{B}$ in order to create a new randomness $\mathcal{B}^x$ independent of $\bar{B}$.

2. Then, the main part of the proof consists in defining a proper **semi-group**, i.e. to be able to start the Liouville Brownian motion simultaneously from all points $x \in \mathbb{S}^2$. In particular, this requires to understand what happens when one starts from atypical points of the free field X:

   - If one starts from a point where the field X is unusually "high", then it could be that the Liouville Brownian motion would remain "stuck" there (this would correspond in Picture 1.7 to a simple random walk starting from a very dense area).
   - On the other hand, if one starts from a point where the field X is unusually "low", then the Liouville Brownian motion might immediately blow up.

   We will say a few words in Section 2 on how to control the Liouville Brownian motion "uniformly" in $x \in \mathbb{S}^2$.

3. Finally, we prove in [**G13**] that the semi-group thus obtained is **Feller** and preserves the **Liouville measure** M_γ.

Initially, we were hoping to extract the **Liouville quantum metric** (item (B) above) out of our Liouville semi-group. The plan was that the Liouville semi-group should lead to an interesting **Dirichlet form** which in turn should be associated to a natural metric using the theory developed for example in [Sto10, Stu98]. But we realized in [**G14**] that this program did not quite work. Yet, using the classical results from Fuskushima [FOT94], we could identify the explicit Dirichlet form of the process and with some additional work, we proved in [**G14**] the existence of a **Liouville heat kernel** associated to (P^X_t). See Section 3 in Chapter 8.

Chapter 2

Scaling limit of near-critical percolation in the plane

Based on joint works with
Oded Schramm and Gábor Pete

In this Chapter, I will give a glimpse of the techniques elaborated in [G4, G5] in order to prove that near-critical and dynamical percolation have a scaling limit (Theorems 1.3 and 1.4). Let us recall Definition 1.1 of the near-critical coupling (See Definition 1.2 for dynamical percolation):

Definition 2.1. *We define the* **near-critical coupling** $(\omega_\eta^{\mathsf{nc}}(\lambda))_{\lambda\in\mathbb{R}}$ *to be the following process:*

(i) Sample $\omega_\eta = \omega_\eta^{\mathsf{nc}}(\lambda = 0)$ according to $\mathbb{P}_\eta$, the law of critical percolation on $\eta\mathbb{T}$, the ***triangular lattice*** *with mesh η.*

(ii) As λ increases, closed (white) hexagons switch to open (black) at exponential rate $r(\eta)$, where

$$r(\eta) := \eta^2 \alpha_4^\eta(\eta, 1)^{-1} (= \eta^{3/4+o(1)}) .$$

(iii) As λ decreases, open (black) hexagons switch to closed (white) at rate $r(\eta)$.

As such, for any $\lambda \in \mathbb{R}$, the near-critical percolation $\omega_\eta^{\mathsf{nc}}(\lambda)$ corresponds exactly to a percolation configuration on the rescaled graph $\eta\mathbb{T}$ with parameter

$$\begin{cases} p = p_c + 1 - e^{-\lambda\, r(\eta)} & \text{if } \lambda \geq 0 \\ p = p_c - (1 - e^{-|\lambda|\, r(\eta)}) & \text{if } \lambda < 0 \,, \end{cases}$$

We are interested in the scaling limit of this monotone coupling $(\omega_\eta^{\mathsf{nc}}(\lambda))_{\lambda\in\mathbb{R}}$ as the mesh $\eta \searrow 0$. Since we are looking for a **convergence in law** as $\eta \to 0$, we need to precise what our topological setup is. This is the purpose of the next section.

Contents

1. — Topological framework: the Schramm-Smirnov space $\mathscr{H}$

We are in the common situation where one considers some random object on a discrete structure (here $\eta\mathbb{T}$) which in some sense shrinks to the continuum ($\eta \to 0$). The most well-known example of such a situation is the (interpolated) simple Random Walk $S_\eta(t)$ defined on the space-time product $\sqrt{\eta}\,\mathbb{Z} \times \eta\mathbb{N}$. In order to identify a convergence in law of these random walks to the limiting **Brownian motion** $B(t)$, a natural way to proceed is to view the discrete objects $\{S_\eta(t)\}_\eta$ as well as their limits in the same natural topological space which will then provide the functional setup for the convergence in law. In the case of these random walks, a good choice is the functional space $\mathcal{C}([0,1], \mathbb{R}, \|\cdot\|_\infty)$ or the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}_+$. Indeed, it is well known that the sequence $\{S_\eta(t)\}_\eta$ is **tight** in each of these spaces.

Going back to our critical percolation $\omega_\eta \sim \mathbb{P}_\eta$ on $\eta\mathbb{T}$, one would like to find a convenient topological space in which

1. $\{\omega_\eta\}_\eta$ would be tight.
2. And furthermore $\{\omega_\eta\}_\eta$ would converge to a limiting interesting object.

A minute of thoughts shows that it is not so simple to find an adequate space which would include discrete percolation configurations together with their "scaling limits". Indeed, as we shall see in the next subsection, many different topological spaces have been suggested in the past. The latter one, the **Schramm-Smirnov** space $\mathscr{H}$ introduced in [SchSm11], will be our choice.

Before listing these various approaches, let us discuss the first idea that naturally comes to mind: a critical percolation configuration ω_η on $\eta\mathbb{T}$ can be realized as an i.i.d black/white coloring of the plane by small hexagons. Viewed this way, one can think of ω_η as a discrete two-dimensional **white noise**. More precisely, let

$$X_\eta := \eta \sum_{x \in \eta\mathbb{T}} \sigma_x \delta_x \,,$$

where $\sigma_x := 1$ if the corresponding tile $x \in \eta\mathbb{T}$ is black and -1 otherwise. As such, X_η is a random distribution which belongs for example to the Sobolev space $\mathcal{H}^{-1-\epsilon}$. Furthermore it is a standard fact that $\{X_\eta\}_\eta$ is **tight** in $\mathcal{H}^{-1-\epsilon}$ and converges in law as $\eta \to 0$ to the **two-dimensional Gaussian white noise** X_∞. One might thus hope that the Gaussian process X_∞ is a natural candidate for the scaling limit of critical percolation. But, as it is suggested by the study of the **noise sensitivity** of percolation in [BKS99], a stable object such as X_∞ cannot possibly encode limits of "crossings events" in critical percolation. Another way to see why the embedding in $\mathcal{H}^{-1-\epsilon}$ is not the right point of view for encoding critical percolation is the fact that by changing a few pivotal tiles in ω_η, the large scale connectivity properties of ω_η might change drastically while the associated distribution $X_\eta \in \mathcal{H}^{-1-\epsilon}$ will remain almost unchanged.

1.1. — A brief history of the topological frameworks for percolation

Here is a list of different setups that have been designed in the past to encode planar percolation configurations. It is inspired from the exhaustive list provided in [SchSm11].

1. One might view a critical percolation configuration ω_η as a collection of compacts subsets of the compactified plane $\hat{\mathbb{C}} = \mathbb{R}^2 \cup \{\infty\}$. The space of such collections of compact sets may be endowed with some version of the Hausdorff topology. The main issue with this natural approach is that as $\eta \to 0$, it can be seen that some very important information is lost at the scaling limit: namely the color of "half" of the macroscopic pivotal points. See [SchSm11] for a discussion on this.

2. To overcome the above issue, Aizenman suggested in [Ai98] to view a percolation configuration ω_η as the collection of all (not necessarily simple) open paths which lie inside open clusters. This representation is now referred to as **Aizenman's web**. The relevant topology is also given by a version of Hausdorff's topology on the space of collections of continuous paths. A tightness criterion for this topological setup is proved by Aizenman and Burchard in [AB99]. Uniqueness of the scaling limit still remains to be proved in this functional setup.

3. In [CN06], Camia and Newman encode each percolation configuration ω as a collection of oriented loops $\{\gamma_i\}_i$ which represent the interfaces between primal and dual clusters. The topology is also based on a version of the Hausdorff topology. Based on [Sm01], they prove in this topological framework a scaling limit result as the mesh $\eta \searrow 0$.

4. A similar approach suggested by Sheffield in [Sh09] consists in encoding a percolation configuration using a **branching exploration tree**. This approach is not restricted to percolation and can be used for other statistical physics models such as FK percolation for example. This approach is well designed for the so-called **conformal loop ensembles** CLE_κ.

5. Some setups used successfully for other critical models may possibly be used also with percolation. For example one could try to encode a percolation configuration via an **height function** similarly as one does with **dimer models** and then pass to the limit as $\eta \searrow 0$. This point of view has not been investigated yet. It is not clear though, assuming a scaling limit exists in this setting, whether interfaces and connectivity properties would be *measurable* with respect to the limiting height function. I.e. one could be in the same situation described above where X_η converges in law to $X_\infty \in \mathcal{H}^{-1-\epsilon}$ but there is no "percolation information" left in X_∞.

6. Another very fruitful approach for example in the case of the Ising model is the use of **correlation functions** to describe scaling limits. See the recent work by Chelkak, Hongler, Izyurov [CHI12] on the scaling limit of correlation functions for the magnetization field of the critical Ising model. For critical percolation, correlation functions would correspond to the following n-point functions:

$$\phi(z_1, \ldots, z_n) := \lim_{\eta \to 0} \frac{1}{\alpha_1(\eta, 1)^n} \mathbb{P}_\eta\big[z_1^\eta, \ldots, z_n^\eta \text{ are all connected}\big] .$$

7. Finally, one has the **Schramm-Smirnov** topological space $(\mathscr{H}, \mathcal{T})$ introduced below.

1.2. — The Schramm-Smirnov topological space $\mathscr{H}$

The idea in [SchSm11] is in some sense to consider a percolation configuration ω_η as the set of all the "quads" that are crossed (or traversed) by the configuration ω_η. We start by defining precisely what we mean by a **quad**.

Definition 2.2 (The space of quads $\mathcal{Q}$). *A* **quad** *in the complex plane $\mathbb{C}$ can be considered as a homeomorphism Q from $[0,1]^2$ into $\mathbb{C}$. A* **crossing** *of a quad Q is a connected closed subset of $[Q] := Q([0,1]^2)$ that intersects both $\partial_1 Q := Q(\{0\} \times [0,1])$ and $\partial_3 Q := Q(\{1\} \times [0,1])$ (let us also define $\partial_2 Q := Q([0,1] \times \{0\})$ and $\partial_4 Q := Q([0,1] \times \{1\})$). The space of all quads is denoted by $\mathcal{Q}$ and is equipped with the following metric: $d_{\mathcal{Q}}(Q_1, Q_2) := \inf_\phi \sup_{z \in \partial[0,1]^2} |Q_1(z) - Q_2(\phi(z))|$, where the infimum is over all homeomorphisms $\phi : [0,1]^2 \to [0,1]^2$ which preserve the 4 corners of the square.*

From the point of view of crossings, there is a natural partial order on $\mathcal{Q}$: we write $Q_1 \leq Q_2$ if any crossing of Q_2 contains a crossing of Q_1. See Figure 2.1. A subset $S \subset \mathcal{Q}$ is called **hereditary** if whenever $Q \in S$ and $Q' \in \mathcal{Q}$ satisfies $Q' \leq Q$, we also have $Q' \in S$. The following definition is inspired by the Dedekind cuts.

Definition 2.3 (The Schramm-Smirnov space $\mathscr{H}$). *We define the* **Schramm-Smirnov space** $\mathscr{H}$ *to be the collection of all* ***closed*** *hereditary subsets of $\mathcal{Q}$.*

Now, notice that any discrete percolation configuration ω_η of mesh $\eta > 0$ can be viewed as a point in $\mathscr{H}$ in the following manner. Consider ω_η as a union of the topologically closed percolation-wise open hexagons in the plane. It thus naturally defines an element $S(\omega_\eta)$ of $\mathscr{H}$: the set of all quads for which ω_η contains a crossing. By a slight abuse of notation, we will still denote by ω_η the point in $\mathscr{H}$ corresponding to the configuration ω_η.

Since configurations ω_η in the complex plane $\mathbb{C}$ are now identified as points in the space $\mathscr{H}$, it follows that critical percolation induces a probability measure on $\mathscr{H}$, which will be denoted by $\mathbb{P}_\eta$.

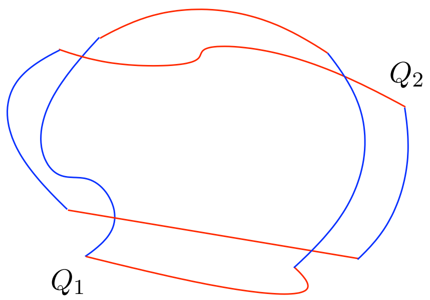


Figure 2.1: Two quads, $Q_1 \leq Q_2$.

In order to study the *scaling limit* of $\omega_\eta \sim \mathbb{P}_\eta$, a topology on the space $\mathscr{H}$ is introduced in [SchSm11] for which the measures $\mathbb{P}_\eta$ will converge weakly as $\eta \to 0$. It is called the **quad-crossing topology** and is similar to the **Fell topology** which defines a natural topology on the space $\mathcal{F}_X$ of all closed subsets of a topological space X: let us then consider the following subsets of $\mathscr{H}$:

(i) For any quad $Q \in \mathcal{Q}$, let

$$\boxminus_Q := \{\omega \in \mathscr{H} : Q \in \omega\} \tag{2.1}$$

(ii) For any open $U \subset \mathcal{Q}$, let

$$\boxdot_U := \{\omega \in \mathscr{H} : \omega \cap U = \emptyset\}\,. \tag{2.2}$$

It is easy to see that these subsets of $\mathscr{H}$ have to be considered closed if one wants $\mathscr{H}$ to be compact. This motivates the following definition from [SchSm11].

Definition 2.4 (The quad-crossing topology, [SchSm11]). *We define $\mathcal{T}$ to be the minimal topology on $\mathscr{H}$ that contains every $\boxminus_Q^c$ and $\boxdot_U^c$ as open sets.*

1.3. — Properties and metrizability of $(\mathscr{H}, \mathcal{T})$

The following theorem is proved in [SchSm11].

Theorem 2.1 (Theorem 3.10 in [SchSm11]). *The topological space $(\mathscr{H}, \mathcal{T})$ is compact, Hausdorff, and metrizable.*

Furthermore, for any dense $\mathcal{Q}_0 \subset \mathcal{Q}$, the events $\{\boxminus_Q : Q \in \mathcal{Q}_0\}$ generate the Borel σ-field of $\mathscr{H}$.

In particular, the space $\mathscr{H}$ is a Polish space.

This compactness property is very convenient since it implies readily the existence of subsequential scaling limits.

Furthermore, since the topological space $(\mathscr{H}, \mathcal{T})$ happens to be metrizable, it would be convenient to have at our disposal a natural and explicit metric on $\mathscr{H}$ which would induce the quad-crossing topology $\mathcal{T}$. The following one, $\tilde{d}_{\mathscr{H}}$, seems to be a good candidate since it is invariant under translations.

For any $\omega, \omega' \in \mathscr{H}$, define

$$\tilde{d}_{\mathscr{H}}(\omega, \omega') := \inf_{\epsilon>0 \text{ such that}} \left\{ \begin{array}{l} \forall Q \in \omega,\, \exists Q' \in \omega' \;\text{ with }\; d_{\mathcal{Q}}(Q, Q') < \epsilon \\ \text{and} \\ \forall Q' \in \omega',\, \exists Q \in \omega \;\text{ with }\; d_{\mathcal{Q}}(Q, Q') < \epsilon \end{array} \right\}.$$

As such, $(\mathscr{H}, \tilde{d}_{\mathscr{H}})$ is clearly a metric space. Unfortunately, it turns out that the topology on $\mathscr{H}$ induced by $\tilde{d}_{\mathscr{H}}$ is strictly finer than the topology $\mathcal{T}$.

We did not succeed in finding a simple and explicit metric compatible with the topology $\mathcal{T}$. (One possible way is to go through Urysohn's metrization theorem proof, but that does not lead to a nice and explicit metric). We thus relied in [G5] on some non-explicit metric $d_{\mathscr{H}}$ which induced the above topology $\mathcal{T}$:

Definition 2.5. *Let us fix once and for all a metric $d_{\mathscr{H}}$ on $\mathscr{H}$ which is such that it induces the topology $\mathcal{T}$ on $\mathscr{H}$. In particular, the space $(\mathscr{H}, d_{\mathscr{H}})$ is a compact metric space. It is also a Polish metric space. Since by compactness,* $\mathrm{diam}(\mathscr{H}) < \infty$, *we will assume without loss of generality that* $\mathrm{diam}_{d_{\mathscr{H}}}(\mathscr{H}) = 1$.

Since $d_{\mathscr{H}}$ is not explicit, we will need to find some explicit and quantitative criteria which will tell us whenever two configurations $\omega, \omega' \in \mathscr{H}$ are $d_{\mathscr{H}}$-close or not. This will be discussed in Subsection 1.7.

1.4. — The critical slice $\omega_\infty \sim \mathbb{P}_\infty$

This setup we just described allows us to think of $\omega_\eta \sim \mathbb{P}_\eta$ as a random point in the compact metric space $(\mathscr{H}, d_{\mathscr{H}})$. Now, since Borel probability measures on a compact metric space are always tight, we have subsequential scaling limits of $\mathbb{P}_\eta$ on $\mathscr{H}$, as the mesh $\eta_k \to 0$, denoted by $\mathbb{P}_\infty = \mathbb{P}_\infty(\{\eta_k\})$.

One of the main results proved in [SchSm11] is the fact that any subsequential scaling limit $\mathbb{P}_\infty$ is a **noise** in the sense of Tsirelson (see [Tsi04]). It is not proved in [SchSm11] that there is a **unique** such subsequential scaling limit but as it is explained along section 2.3 in [**G4**], the uniqueness property follows from the work [CN06]. More precisely, [CN06] proves the uniqueness of subsequential scaling limits in a different topological space than $(\mathscr{H}, d_{\mathscr{H}})$, but it follows from their proof that $\omega \in \mathscr{H}$ is *measurable* with respect to their notion of scaling limit (where a percolation configuration, instead of being seen as a collection of *quads*, is seen as a collection of nested *loops*). See [**G4**], section 2.3, for a more detailed discussion. In particular one has:

Theorem 2.2 ([Sm01, CN06, G4]). *Critical site percolation on $\eta\mathbb{T}$, $\omega_\eta \sim \mathbb{P}_\eta$ converges in law as $\eta \searrow 0$ to a continuum percolation $\omega_\infty \sim \mathbb{P}_\infty$. The convergence in law holds in the space $(\mathscr{H}, d_{\mathscr{H}})$.*

As explained carefully in [SchSm11, **G4**], the choice of the space $\mathscr{H}$ (or any other setup for the scaling limit) already poses restrictions on what events one can work with. Note, for instance, that $\mathcal{A} := \{\exists$ neighborhood U of the origin $0 \in \mathbb{C}$ s.t. all quads $Q \subset U$ are crossed$\}$ is clearly in the Borel σ-field of $(\mathscr{H}, \mathcal{T})$, and it is easy to see that $\mathbb{P}_\infty[\mathcal{A}] = 0$, but if the sequence of η-lattices is such that 0 is always the center of an hexagonal tile, then $\mathbb{P}_\eta[\mathcal{A}] = 1/2$.

With such an example in mind, it is natural to wonder how to effectively measure crossing events, multi-arms events and so on. Since the crossing event $\boxminus_Q$ is a Borel set, it is measurable and $\mathbb{P}_\infty[\boxminus_Q]$ is thus well-defined. Yet, one still has to check that

$$\mathbb{P}_\eta[\boxminus_Q] \to \mathbb{P}_\infty[\boxminus_Q]\,, \text{ as } \eta \to 0\,,$$

which will ensure that $\mathbb{P}_\infty[\boxminus_Q]$ is given by Cardy's formula. This property was proved in [SchSm11]. (Note that since $\boxminus_Q$ is a closed set, one inequality is obvious). More precisely they prove the following result.

Theorem 2.3 ([SchSm11], Corollary 5.2). *For any quad $Q \in \mathcal{Q}$,*

$$\mathbb{P}_\infty[\partial \boxminus_Q] = 0\,.$$

In particular, one indeed has

$$\mathbb{P}_\eta[\boxminus_Q] \to \mathbb{P}_\infty[\boxminus_Q]\,,$$

as $\eta \to 0$, by weak convergence of $\mathbb{P}_\eta$ to $\mathbb{P}_\infty$.

In the next subsection, we define Borel sets in $(\mathscr{H}, d_{\mathscr{H}})$ which correspond to the so-called mutli-arms events. They were introduced and studied in our work [G4] where an analog of the above Theorem 2.3 was proved. See Lemma 2.1 below.

1.5. — Measurability of arms events ([G4])

Let $A = (\partial_1 A, \partial_2 A)$ be any non-degenerate smooth annulus of the plane (see [G4]). We wish to define events $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \ldots, \mathcal{A}_j$ which belong to the Borel sigma-field of $(\mathscr{H}, d_{\mathscr{H}})$ and which are such that for the discrete percolation configurations $\omega_\eta \sim \mathbb{P}_\eta \in (\mathscr{H}, d_{\mathscr{H}})$, $1_{\mathcal{A}_i}(\omega_\eta)$ coincides with the indicator function that ω_η has j (alternate) arms in the annulus A. Let us give a precise definition below in the case where $j = 4$ (which is the most relevant case in this chapter).

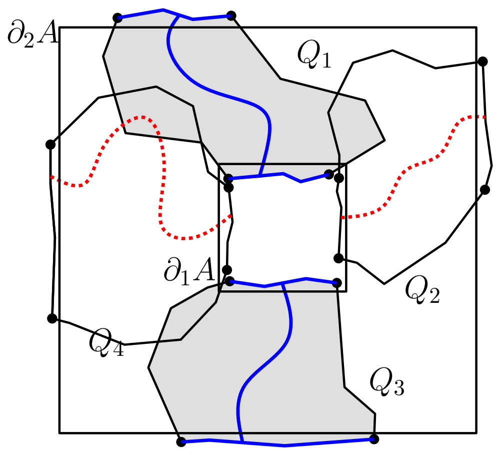


Figure 2.2: Defining the alternating 4-arm event using quads crossed or not crossed.

Definition 2.6 (Definition of the 4 arms event). *Let $A = (\partial_1 A, \partial_2 A)$ be a piecewise smooth annulus in the plane. We define the* **alternating 4-arm event** *in A as $\mathcal{A}_4 = \mathcal{A}_4(A) = \bigcup_{\delta > 0} \mathcal{A}_4^\delta$, where $\mathcal{A}_4^\delta$ is the existence of quads Q_i, $i = 1, 2, 3, 4$, with the following properties (See figure 2.2):*

(i) Q_1 and Q_3 (resp. Q_2 and Q_4) are disjoint and are at distance at least δ from each other.

(ii) For all $i \in \{1, \ldots, 4\}$, the paths $Q_i(\{0\} \times [0,1])$ (resp. $Q_i(\{1\} \times [0,1])$ lie inside (resp. outside) $\partial_1 A$ (resp. $\partial_2 A$) and are at distance at least δ from the annulus A and from the other Q_j's.

(iii) The four quads are ordered cyclically around A according to their indices.

(iv) For $i \in \{1, 3\}$, $\omega \in \boxminus_{Q_i}$.

(v) For $i \in \{2, 4\}$, $\omega \in \boxminus^c_{\overset{\frown}{Q_i}}$,

where if Q is a quad in $\mathcal{Q}$ (i.e. an homeomorphism from $[-1,1]^2$ into $\mathbb{C}$), then $\overset{\frown}{Q}$ denotes the rotated quad by $\pi/2$, i.e.

$$\overset{\frown}{Q} := Q \circ e^{i\pi/2} \,. \tag{2.3}$$

Remark 2.1. Note that by construction, $\mathcal{A}_4 = \mathcal{A}_4(A)$ is a *measurable* event. In fact, it is easy to check that it is an open set for the quad-topology $\mathcal{T}$.

Also, the definitions of general **(mono or polychromatic) k-arm events** in A are analogous: see [**G4**] for more details.

The following result is proved in [**G4**]. This is an analog of the above Theorem 2.3:

Lemma 2.1 (Lemma 2.4. and Corollary 2.10 in [G4]). *Let A be a piecewise smooth topological annulus (with finitely many non-smooth boundary points). Then the 1-arm, the alternating 4-arm and any polychromatic 6-arm event in A, denoted respectively by $\mathcal{A}_1$, $\mathcal{A}_4$ and $\mathcal{A}_6$ satisfy*

$$\lim_{\eta\to 0} \mathbb{P}_\eta[\mathcal{A}_i] = \mathbb{P}_\infty[\mathcal{A}_i] \,.$$

Moreover, in any coupling of the measures $\{\mathbb{P}_\eta\}$ and $\mathbb{P}_\infty$ on $(\mathscr{H}, \mathcal{T})$ in which $\omega_\eta \to \omega$ a.s. as $\eta \to 0$, we have

$$\mathbb{P}\Big[\{\omega_\eta \in \mathcal{A}_i\}\Delta\{\omega \in \mathcal{A}_i\}\Big] \to 0 \qquad (as\ \eta \to 0) \,. \tag{2.4}$$

Finally, for any exponent $\gamma < 1$, there is a constant $c = c_{A,\gamma} > 0$ such that, for any $\delta > 0$ and any $\eta > 0$:

$$\mathbb{P}_\eta\big[\mathcal{A}_4^\delta \mid \mathcal{A}_4\big] \geq 1 - c\,\delta^\gamma \,. \tag{2.5}$$

The proof of this result is divided into two parts. First, one needs to make sure that on the discrete level, the following scenario is very unlikely to happen:

(i) A four-arms event is realized for ω_η.

(ii) But it is hardly detected by quads in the sense of Definition 2.6 (namely, one would need a "vanishing" δ).

The illustration 2.3 gives an artistic view of what needs to be avoided. The proof of this in [**G4**] unsurprisingly relies on the absence of macroscopic *six-arms* events in critical percolation. This way, one obtains the quantitative bound (2.5).

The second part of the proof is more topological and shows that if ω_η satisfies $\mathcal{A}_4^\delta$ and if ω_∞ is $d_{\mathscr{H}}$-close to ω_η, then ω_∞ necessarily satisfies $\mathcal{A}_4^{\delta/2}$. This part is not as straightforward as one might initially think and is in the spirit of the explicit uniform structure outlined in Subsection 1.7 below.

1.6. — Space and topology for càdlàg trajectories in $\mathscr{H}$ ([G5])

Our near-critical coupling $\lambda \mapsto \omega_\eta^{\mathrm{nc}}(\lambda)$ as defined in Definition 2.1 is a càdlàg process in the metric space $(\mathscr{H}, d_{\mathscr{H}})$ (it is also the case of the rescaled dynamical percolation process introduced in Definition 1.2).

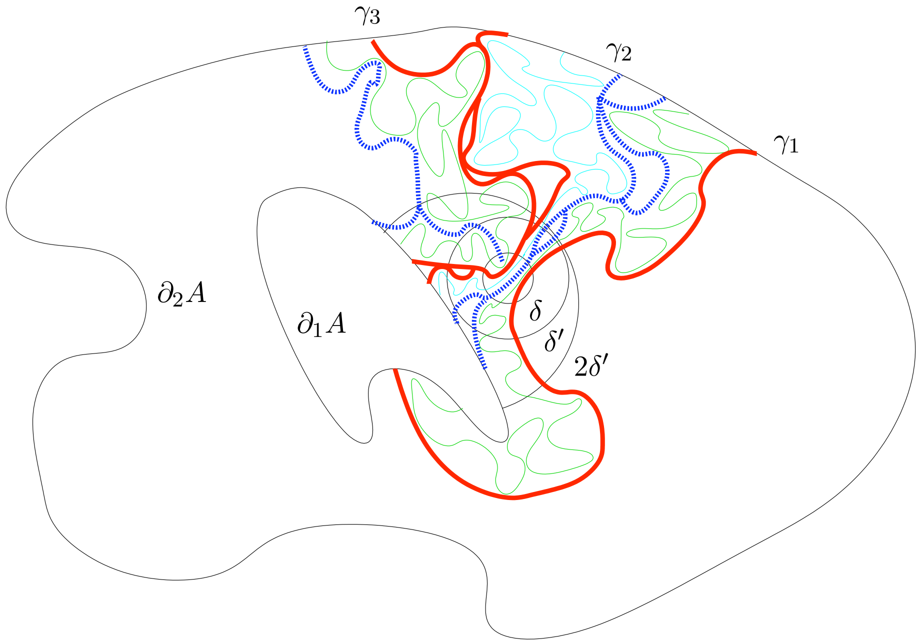


Figure 2.3: Illustration of the proof of convergence in [G4] for the event $\mathcal{A}_4$

Since $(\mathscr{H}, d_{\mathscr{H}})$ is a Polish space, one can rely on the following classical setup. Recall that if (X,d) is a Polish space and if $\mathcal{D}_X = \mathcal{D}_X[0,1]$ denotes the space of càdlàg functions from $[0,1]$ to X, then one can define a natural metric d_{Sk} on $\mathcal{D}_X$ for which $(\mathcal{D}_X, d_{\mathsf{Sk}})$ is a Polish space. This metric is usually known under the name of **Skorohod metric**. Let us summarize these facts in the following classical Proposition

Proposition 2.1 (See for example [EK86], Chapter 3.5). *Let (X,d) be a Polish metric space (i.e. a complete separable metric space). Let $\mathcal{D}_X = \mathcal{D}_X[0,1]$ be the space of càdlàg functions $[0,1] \to X$. Then $\mathcal{D}_X$ is a Polish metric space under the Skorohod metric d_{Sk} defined as follows: for any càdlàg processes $x, y : [0,1] \to X$, define*

$$d_{\mathsf{Sk}}(x,y) := \inf_{\lambda \in \Lambda} \left\{ \|\lambda\| \vee \sup_{0 \le u \le 1} d_X(x(u), y(\lambda(u))) \right\}, \tag{2.6}$$

where the infimum is over the set Λ of all strictly increasing continuous mappings of $[0,1]$ onto itself and where

$$\|\lambda\| := \sup_{0 \le s < t \le 1} \left| \log \frac{\lambda(t) - \lambda(s)}{t-s} \right|. \tag{2.7}$$

Hence, this motivates the following definition:

Definition 2.7. *For any $T > 0$, let $\mathsf{Sk}_T := \mathcal{D}_{\mathscr{H}}[0,T]$ be the space of càdlàg processes from $[0,T]$ to $\mathscr{H}$ and following Proposition 2.1, let*

$$d_{\mathsf{Sk}_T}(\omega(t), \tilde\omega(t)) := \inf_{\lambda \in \Lambda_T} \left\{ \|\lambda\| \vee \sup_{0 \le u \le T} d_{\mathscr{H}}(\omega(u), \tilde\omega(\lambda(u))) \right\},$$

Here we used the same notations as in Proposition 2.1 (at least their natural extensions to $[0,T]$). When the context will be clear, we will often omit the subscript T in the notation d_{Sk_T}.

With the following straightforward extension to $\mathbb{R}_+$ and $\mathbb{R}$, we have now made precise the statement of our main result, Theorem 1.4:

Lemma 2.2. *Let $\mathsf{Sk} = \mathsf{Sk}_{(-\infty,\infty)}$ (resp. $\mathsf{Sk}_{[0,\infty)}$) be the space of càdlàg processes from $\mathbb{R}$ (resp. $[0,\infty)$) to $\mathscr{H}$. Then, if we define*

$$d_{\mathsf{Sk}}(\omega(\lambda),\tilde{\omega}(\lambda)) := \sum_{k\geq 1} \frac{1}{2^k} d_{\mathsf{Sk}_{[-k,k]}}(\omega,\tilde{\omega})\,, \tag{2.8}$$

this gives us a Polish space $(\mathsf{Sk}, d_{\mathsf{Sk}})$ (and analogously for $\mathsf{Sk}_{[0,\infty)}$).

1.7. — An explicit uniform structure on $(\mathscr{H},\mathcal{T})$ ([G5])

There is an hidden issue in the way we defined our Skorohod space of càdlàg trajectories in $\mathscr{H}$: recall that the underlying metric $d_{\mathscr{H}}$ used to construct d_{Sk} is non-explicit (Definition 2.5). In order to prove Theorem 1.4 on the weak convergence of $\lambda \mapsto \omega_\eta^{\mathsf{nc}}(\lambda)$ to $\lambda \mapsto \omega_\infty^{\mathsf{nc}}(\lambda)$, we will need to overcome this issue by finding an explicit and convenient way to ensure that two càdlàg trajectories are close to each other in $\mathscr{H}$.

In general, when dealing with càdlàg processes on a topological space (X,τ), just having a topology τ on X is not enough for comparing two càdlàg trajectories on X similarly as in equation (2.6). An additional notion of *uniformity* is needed on the space X and this brings us to the classical notion of **uniform structure**:

Definition 2.8. *A* **uniform structure** *on a topological space (X,τ) is a given family Φ of* **entourages***, which are subsets of $X \times X$. The uniform structure Φ needs to satisfy a few properties (such as symmetry, a certain type of associativity and so on) and needs to generate in a certain sense the topology τ. See [Tu40] for example for an introduction on uniform spaces.*

If τ is generated by a metric d_X, then the canonical uniform structure on the metric space (X,d_X) is generated by the *entourages* of the form $U_a := \{(x,y) \in X \times X,\ d_X(x,y) < a\}, a > 0$. Furthermore, the following useful fact for us is known (see for example[Tu40]).

Proposition 2.2. *If (X,τ) is a compact Hausdorff topological space, then there is a* ***unique*** *uniform structure on (X,τ) compatible with the topology τ.*

Since $(\mathscr{H},\mathcal{T})$ is itself a compact Hausdorff topological space, we are thus left with finding a convenient and explicit uniform structure compatible with the quad-crossing topology $\mathcal{T}$. Let us briefly explain how one proceeds in [G5] to which we refer for more details. The intuitive idea in [G5] is that two configurations $\omega_1,\omega_2 \in \mathscr{H}$ are "uniformly" close to each other (in the sense that the criterion will not depend on the location of ω_i in the space $\mathscr{H}$) if, up to some small perturbations, they share the same crossing properties on "large-enough" quads. In [G5], the following dense family of quads in $\mathcal{Q}$ is used:

Definition 2.9 (A dyadic family of quads). *For any $k \geq 1$, let $(Q_n^k)_{1\leq n\leq N_k}$ be the **finite** family of all quads which are* polygonal quads *in $[-k,k]^2 \cap 2^{-k}\mathbb{Z}^2$, i.e. their boundary ∂Q_n^k is included in $[-k,k]^2 \cap 2^{-k}\mathbb{Z}^2$ and the four marked vertices are vertices of $2^{-k}\mathbb{Z}^2$.*

We will denote by $\mathcal{Q}^k$ this family of quads. Notice that $\mathcal{Q}^k \subset \mathcal{Q}^{k+1}$ and that the family $\mathcal{Q}_{\mathbb{N}} := \bigcup_k \mathcal{Q}^k$ is dense in the space of quads $(\mathcal{Q}, d_{\mathcal{Q}})$. In particular, Theorem 2.1 implies that the events $\{\boxminus_Q : Q \in \mathcal{Q}_{\mathbb{N}}\}$ generate the Borel σ-field of $\mathscr{H}$.

We will not give the precise definition of the uniform structure introduced in [G5], but it goes roughly as follows:

Proposition 2.3. (See Proposition 3.9. in [G5] for a more precise statement). *There is a function $r \mapsto \mathfrak{k}(r) \in \mathbb{N}^*$ which satisfies the following property: if any pair of configurations $\omega, \omega' \in \mathscr{H}$ share the same crossing properties for all the quads in $\mathcal{Q}^{\mathfrak{k}(r)}$ (up to some small "perturbation" of order $2^{-\mathfrak{k}(r)}$), then they are such that*

$$d_{\mathscr{H}}(\omega, \omega') \leq r\,.$$

Without entering into more details, the content of this Proposition basically says that it is enough to control crossing events on finitely many quads in order to compare two càdlàg trajectories in $\mathscr{H}$.

2. — Pivotal measures

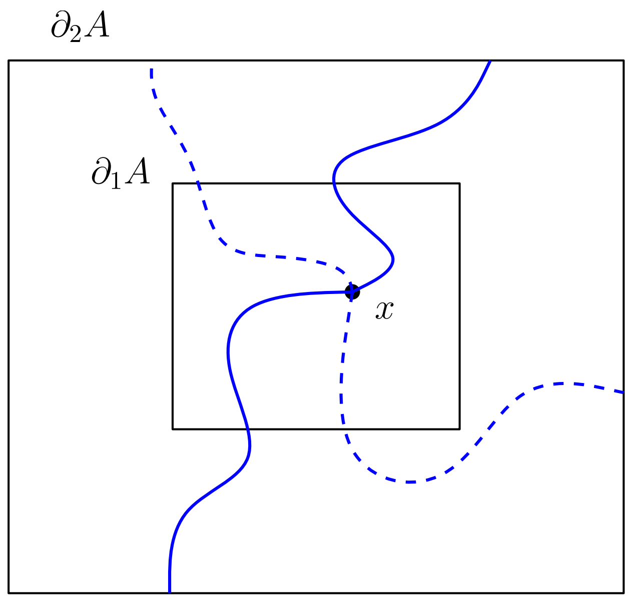
2.1. — Setup and main result

In what follows, $A = (\partial_1 A, \partial_2 A)$ will be a piecewise smooth annulus with compact inside face denoted by Δ. The purpose of our work [G4] is to study the scaling limit of suitably renormalized counting measures on the set of **A-important points** where the latter points are defined as follows:

Definition 2.10. *For any $\eta > 0$, a point $x \in \eta\mathbb{T} \cap \Delta$ is called **A-important** for the configuration ω_η if one can find four alternating arms in ω_η from x to the exterior boundary $\partial_2 A$. See figure 2.4*

Definition 2.11 (Pivotal measure μ^A). *Let us introduce the following counting measure on the set of A-important points:*

$$\mu^A = \mu^A(\omega_\eta) := \sum_{\substack{x \in \eta\mathbb{T}\cap\Delta \\ x \text{ is } A\text{-important}}} \delta_x\, r(\eta)\,,$$



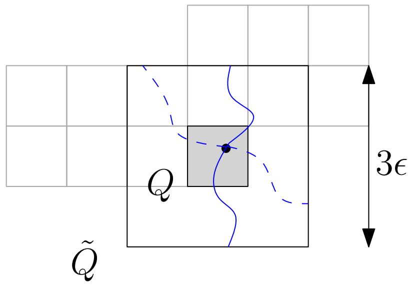


Figure 2.4: On the left picture, a point x which is A-important for the annulus $A = (\partial_1 A, \partial_2 A)$. On the right, a point which is ϵ-important, i.e. in $\mathcal{P}^\epsilon$

where $r(\eta) := \eta^2 \alpha_4^\eta(\eta, 1)^{-1}$ *was defined in* (1.3).

The main Theorem proved in [G4] may be stated as follows:

Theorem 2.4 (Theorem 1.1 in [G4]). *For any annulus A as above, there is a measurable map μ^A from $(\mathscr{H}, d_{\mathscr{H}})$ into the space $\mathfrak{M}$ of finite Borel measures on Δ such that*

$$(\omega_\eta, \mu^A(\omega_\eta)) \xrightarrow{(d)} (\omega_\infty, \mu^A(\omega_\infty)),$$

as the mesh $\eta \to 0$. The topology on $\mathfrak{M}$ is the topology of weak convergence (see the Prohorov metric $d_{\mathfrak{M}}$ in (**??**)*) and the above convergence in law holds under the product topology induced by $d_{\mathscr{H}}$ and $d_{\mathfrak{M}}$.*

For each $\epsilon > 0$, let us consider the grid $\epsilon\mathbb{Z}^2$. To each such square Q, we associate the square $\tilde{Q}$ of side-length 3ϵ centered around Q and we consider the annulus A_Q so that $\partial_1 A_Q = \partial Q$ and $\partial_2 A_Q = \partial \tilde{Q}$. See figure 2.4.

Definition 2.12. *For any $\eta > 0$, We define the set $\mathcal{P}^\epsilon = \mathcal{P}^\epsilon(\omega_\eta)$ to be the set of points $x \in \eta\mathbb{T}$, which are such that x belongs to an ϵ-square Q in the grid $\epsilon\mathbb{Z}^2$ and x is A_Q-important for the configuration ω_η. The points in $\mathcal{P}^\epsilon$ are called* **ϵ-important points**.

Furthermore, we will denote by $\mu^\epsilon = \mu^\epsilon(\omega_\eta)$ the Pivotal measure on these ϵ-important points, namely:

$$\mu^\epsilon = \mu^\epsilon(\omega_\eta) := \sum_{x \in \mathcal{P}^\epsilon(\omega_\eta)} \delta_x \, r(\eta).$$

Theorem 2.4 above clearly implies the following result on the scaling limit of $\mu^\epsilon(\omega_\eta)$:

Corollary 2.5. *For any $\epsilon > 0$, there is a measurable map μ^ϵ from $(\mathscr{H}, d_{\mathscr{H}})$ into the space of σ-finite Borel measures on $\mathbb{C}$, such that*

$$(\omega_\eta, \mu^\epsilon(\omega_\eta)) \xrightarrow{(d)} (\omega_\infty, \mu^\epsilon(\omega_\infty))\,,$$

under (a straightforward extension of) the above product topology.

Indeed, one would need to be a bit more precise since the measures considered here are not finite, only σ-finite, but the result follows easily by restricting μ^ϵ to larger and larger compact windows.

2.2. — Idea of the proof of Theorem 2.4

We wish to prove that the random finite Borel measures $\mu^A(\omega_\eta)$ on Δ have a scaling limit as $\eta \searrow 0$ and furthermore that the limiting random measure can be recovered from the information contained in ω_∞.

— ***Tightness*** —

Let us start by showing the following Lemma:

Lemma 2.3. *The sequence of random variables $\{(\omega_\eta, \mu^A(\omega_\eta))\}_\eta$ in the space $\mathscr{H} \times \mathfrak{M}$ is* **tight**.

Proof:
Since $(\mathscr{H}, d_{\mathscr{H}})$ is compact, the proof of the lemma easily reduces to showing that

$$\limsup_{\eta\to 0} \mathbb{E}\big[\mu_\eta^A(\Delta)\big] < \infty\,.$$

This upper bound follows very easily from the definition of the measure μ_η^A. Indeed, let $d > 0$ be the distance between $\partial_1 A$ and $\partial_2 A$: a point in Δ has to be d-important in order to be A-important. Therefore, if $d' = d \wedge 1/2$:

$$\begin{aligned}
\mathbb{E}\big[\mu_\eta^A(\Delta)\big] &= \sum_{x\in\Delta} \mathbb{P}\big[x \text{ is } A\text{-important}\big]\,\eta^2\,\alpha_4^\eta(\eta,1)^{-1} \\
&\le \sum_{x\in\Delta} \alpha_4^\eta(\eta,d')\,\eta^2\,\alpha_4^\eta(\eta,1)^{-1} && (2.9)\\
&\asymp \operatorname{area}(\Delta)\,\alpha_4^\eta(d',1)^{-1} && (2.10)\\
&\asymp \operatorname{area}(\Delta)\,\alpha_4(d',1)^{-1} < \infty. && (2.11)
\end{aligned}$$

To go from (2.9) to (2.10), we used the observation that $\lim_{\eta\to 0} \big|\Delta \cap \eta\mathbb{T}\big|\,\eta^2 \asymp \operatorname{area}(\Delta)$ as well as the following well-known **quasi-multiplicativity property**:

Proposition 2.4 (Quasi-multiplicativity). *The four-arm event for critical percolation on the triangular lattice $\eta\mathbb{T}$ satisfies the following property: for any radii $\eta \le r_1 \le r_2 \le r_3$,*

$$\frac{1}{C}\alpha_1^\eta(r_1,r_2)\alpha_1^\eta(r_2,r_3) \le \alpha_1^\eta(r_1,r_3) \le C\,\alpha_1^\eta(r_1,r_2)\alpha_1^\eta(r_2,r_3)\,, \tag{2.12}$$

for some universal constant $C>0$ (independent of the mesh η).

Finally, to go from (2.10) to (2.11), we used the fact the quantity $\alpha_4^\eta(d',1)$ (which depends on η) converges as $\eta \searrow 0$ to the macroscopic probability $\alpha_4(d',1)$ (computed up to constants in [SW01]). □

It follows from this tightness property that there exists some subsequential scaling limit (ω_∞,μ^A) along some subsequence $\{\eta_k\}_{k>0}$, where η_k goes to 0. We now wish to show that this μ^A can actually be recovered from ω_∞, which is the unique subsequential scaling limit of $\{\omega_\eta\}_{\eta>0}$, as we already know from Subsection 1.4. Consequently, the pair (ω_∞,μ^A) will also be unique.

— ***Strategy to prove uniqueness*** —

Once again, we will be inspired from the following analogous and classical situation: consider the simple random walk $S_\eta(t)$ on $\sqrt{\eta}\mathbb{Z}\times\eta\mathbb{N}$ together with the renormalized counting measure on its set of zeros:

$$M(S_\eta) := \sum_{t\in\eta\,\mathbb{N}:\,S_\eta(t)=0} \sqrt{\eta}\,\delta_t\,.$$

Notice the clear analogy with the definition of the measure $\mu^A(\omega_\eta)$ in Definition 2.11. It is a classical fact that the coupling $(S_\eta(\cdot), M(S_\eta))$ converges in law to $(B(\cdot), M)$ where $B(\cdot)$ is a standard **Brownian motion** and M is the **local time measure at 0** (sometimes denotes as $dL^{x=0}$) of the Brownian motion. Furthermore, the local time measure M is a (measurable) function of the Brownian motion $B(\cdot)$. The latter property (which is the analog of our desired $\mu^A=\mu^A(\omega_\infty)$) corresponds to the fact that for any time $T>0$, $M([0,T])$ is well approximated (as $\alpha\to 0$) by

$$\frac{1}{2\alpha}\int_0^T 1_{|B_s|\le\alpha}ds\,.$$

Similarly, $M([0,T])$ can be recovered by counting the number of **Brownian excursions** which leave the interval $[-\alpha,\alpha]$ and by letting $\alpha\to 0$.

To gain some intuition and set up some notations, we will first analyze the easy case of the random walk. Say we wish to approximate the number of zeros of S_η on the interval $[0,1]$ by using only **macroscopic information** (i.e. information which passes to the limit when $S_\eta(\cdot)\to B(\cdot)$ under the topology of uniform convergence on compact sets). We will not follow the optimal route here but rather a strategy that will be applicable to the pivotal measure $\mu^A(\omega_\eta)$.

Fix some small $\alpha>0$ and divide the unit interval $[0,1]$ into $N=\alpha^{-1/2}$ intervals $I_i=[\frac{i-1}{\sqrt{\alpha}},\frac{i}{\sqrt{\alpha}}]$, $i=1,\dots,N$. Decompose

$$X = X(S_\eta) := |\{t\in\eta\mathbb{N}\cap[0,1] : S_\eta(t)=0\}|$$

into $\sum_{1\le i\le N} X_i$ where for each i, X_i stands for the number of zeros in the i-th interval I_i, i.e.

$$X_i = |\{t\in\eta\mathbb{N}\cap I_i : S_\eta(t)=0\}|$$

(We neglect the boundary issues here). Now, for each $1 \le i \le N$, let Y_i be one if the random walk S_η intersects the mesoscopic block $[-\alpha, \alpha] \times I_i$ and 0 otherwise. This way, the quantity $Y = Y^\alpha(S_\eta) := \sum_i Y_i$ corresponds to the number of intervals I_i where the random walk came α-close to the origin. Clearly this quantity Y is measurable at the scaling limit, in the sense that $Y(S_\eta) \to Y(B)$ in law. Furthermore, it is clear that if the 100-th block and the 1000-th block are both intersected by the random walk, then X_{100} and X_{1000} should be more or less independent. This way, the desired quantity X should be well approximated by Y times a certain deterministic quantity $\beta = \beta(\alpha, \eta)$ which should represent the average number of zeros in a "typical" block i knowing that $Y_i = 1$. For example one could choose

$$\beta = \beta(\alpha, \eta) := \mathbb{E}\big[X_N \bigm| Y_N = 1\big]$$

(the proportional factor should not be defined by the first block $\mathbb{E}\big[X_1\big] = \mathbb{E}\big[X_1 \bigm| Y_1 = 1\big]$ which is very atypical since our walk starts at 0). More quantitatively, the proof that the renormalized number of zeros in $[0,1]$ weakly converges to a measurable quantity of the Brownian motion can be divided into the following three steps:

(A) For each $\alpha > 0$, there is a measurable functional of the Brownian motion $Y = Y^\alpha(B)$ s.t. $Y(S_\eta)$ converges in law to $Y(B)$.

(B) $\mathbb{E}\big[(X - \beta Y)^2\big] = o(\mathbb{E}\big[X^2\big])$, uniformly as $0 < \eta < \alpha$ converge to 0.

(C) $\beta(\eta, \alpha) \sim c\sqrt{\frac{\alpha}{\eta}}$, as η/α and α go to zero for some universal constant $c > 0$.

Remark 2.2. Needless to say, there are simpler ways to prove the convergence in law of $(S_\eta(\cdot), M(S_\eta))$ to $(B(\cdot), M)$ but they would rely on the **strong Markov property** which will not have an analog in our case.

Back to our pivotal measure $\mu^A = \mu^A(\omega_\eta)$, let us explain what the strategy is in [G4] in order to prove Theorem 2.4. As in the above case of the random walk, let us focus ourselves on the total mass

$$\begin{aligned} \mu_\eta^A(\Delta) &:= r(\eta)\, |\{x \in \eta\mathbb{T} \cap \Delta : x \text{ is } A\text{-important for } \omega_\eta\}| \\ &= \frac{\eta^2}{\alpha_4^\eta(\eta, 1)} |\{x \in \eta\mathbb{T} \cap \Delta : x \text{ is } A\text{-important for } \omega_\eta\}| \end{aligned}$$

Similarly as with the random walk, fix some "small" mesoscopic scale $\eta \ll \alpha \ll 1$ and decompose the inside face Δ into $N = O(\alpha^{-2})$ squares $Q_i, i = 1, \ldots, N$ along the square grid $\alpha\,\mathbb{Z}^2$. The total number $X = X(\omega_\eta)$ of A-important points in $\eta\mathbb{T} \cap \Delta$ is decomposed into

$$X = \sum_{i=1}^N X_i\,,$$

where the X_i are the number of A-pivotal points in each square Q_i (we neglect the presence of pivotal points on the edges of the squares Q_i). Using the same notations as in the RW case, let Y_i be the indicator function that there is a four-arms event from Q_i to $\partial_2 A$ (in particular $X_i = 0$ if $Y_i = 0$) and let

$$Y = Y^\alpha(\omega_\eta) := \sum Y_i\,,$$

be the total number of α-square which satisfy a four-arms event up to $\partial_2 A$. To prove (part of) Theorem 2.4, one would like to show that knowing $Y = Y^\alpha(\omega_\eta)$, one can guess with

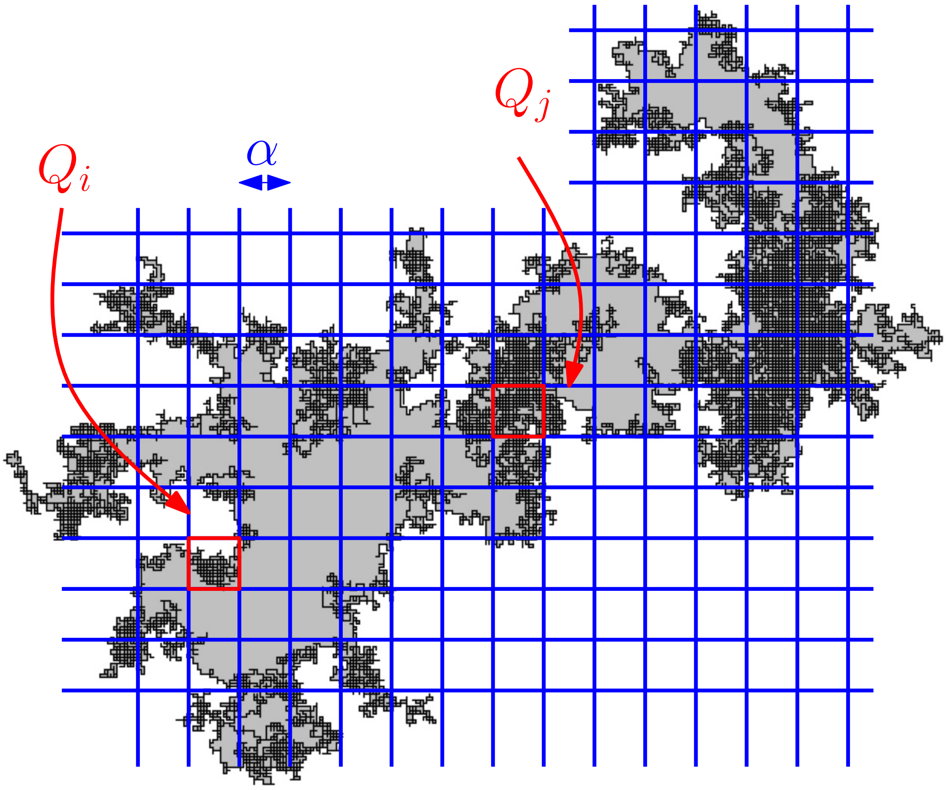


Figure 2.5: Assume this fractal picture (a Brownian loop) represents the set of Pivotal points. (This requires some imagination since the latter one is a totally disconnected Cantor like set). Then its total number of points should be well approximated by the number of mesoscopic squares Q_i times a deterministic constant β which depends on both $\eta \ll \alpha$.

good precision what $X = X(\omega_\eta)$ is. More precisely, we wish to show an analog of the above item (B), i.e. that

$$\mathbb{E}\big[(X - \beta Y)^2\big] = o(\mathbb{E}\big[X\big]^2)\,, \tag{2.13}$$

uniformly as $0 < \eta < \alpha$ go to zero for some well chosen proportional factor $\beta = \beta(\alpha, \eta)$. By Cauchy-Schwarz, if the estimate (2.13) holds, then $|\mathbb{E}\big[X - \beta Y\big]|$ needs to be $o(\mathbb{E}\big[X\big])$. Notice now that the first moment can be rewritten as follows

$$\begin{aligned}\mathbb{E}\big[X - \beta Y\big] &= \sum_i \mathbb{E}\big[X_i - \beta Y_i\big] \\ &= \sum_i \mathbb{P}\big[Y_i = 1\big]\big(\mathbb{E}\big[X_i \mid Y_i = 1\big] - \beta\big)\,,\end{aligned}$$

since $Y_i = 0$ implies $X_i = 0$. Now, for any i, $\mathbb{E}\big[X_i \mid Y_i = 1\big]$ is the expected number of A-important points inside Q_i knowing that there are already four arms from Q_i to $\partial_2 A$. An important part of [G4] (Section 3) consists in establishing a **coupling property** (Proposition 3.1) whose content is illustrated in Figure 2.6. Its full strength is not needed for the first moment but it implies in particular that all the conditional expectations $\mathbb{E}\big[X_i \mid Y_i = 1\big]$ are more or less equal to a deterministic quantity $\beta = \beta(\alpha, \eta)$ which for simplicity and for symmetry reasons is chosen to be the one illustrated on the left of Figure

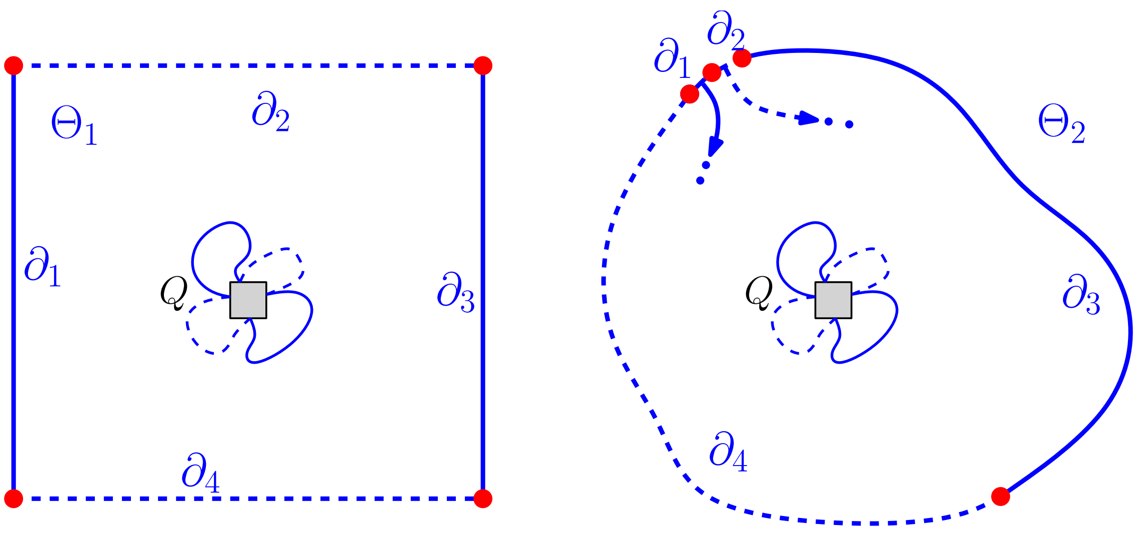


Figure 2.6: Consider two macroscopic quads Θ_1 and Θ_2 with four marked arcs $\partial_1, \ldots, \partial_4$. Let ν_{Θ_1} and ν_{Θ_2} be respectively the laws of critical percolation inside Θ_1 and Θ_2 conditioned on having four alternative arms from Q to the alternate arcs of Θ_1 and Θ_2. The content of Proposition 3.1. in [**G4**] states that one can **couple** the conditional probability measures ν_{Θ_1} and ν_{Θ_2} so that "viewed from Q", the configurations look the same with high probability. An important aspect of Proposition 3.1 for the proof is that it handles very degenerate situations such as Θ_2 illustrated here.

2.6 (i.e. the expected number of Θ_1 pivotal points inside Q when Q is conditioned to have a four-arm event up to Θ_1).

We sketched this first moment analysis only to highlight where the β factor might come from. What we really need is the second moment. It can be written as

$$\begin{aligned}\mathbb{E}\big[(X-\beta Y)^2\big] &= \sum_{i,j} \mathbb{E}\big[\big(X_i-\beta Y_i\big)\big(X_j-\beta Y_j\big)\big] \\ &= \sum_{i,j} \mathbb{E}\big[Y_i Y_j\big]\mathbb{E}\big[\big(X_i-\beta\big)\big(X_j-\beta\big) \bigm| Y_i=1,\, Y_j=1\big]\,. \end{aligned} \tag{2.14}$$

In this sum over squares Q_i, Q_j, there are relatively few pairs of nearby squares (say, whose distance $d(Q_i,Q_j)$ is less than r, with $\alpha \ll r \ll 1$). Let us neglect these near-diagonal terms and for any two squares Q_i, Q_j such that $d(Q_i,Q_j) \geq r \gg \alpha$, if one conditions on $\{Y_i = 1, Y_j = 1\}$, i.e., on the event that both Q_i and Q_j are A-important, then, again by the coupling argument (Figure 2.6), one expects that the configuration seen inside Q_i should be almost independent of what is seen inside Q_j. This should lead to $\mathbb{E}\big[\big(X_i-\beta\big)\big(X_j-\beta\big) \bigm| Y_i=1,\, Y_j=1\big] \approx \mathbb{E}\big[X_i-\beta \bigm| Y_i=1\big]\mathbb{E}\big[X_j-\beta \bigm| Y_j=1\big]$. Now, since β is precisely chosen to match well with these first moment quantities, one should indeed obtain a small second moment in (2.14).

However, in order to apply the coupling proposition from section 3 in [**G4**], there are some issues about the above conditioning. Conditioned on the entire configuration outside Q_i and Q_j in a way that makes $\{Y_i = Y_j = 1\}$ possible, the value of X_j and Y_j might not at all be independent of the configuration inside Q_i. See Figure 2.7. This shows that somehow the configurations inside Q_i and Q_j interfere with each other in a nontrivial way, which is bad news for applying the coupling result illustrated in Figure 2.6. Additional intermediate scales and further conditionings are needed to handle these interferences (it is

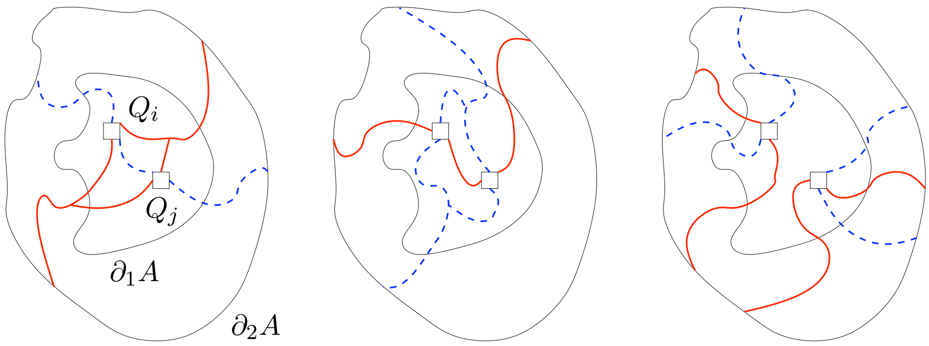


Figure 2.7: Assume that all disjoint arms are shown between Q_i, Q_j and $\partial_2 A$. Then, in the first example, $X_j > 0$ requires a blue (dashed) connection within Q_i, while, in the second one, it requires a red (solid) connection. In the third example, X_i and X_j are independent.

important here to be able to control degenerate conditionings such as the one illustrated on the right of Figure 2.6). We will not describe these here and we refer to the discussion at the beginning of Section 4 in [G4]. We will not say more on the rather lengthy proof of the second moment estimate (2.13) in [G4]. Let us conclude this sketch with the analogs of items (A) and (C) in the RW case:

- For the analog of item (A), Definition 2.6 enables us to define a measurable functional of the continuum percolation $Y^\alpha = Y^\alpha(\omega_\infty)$ and Lemma 2.1 ensures the limit in law of $Y^\alpha(\omega_\eta)$ to this $Y^\alpha(\omega_\infty)$.
- For the analog of item (C), one needs to study how the quantity $\beta(\alpha,\eta)r(\eta)$ behaves as η/α and η go to zero, where $\beta(\alpha,\eta)$ is the above conditional expectation. This step is important since the second moment estimate (2.13) handles only the non-renormalized quantity X and we are interested in $\mu_\eta^A(\Delta)$. This part is a bit technical since the probabilities $\alpha_4(r,R)$ of four-arm events are only known so far **up to logarithmic corrections** (see [SW01]). To overcome this we prove in [G4] the following result which we believe is interesting in its own:

Proposition 2.5 (Proposition 4.9 in [G4]). *For any fixed $r>0$,*

$$\lim_{\eta\to 0}\frac{\alpha_4^\eta(\eta,r)}{\alpha_4^\eta(\eta,1)} = \lim_{\epsilon\to 0}\frac{\alpha_4(\epsilon,r)}{\alpha_4(\epsilon,1)} = r^{-5/4}\,. \tag{2.15}$$

Remark 2.3. Recall the above Remark 2.2. It might seem strange that one cannot rely also on a strong Markov property here since percolation is by essence made of many independent coin flips. There is indeed an easy spatial Markov property in the setting of percolation, but the big difference with the one-dimensional setting of the RW is that to know whether a point x is A-important or not, one needs to discover "most" of the percolation configuration around, while in the RW case, one can use simple stopping times to detect whether a time is a zero or not.

Remark 2.4. Another important contribution of [G4] that we will not discuss here is that these pivotal measure satisfy a **conformal covariance** property (in $|\phi'(z)|^{3/4}$). This is proved in [G4] by using a version of the above observable $Y = Y^\alpha(\omega_\infty)$ which is invariant by rotation.

Remark 2.5. Finally, let us stress that the sketch of proof presented here also works for limits of counting measures on other geometrical objects of interest in critical percolation, for example an "area"-measure on macroscopic clusters and a "length"-measure on interfaces such as SLE_6. See [G4]. The latter length measure is the first **natural parametrization** of an SLE curve arising from a discrete system. See [LS11] where natural parametrizations of SLE curves are established (without links with a discrete system).

3. — Cut-off trajectories in the continuum

Our goal in this section is to explain how to add some infinitesimal mass to the **critical slice** $\omega_\infty = \omega_\infty^{\mathsf{nc}}(\lambda = 0)$ in order to obtain an ϵ-approximation of what will be our limiting object, namely the càdlàg path $\lambda \mapsto \omega_\infty^{\mathsf{nc}}(\lambda)$. In this section, we will only work at the continuum limit (besides some sketch of proofs which will rely on the properties of the discrete percolations). We proceed as follows:

1. Fix some small parameter $\epsilon > 0$.
2. Sample the critical slice $\omega_\infty \sim \mathbb{P}_\infty$.
3. Get the pivotal measure $\mu^\epsilon = \mu^\epsilon(\omega_\infty)$ from Corollary 2.5 . Use this measure to select the macroscopic (ϵ) pivotal points which will change: this amounts to sample a Poisson Point Process on $\mathbb{C} \times \mathbb{R}$ whose intensity measure is $d\mu^\epsilon(x) \times d\lambda$. This Poisson Point Process is denoted by $\mathsf{PPP} = \mathsf{PPP}(\mu^\epsilon(\omega_\infty))$ and it is not hard to show that it is a.s. locally finite (basically by the first moment analysis in Lemma 2.3).
4. Given $\omega_\infty = \omega_\infty^{\mathsf{nc}}(\lambda = 0)$ and PPP, "update" the initial critical slice as λ increases (or decreases) according to the information provided by the Poisson point process PPP. Call the process thus obtained $\lambda \mapsto \omega_\infty^{\mathsf{nc},\epsilon}(\lambda)$.

> Notice that the last step is straightforward in the discrete but is much less clear in the continuum. In fact this is probably the most resisting problem we faced in this project. Let me explain briefly what the difficulty is and how we managed to deal with it.

Recall we are given a critical slice $\omega_\infty \sim \mathbb{P}_\infty$ and a locally finite cloud of points PPP (it is important to keep in mind that PPP is far from being independent of ω_∞ since it is made of ϵ-important points of ω_∞). The main difficulty lies in the fact that updating a configuration ω_∞ in the abstract space $\mathscr{H}$ requires in principle to follow the status of all crossing events $\boxminus_Q$ for all quads $Q \in \mathcal{Q}$ (see Section 1 for these notations) and a single pivotal switch will have an effect on infinitely many such crossing events. As such one needs to "organize" the information efficiently. The first step is to notice that under some natural consistency conditions (See Lemma 7.4 in [G5]), it is enough to follow the status of countably many quads $Q \in \mathcal{Q}_\mathbb{N}$. (Surprisingly, these conditions are not so easy to check in the continuum, see Figure 7.1 in [G5]). We are thus left with the following problem: given a fixed quad $Q \in \mathcal{Q}_\mathbb{N}$ and a level $\lambda \in \mathbb{R}$, can one decide based on ω_∞ and PPP whether the process one is building should cross or not the quad Q at level λ?

To answer the above problem, to each level $\lambda \in \mathbb{R}$ and each quad $Q \in \mathcal{Q}_\mathbb{N}$, we define in [G5] a kind of graph structure (with two types of edges, primal and dual ones), called a **network**, whose vertices are the points in $\mathsf{PPP}_\lambda = \mathsf{PPP} \cap (Q \times [0, \lambda])$ (we assume here that $\lambda > 0$). The purpose of this network denoted by $\mathsf{N}_{Q,\lambda} = \mathsf{N}_{Q,\lambda}(\omega_\infty, \mathsf{PPP}_\lambda)$ is to represent the connectivity properties of the configuration ω_∞ within $Q \setminus \mathsf{PPP}_\lambda$. In order to be measurable w.r.t ω_∞, this network is obtained as a limit of **mesoscopic networks** $\mathsf{N}^r_{Q,\lambda}$. See Theorem

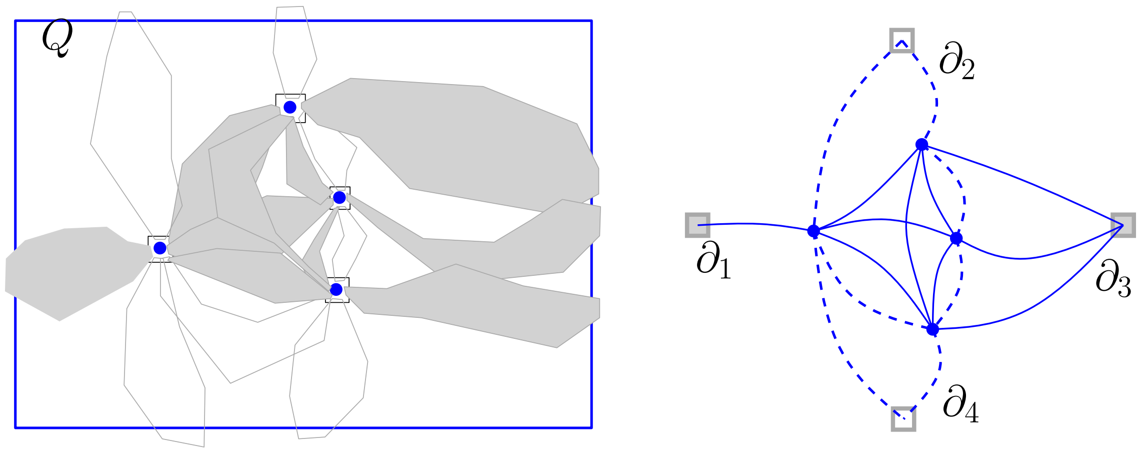


Figure 2.8: On the right we represented the graph structure (**Network**) associated to the continuum percolation on the left, where each edge (primal or dual) corresponds to a certain quad-crossing. These quads come to a mesoscopic distance close to their associated vertices. These graph structures carry the minimal information needed to recover connectivity properties. In particular, these graphs are in general NOT planar. See [**G5**].

6.14 in [**G5**] where a coupling with discrete percolations is used to prove the convergence of the mesoscopic networks. See also the illustration in Figure 2.8 to get an idea of what these networks are. Once we have at our disposal such a structure $\mathsf{N}_{Q,\lambda} = \mathsf{N}_{Q,\lambda}(\omega_\infty, \mathsf{PPP}_\lambda)$, one can answer the above question. As such we obtain in [**G5**] a well-defined càdlàg process $\lambda \mapsto \omega_\infty^{\mathsf{nc},\epsilon}(\lambda)$ in the space $\mathscr{H}$.

4. — No cascades from the microscopic scales

We have just introduced a càdlàg cut-off trajectory in the continuum. There is a natural analogous process in the discrete which we shall denote by $\lambda \mapsto \omega_\eta^{\mathsf{nc},\epsilon}(\lambda)$. Its definition is straightforward: it is the same process as the near-critical coupling introduced in Definition 2.1 except that the only sites which are updated are the one which are initially ϵ-important (i.e. for the configuration $\omega_\eta = \omega_\eta^{\mathsf{nc}}(\lambda = 0)$). This càdlàg process in $\mathscr{H}$ is denoted by $\lambda \mapsto \omega_\eta^{\mathsf{nc},\epsilon}(\lambda)$.

As discussed in the introduction, one might be afraid that the process $\omega_\eta^{\mathsf{nc},\epsilon}(\cdot)$ soon diverges away from the "true" process $\omega_\eta^{\mathsf{nc}}(\cdot)$. This would happen if there were points initially "invisible" (i.e. not macroscopically important) that would gain some importance along the process (i.e. as λ increases or decreases). The following **stability result** is established in [**G5**] by showing that such "cascades of importance" are unlikely to happen. See Figure 1.3 for an example of such a possible cascade, but one could also imagine more complicated (multi-scales) cascades.

Proposition 2.6 (Proposition 8.1 in [G5]). *There exists a continuous function ψ : $[0,1] \to [0,1]$, with $\psi(0) = 0$ such that uniformly in $0 < \eta < \epsilon$,*

$$\mathbb{E}\big[d_{\mathsf{Sk}}(\omega_\eta(\cdot), \omega_\eta^\epsilon(\cdot))\big] \le \psi(\epsilon)\,,$$

where the Skorohod distance d_{Sk} was introduced in Lemma 2.2.

This Proposition handles in particular simple cascades as pictured in Figure 1.3 but also more complex ones such as multi-scale cascades and so on. The proof of this result has the same flavour as the main proof in [Ke87] but is more general and significantly shorter.

5. — End of the sketch

The last ingredient to conclude the proof of our main theorem is the fact that $\lambda \mapsto \omega_\eta^{\mathsf{nc},\epsilon}(\lambda)$ converges in distribution to $\lambda \mapsto \omega_\infty^{\mathsf{nc},\epsilon}(\lambda)$. This convergence is not the main technical difficulty in [G5] (to which we refer).

Summarizing, we thus introduced cut-off processes $\omega_\eta^{\mathsf{nc},\epsilon}(\cdot)$ and $\omega_\infty^{\mathsf{nc},\epsilon}(\cdot)$ which satisfy:

(i) $\omega_\eta^{\mathsf{nc}}(\cdot) \approx \omega_\eta^{\mathsf{nc},\epsilon}(\cdot)$ with high probability under the Skorohod distance d_{Sk}. See Proposition 2.6.

(ii) $\omega_\eta^{\mathsf{nc},\epsilon}(\cdot) \xrightarrow{(d)} \omega_\infty^{\mathsf{nc},\epsilon}(\cdot)$ under the Skorohod topology given by d_{Sk} (Definition 2.7).

Using these two facts, it is not hard to show that $\omega_\infty^{\mathsf{nc},\epsilon}(\cdot)$ converges in probability as $\epsilon \to 0$ to our desired near-critical càdlàg process $\lambda \mapsto \omega_\infty^{\mathsf{nc}}(\lambda)$ and that this process is the limit in law of $\lambda \mapsto \omega_\eta^{\mathsf{nc},\epsilon}(\lambda)$. This concludes our sketch of proof of our main Theorem 1.4.

6. — Miscellaneous

Let us end this Chapter with a concise list of properties satisfied by the limiting objects $\lambda \mapsto \omega_\infty^{\mathsf{nc}}(\lambda)$ and $t \mapsto \omega_\infty(t)$ constructed in [G5]. We also list some related models that can be analyzed in the same fashion. See [G5], Sections 10 to 13 in [G5] for a much more detailed exposure.

1. **Markov property:** as mentioned above, our limiting processes are still Markovian in the continuum (this was far from obvious to start with). Let us highlight that the Markov property is harder to prove in the near-critical setting. Indeed one needs in this case to extend the above sketch when starting from a non-critical slice (and this is non-trivial in particular for the construction of pivotal measures). See Theorem 1.7 in the Introductory Chapter and Section 11 in [G5].

2. Even though $\omega_\infty^{\mathsf{nc}}(\cdot)$ and $\omega_\infty(\cdot)$ are Markovian, it can be shown that they are **Non-Feller** processes!

3. **"Relativistic" invariance:** the conformal map of a dynamical percolation $\omega_\infty(\cdot)$ from a domain D to a domain $\tilde D$ is again a dynamical percolation process $\tilde\omega_\infty(\cdot)$ in $\tilde D$ but whose "spatial clocks" depend on $|\phi'(z)|^{-3/4}$. See Theorem 10.3 in [G5] for a precise statement.

4. **Correlation length:** We show in Section 10 in [G5] that there is a.s. an **infinite cluster** in $\omega_\infty^{\mathsf{nc}}(\lambda)$ as soon as $\lambda > 0$. Furthermore, one can define the following two notions of correlation lengths:

$$\begin{cases} L_1(\lambda) & := \inf\{r > 0 : \mathbb{P}\big[\omega_\infty^{\mathsf{nc}}(\lambda) \in \bigcap_{R>r} \mathcal{A}_1(r,R)\big] > 1/2\} \\ L_2(\lambda) & := \inf\{r > 0 : \mathbb{P}\big[\omega_\infty^{\mathsf{nc}}(\lambda) \text{ crosses } [0,2r] \times [0,r]\big] > 0.99\} \end{cases} \tag{2.16}$$

Ii is proved in [G5] that there exist two constants $c_1, c_2 \in (0,\infty)$, s.t.

$$\begin{cases} L_1(\lambda) = c_1 \lambda^{-4/3} \\ L_2(\lambda) = c_2 \lambda^{-4/3} \end{cases} \tag{2.17}$$

5. **Noise-sensitivity:** we show in [**G5**] that the càdlàg process $t \mapsto \omega_\infty(t)$ is noise sensitive and a.s. has **exceptional times** with an **infinite cluster**. This is a serious hint that $t \mapsto \omega_\infty(t)$ should be **ergodic** even though we could not prove this. See Remark 12.3 in [**G5**].

   Let us point out that this single property is the ONLY link between the works [**G4**, **G5**] and the work [**G3**]. This has often been a source of confusion.

6. **Gradient percolation:** using exactly the same techniques, one can generalize our near-critical scaling limit to a near-critical coupling $\omega_\infty^{\mathsf{nc},\phi}(\lambda)$, where the near-critical parameter at $z \in \mathbb{C}$ is now $\lambda\phi(z)$ for any continuous spatial function $\phi : \mathbb{C} \to \mathbb{R}$. By taking $\phi(z) := \mathrm{Im}(z)$, we obtain a scaling limit for the model of gradient percolation introduced in [N08b]. See Section 13 in [**G5**].

7. **Wulff crystal:** in [Dum13], the author relies on our main result in [**G5**] in his proof that the Wulff crystal (see [Cer06]) for supercritical percolation on the triangular lattice converges to a ball as $p > p_c$ tends to $p_c(\mathbb{T}) = 1/2$. (See also [CM10] which proves the analogous result in the case of the Ising model).

8. **Massive** SLE_6**:** in Section 13.3 in [**G5**] we investigate what should be the Loewner driving function of the so-called **massive** SLE_6 which correspond to the "exploration paths" of $\omega_\infty^{\mathsf{nc}}(\lambda)$ with $\lambda \neq 0$. We expect the driving function to be of the form

$$dW_t = \sqrt{6}\, dB_t + dA_t\,,$$

   where B_t is Brownian motion and A_t is a monotone drift, increasing for $\lambda > 0$, decreasing for $\lambda < 0$. More precisely we conjecture the following expression for A_t:

$$\begin{aligned} dA_t = &= c\,\lambda\, |d\gamma_t|^{3/4}\, |dW_t| \\ &= c'\,\lambda\, |d\gamma_t|^{3/4}\, |dt|^{1/2}\,. \end{aligned}$$

   Massive SLE_κ are also studied in [MS10] for values of $\kappa \in [0,4] \cup \{16/3\} \cup \{8\}$.

Chapter 3

Scaling limit of the Minimal Spanning Tree in the plane

Based on a joint work with
Oded Schramm and Gábor Pete

In this Chapter, we wish to highlight the application of the previous Chapter (near-critical percolation) to the **Minimal Spanning Tree (MST)** in the plane. Much less details will be given.

1. — Minimal spanning tree on the triangular grid $\mathbb{T}$

Since we have a proof of the existence and properties of the scaling limit of the near-critical ensemble only for site percolation on the triangular lattice $\mathbb{T}$, if we want to use that to build the MST scaling limit, we need a version of the MST that uses Unif$[0,1]$ vertex labels $\{V(x)\}$ on $\mathbb{T}$. We proceed as follows: let us assign to each edge $e = (x, y)$ in $\mathbb{T}$ the vector label

$$U(e) := \big(V(x) \vee V(y), V(x) \wedge V(y)\big)\,, \tag{3.1}$$

and consider (for example) the lexicographic ordering on these vectors to determine uniquely the MST. See Figure 3.1. With a slight abuse of terminology, this is what we will call the MST on the lattice $\mathbb{T}$.

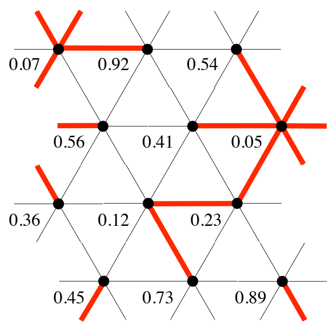


Figure 3.1: The minimal spanning tree associated to vertex labels of the triangular lattice $\mathbb{T}$, with a periodic boundary condition. (The numbers are labels of the **sites**)

Let us make an important remark here:

Remark 3.1. The use of the lexicographic ordering for the vector labels (3.1) may seem quite arbitrary. In fact, there exist initial vertex labels for which different natural ordering DO lead to very different global structures! Fortunately, with the Unif$[0,1]$ labels, for any rule to construct the MST on $\mathbb{T}$ that ensures that any two p-clusters are connected by a unique path of this MST (this would not be the case for example with $U(e) := V(x)+V(y)$), our approximation of the macroscopic structure of the MST using the near-critical ensemble will work with large probability, and hence the scaling limit will be the same.

Now that the model has been made more precise, let us recall our main theorem:

Theorem 3.1 (Limit of MST_η in $\mathbb{C}$, [G6]). *As $\eta \to 0$, the above spanning tree MST_η on $\eta\mathbb{T}$ converges in distribution (under the setup introduced in [AB+99]) to a unique scaling limit MST_∞ that is invariant under translations, scalings, and rotations.*

2. — Setup and Kruskal's algorithm in the continuum

The setup we use in [**G6**] is the setup which was introduced in [AB+99] in order to prove subsequential scaling limits for the UST and MST on $\mathbb{Z}^2$. We will not make this setup explicit here. See [AB+99, **G6**]. In rough terms, a spanning tree T in their setup is viewed as a closed collection of all the sub-trees with finitely many leaves contained in T. Furthermore they rely on some version of the Hausdorff distance. As such they define a Polish space which thus includes "continuous" spanning trees which are spanning $\mathbb{R}^2$. These limiting points are not quite trees in the usual sense, for example there might exist points $x, y \in \mathbb{C}$ which are connected by two distinct arcs. See [AB+99].

Let us now explain how to build a tree MST_∞ out of the near-critical coupling $(\omega_\infty^{\mathsf{nc}}(\lambda))_{\lambda\in\mathbb{R}}$. In some sense we wish to use Kruskal's algorithm in the continuum. Here is intuitively how it goes:

1. We start at "$\lambda = -\infty$".

2. As λ increases, we "grow" a spanning forest depending on the way λ-clusters are merging in $\omega_\infty^{\mathsf{nc}}(\lambda)$.

Obviously, this procedure is way too naive. Here is a more realistic one:

1. First of all, Kruskal's altorithm is not suitable to an infinite setting. Therefore we first restrict ourselves to a compact case: for all $M > 0$, we define a limiting tree MST_∞^M on the torus $\mathbb{T}_M^2$ and we then let $M \to \infty$ to obtain MST_∞ (this non-trivial step uses the **quasi-local property** of MST, see [**G6**]). The choice of the torus also has the advantage to avoid technical boundary issues. In what follows, we fix $M > 0$ and with a slight abuse of notations we will still denote our processes by MST and $\omega_\infty^{\mathsf{nc}}(\lambda)$.

2. Then (in the above compact case), we obtain MST_∞ using approximated spanning trees in a similar fashion as we obtained $\omega_\infty^{\mathsf{nc}}(\lambda)$ using the cut-off processes $\omega_\infty^{\mathsf{nc},\epsilon}(\lambda)$. These approximated spanning trees are denoted $\mathsf{MST}_\infty^{\bar\lambda,\epsilon}$ where $\epsilon > 0$ and $\bar\lambda = (\lambda, \lambda')$ is a pair of near-critical parameters with $\lambda < \lambda'$. As such our approximated trees are indexed by three real parameters. We will explain below how to build the approximated spanning trees $\mathsf{MST}_\infty^{\bar\lambda,\epsilon}$.

3. These approximate spanning trees have a direct analog in the discrete $\mathsf{MST}_\eta^{\bar\lambda,\epsilon}$ and proving that $\mathsf{MST}_\eta^{\bar\lambda,\epsilon}$ tends to $\mathsf{MST}_\infty^{\bar\lambda,\epsilon}$ as $\eta \to 0$ is very analogous to the fact that $\omega_\eta^{\mathsf{nc}}(\lambda)$ was tending to $\omega_\infty^{\mathsf{nc}}(\lambda)$. There is no particular difficulty here compared to [**G5**].

4. Then we prove that as $(\lambda, \lambda', \epsilon) \to (-\infty, \infty, 0)$ (in a suitable manner), then "uniformly in the mesh η" the true MST_η is close to the approximated one $\mathsf{MST}_\eta^{\bar\lambda,\epsilon}$ with high probability. We will say a few words also on this **stability property**.

5. Using items 3 and 4, one easily obtains that $\mathsf{MST}_\infty^{\bar\lambda,\epsilon}$ as a limit as the parameters $(\lambda, \lambda', \epsilon) \to (-\infty, \infty, 0)$ in a suitable manner. This limit is unique (in law) and is denoted MST_∞. It is then straighfoward from what is above to conclude that MST_∞ is the limit in law of MST_η.

The nest two subsections are devoted to the Definition of $\mathsf{MST}_\infty^{\bar\lambda,\epsilon}$ and the stability property as $(\lambda, \lambda', \epsilon) \to (-\infty, \infty, 0)$.

3. — Construction of approximated spanning trees $\mathsf{MST}_\infty^{\bar\lambda,\epsilon}$

Let us motivate the approximation procedure in three steps:

1. Starting Kruskal's algorithm in the continuum from $\lambda = -\infty$ is certainly too degenerate. This is the reason for our first parameter $\lambda \ll -1$. Namely, we wish to start Kruskal's algorithm from level λ in order to build a **spanning tree** on the graph G_λ whose vertices are the **clusters** of $\omega_\infty^{\mathsf{nc}}(\lambda)$ and where two such clusters are connected by an edge if they share a **pivotal point**.

2. The above graph G_λ obviously has infinitely many vertices which is not very pleasant. This is the reason for our second parameter $\epsilon > 0$: let us remove from the graph all the clusters that are of diameter less than ϵ. This gives us a new graph $G_{\lambda,\epsilon}$ with finitely many vertices since we are on a compact torus.

3. The last degeneracy is due to the fact that $G_{\lambda,\epsilon}$ still has infinitely many edges (or pivotal points). This is the reason for our last parameter $\lambda' \gg 1$: we only keep pivotal points between λ-clusters whose labels are less than λ' (and are thus in $[\lambda, \lambda']$). This way we obtain an almost surely **finite** graph $G_{\lambda,\lambda',\epsilon}$.

The approximate spanning tree $\mathsf{MST}_\infty^{\bar\lambda,\epsilon}$ we wish to define is precisely the combinatorial Minimal Spanning Tree on the finite graph $G_{\lambda,\lambda',\epsilon}$. I.e. the spanning tree on the graph of λ-clusters of $\omega_\infty^{\mathsf{nc}}(\lambda)$ of diameter at least ϵ connected by pivotals points with (renormalized) labels in $[\lambda, \lambda']$.

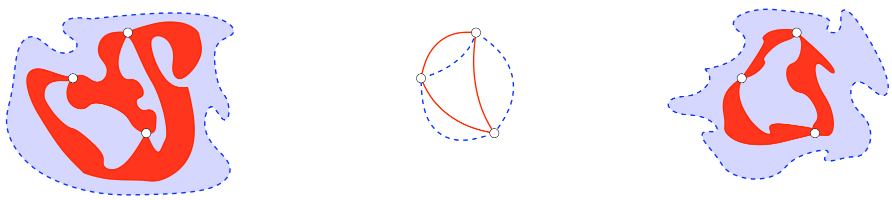


Figure 3.2: The same graph structure in a network (the middle picture) may correspond to very different cluster structures (on the two sides).

There is one important difficulty hidden in this construction, namely the **measurability** of the combinatorial structure $G_{\lambda,\lambda',\epsilon}$, given the configuration $\omega_\infty^{\mathsf{nc}}(\lambda) \in \mathscr{H}$ and the Poisson Point Process $\mathsf{PPP}(\omega_\infty^{\mathsf{nc}}(\lambda), \mu^\epsilon(\omega_\infty^{\mathsf{nc}}(\lambda)))$. Indeed, recall that the combinatorial structures used through [**G5**], i.e. the **Networks** pictured in Figure 2.8 are not necessarily planar graphs. It can be seen that they do not contain enough information to encode the graph $G_{\lambda,\lambda',\epsilon}$. For example a network in itself may completely fail to describe the structure of clusters as is shown in Figure 3.2. Because of this, we needed to add some **extra structure to our networks** in [**G6**]: we added the notion of **routers** in [**G6**] to our earlier notion of **Networks**. See [**G6**] for more details.

4. — Stability property for the approximated MST

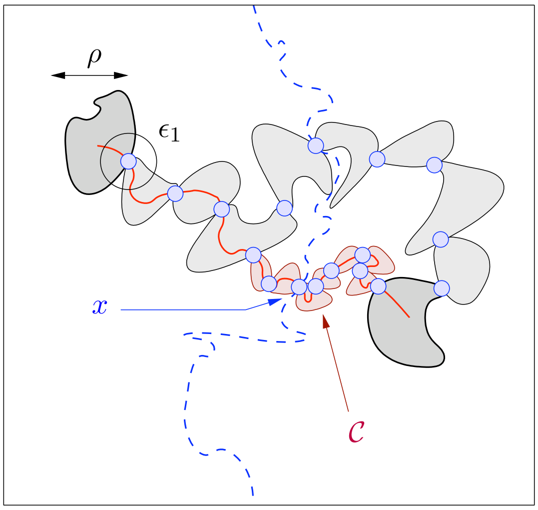


Figure 3.3: The path in MST_η connecting two large λ-clusters does not go through very small λ-clusters, basically because of the near-critical stability of 4-arm probabilities.

There are two main issues in order to show that $\mathsf{MST}_\eta \approx \mathsf{MST}_\eta^{\bar\lambda,\epsilon}$:

1. The first one is that there might exist paths in the true MST_η which would connect two large (i.e. of diameter $\geq \epsilon$) λ-clusters by going (in between) through very tiny λ-clusters as pictured in Figure 3.3. An analysis similar as what is done for the stability property in [**G5**] enables us to show that these situations are unlikely to happen.

2. A more subtle issue is due to the topology from [AB+99]: as $\eta \to 0$ the number of sub-trees in MST_η diverges (there are about η^{-4} 2-trees) while the number of sub-trees in $\mathsf{MST}_\eta^{\bar\lambda,\epsilon}$ remains tight. In the metric from [AB+99] based on the Hausdorff distance, if T and T' are two spanning trees, for ANY sub-tree in T, one needs to find a sub-tree in T' which is close. The issue in comparing MST_η with $\mathsf{MST}_\eta^{\bar\lambda,\epsilon}$ is that there are atypical leaves which are not surrounded by λ-clusters. A **ring Lemma** (which may be useful in other contexts) is proved in [**G6**] to deal with these atypical points. See Figure 3.4 for an illustration of how this "ring Lemma" is used.

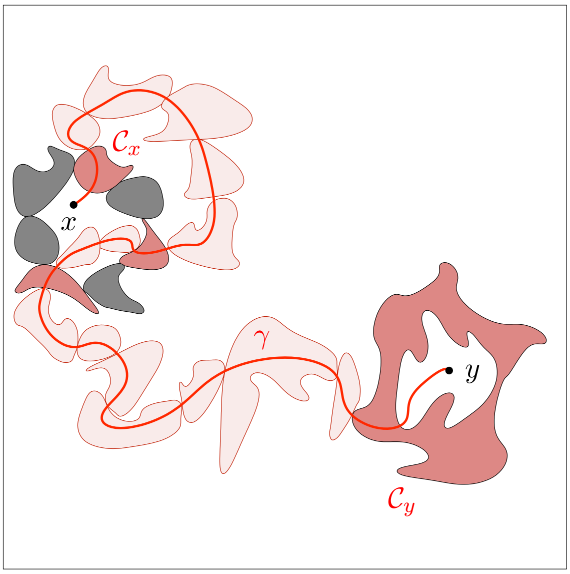


Figure 3.4: Paths in MST_η can be approximated by paths through macroscopic λ-clusters.

5. — Properties of the continuum Minimal Spanning Tree MST_∞

The proof of our main Theorem 3.1, i.e. that MST_η weakly converges to MST_∞, has the following two interesting consequences:

(i) First, it highlights that any "reasonable" model of spanning tree in the plane should have exactly the same scaling limit MST_∞. For example, if Cardy's formula was ever proved in $\mathbb{Z}^2$ for critical percolation, then it would rigorously follow from our proof that $\mathsf{MST}_\infty(\mathbb{Z}^2) = \mathsf{MST}_\infty(\mathbb{T})$. This **universality principle** for MST was of course strongly suspected but it may now be stated as a Theorem. Note that this should also be the case for the Poissonian Minimal Spanning Tree in the plane, but we do not claim that our proof would generalize to this case since a lot of techniques are based on RSW and so on.

(ii) Second, our proof provides some interesting geometric information on MST_∞ which in turn should give information on the large-scale structure of the Minimal Spanning Tree on $\mathbb{T}$.

We list below some of these almost sure properties of MST_∞ established in [**G6**]:

1. The paths in the **trunk** (see [**G6**]) of MST^∞ are all of Hausdorff dimension in $(1+\epsilon, 7/4-\epsilon)$ for some $\epsilon > 0$. (It was known previously from [AB+99] that for any subsequential scaling limits, the dimension lies in $(1+\epsilon, 2-\epsilon)$).

2. There are a.s no points in MST_∞ of degree 5. (The previous bound on the maximal degree, from [AB+99] was some non-explicit $k_0 \in \mathbb{N}$.)

3. There are no **pinching points**. See [**G6**] and figure 3.5 next page.

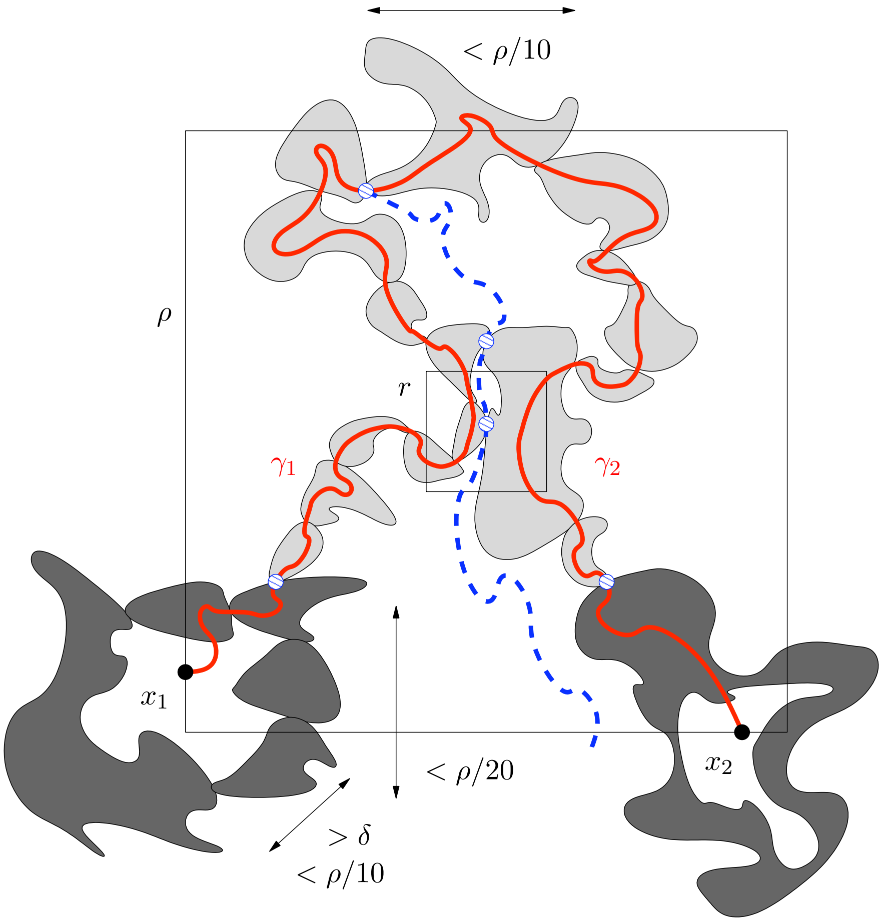
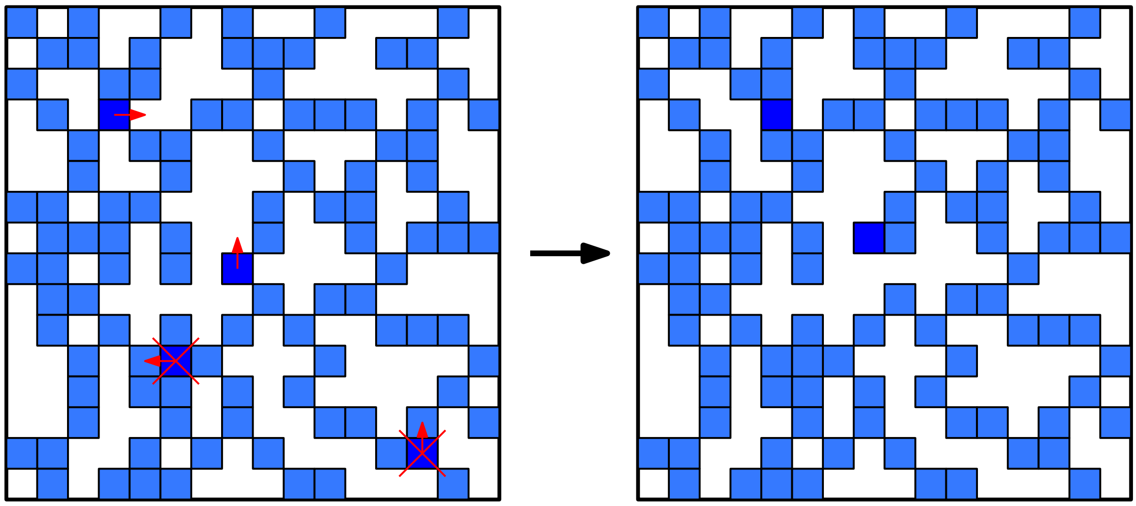


Figure 3.5: Pinching would imply a near-critical 6-arm event.

Chapter 4

Critical percolation under conservative dynamics

Based on a joint work with
Erik Broman and Jeffrey E. Steif



1. — Notion of exclusion sensitivity

Let us start by a precise definition of (standard) **noise sensitivity**. Throughout, $\{V_n\}_{n\in\mathbb{N}}$ will be an increasing sequence of finite sets and the hypercubes $\{-1,1\}^{V_n}$ will be endowed with the uniform measure $\mathbb{P}$ (one could also consider the biased measures $\mathbb{P}_p$ but for simplicity, we will stick to the uniform case $\mathbb{P}=\mathbb{P}_{1/2}$). Furthermore, $t\mapsto\omega(t)$ will denote a dynamics on the configurations in $\{-1,1\}^{V_n}$ which starts at equilibrium ($\omega(0)\sim\mathbb{P}$) and whose bits are updated at rate one. In particular, for any $t>0$, the coupling $(\omega(0),\omega(t))$ has the following law: $\omega(0)=(\omega_i)_{i\in V_n}\sim\mathbb{P}$ and $\omega(t)=(\tilde{\omega}_i)_{i\in V_n}$ where independently for each $i\in V_n$: $\tilde{\omega}_i=\omega_i$ with probability e^{-t} and otherwise $\tilde{\omega}_i$ is resampled to be 1 or -1 with equal probability $1/2$.

Definition 4.1 (goes back to [BKS99]). *A sequence of Boolean functions* $f_n : \{-1,1\}^{V_n} \to \{-1,1\}$ *is* **noise sensitive** *if for any* $\epsilon > 0$*,*

$$\mathrm{Cov}\big[f_n(\omega(0)), f_n(\omega(\epsilon))\big] \underset{n\to\infty}{\longrightarrow} 0\,.$$

The opposite notion of **noise stability** is defined as follows:

Definition 4.2 ([BKS99]). *A sequence of Boolean functions* $f_n : \{-1,1\}^{V_n} \to \{-1,1\}$ *is* **noise stable** *if for any* $\delta > 0$*, there exists* $\epsilon > 0$ *such that for any* $n \in \mathbb{N}$*:*

$$\mathbb{P}\big[f_n(\omega(0)) \neq f_n(\omega(\epsilon))\big] \leq \delta\,.$$

We now wish to introduce a different type of "noising" by letting the configuration $\omega(0) \in \{-1,1\}^{V_n}$ evolve according to some symmetric exclusion process. This latter process is encoded by a symmetric matrix $P = P_n = \{P(i,j)\}_{i,j\in V_n\times V_n}$ which prescribes the rates at which "particles" are exchanged along edges $e = \langle i,j\rangle$. The exclusion process $t \mapsto \omega^P(t)$ is defined as follows: start at $\omega^P(0) \sim \mathbb{P}$ and sample on each edge $e = \langle i,j\rangle$ an independent Poisson Point Process of rate $P(i,j) \geq 0$. Let the configuration $\omega^P(t)$ evolve as follows: when an edge $e = \langle i,j\rangle$ rings, permute the values of the bits at i and j. As such we obtain a **conservative** dynamics such that for any $t \geq 0$, $\omega^P(t) \sim \mathbb{P}$. We will assume throughout that for all $i \in V_n$, $\sum_j P(i,j) \leq 1$. This corresponds to the fact that a particle sitting at site i will not jump at a rate larger than one. Inspired by the above notion of noise sensitivity, we define:

Definition 4.3. *Let* $\{V_n, P_n\}_{n\in\mathbb{N}}$ *be an increasing sequence of finite sets with prescribed rates along the edges. A sequence of Boolean functions* $f_n : \{-1,1\}^{V_n} \to \{-1,1\}$ *is* ***P-exclusion sensitive*** *is for any* $\epsilon > 0$*:*

$$\mathrm{Cov}\big[f_n(\omega^P(0)), f_n(\omega^P(\epsilon))\big] \underset{n\to\infty}{\longrightarrow} 0\,.$$

Let us give a few examples:

Example 4.1. *Let* $V_n = [n]$ *and* $P_n \equiv 1/n$*. Then* $t \mapsto \omega^P(t)$ *is the* **complete graph exclusion dynamics** *on* $[n]$*.*

Example 4.2. *Fix* $d \geq 1$*. Let* $V_n = V_n^d := (\mathbb{Z}/n\mathbb{Z})^d$ *and* $P_n(x,y) = 1$ *if* x,y *are neighbouring sites in the torus* V_n^d *and 0 otherwise.*

One may also easily generalize the above definition to the case of an exclusion process on an infinite graph V as in the following two examples:

Example 4.3. *Fix* $d \geq 1$*. Let* $V = \mathbb{Z}^d$ *and* $P(x,y) := 1_{\|x-y\|=1}$*. In this case,* $t \mapsto \omega^P(t)$ *is well defined and corresponds to the nearest-neighbour symmetric exclusion process on* $\mathbb{Z}^d$*.*

Example 4.4. *Fix $d \geq 1$ and $\alpha > 0$. Let $V = \mathbb{Z}^d$ and $P(x,y) := \frac{1}{\|x-y\|^{d+\alpha}} 1_{x \neq y}$. (Formally the above hypothesis that $\sum_y P(\cdot, y) \leq 1$ is not satisfied here but it would be sufficient to slow down the dynamics by a uniform constant c_d). In this case, $t \mapsto \omega^P(t)$ is a* **medium-range** *symmetric exclusion dynamics on $\mathbb{Z}^d$. The larger α is, the more "localized" the exclusion process is.*

One may apply the last 3 examples to the study of the exclusion-sensitivity of critical percolation by looking at the decorrelation of the crossing events $f_n(\omega^P(t))$ defined in Definition 1.3. Let us recall our main result we wish to discuss in this Chapter:

Theorem 4.1 ([G8]). *Consider critical percolation on the triangular lattice $\mathbb{T}$ under an exclusion dynamics with symmetric Kernel defined for any $x \neq y \in \mathbb{T}$ by*

$$P(x,y) := \frac{1}{\|x-y\|^{2+\alpha}} \,,$$

for some exponent $\alpha > 0$. Then,

$$\mathrm{Cov}\big[f_n(\omega^P(0)), f_n(\omega^P(t))\big] \to 0 \,,$$

as $n \to \infty$. Furthermore, this remains true is $t = t_n \geq n^{-\beta(\alpha)}$ for some exponent $\beta(\alpha) > 0$.

2. — Some general facts about exclusion sensitivity

As we already explained in the introduction, (standard) noise sensitivity does not imply exclusion noise sensitivity. Interestingly the reverse is true, namely:

Theorem 4.2 (Theorem 1.12 in [G8]). *If $\{f_n\}_{n \in \mathbb{N}}$ is P-exclusion sensitive for some sequence of kernels $\{P_n\}$, then $\{f_n\}$ is necessarily noise sensitive.*

This result illustrates that exclusion sensitivity is in general (much) harder to obtain than classical noise sensitivity. Yet there is a particular situation for which one can prove the equivalence between the two notions:

Theorem 4.3 (Theorem 1.14 in [G8]). *If $\{f_n\}$ is a sequence of* **monotone** *Boolean functions $\{-1,1\}^{V_n} \to \{-1,1\}$, then $\{f_n\}$ is noise sensitive if and only if it is* ***complete graph noise sensitive*** *(i.e. for the complete graph exclusion dynamics introduced in Example 4.1).*

Theorem 1.14 is stated for more general functions than monotone ones. Its proof uses a coupling between an i.i.d dynamics and a complete graph exclusion dynamics. The "error" between the two dynamics is analyzed thanks to a criterion for noise sensitivity in the monotone case proved in [BKS99] which asserts that the sum of the influences squared tends to zero ($\mathrm{II}(f_n) := \sum_i \mathrm{I}_{f_n}(i)^2 \to 0$). Note that Theorem 4.1 does not follow from the above Theorem 4.3 even though the crossing events $\{f_n\}$ are monotone. This is due to the fact that the medium-range dynamics from example 4.4 is much more local than the complete-graph dynamics needed in Theorem 4.3.

At least half of [G8] focuses on the “general” theory of exclusion sensitivity of Boolean functions in the spirit of the above two results. We will not say more on this theory and will now focus on the main ideas that enabled us to analyze percolation under a medium-range dynamics (Theorem 4.1).

3. — A bit of spectral analysis of Boolean functions

Any Boolean function $f:\{-1,1\}^m \to \{-1,1\}$ can be viewed as a function in the vector space $L^2(\{-1,1\}^m)$ endowed with the scalar product

$$\langle f,g\rangle := \mathbb{E}\big[fg\big] = 2^{-m} \sum_{\omega\in\{-1,1\}^m} f(\omega)g(\omega)\,.$$

A very natural orthonormal basis of $L^2(\{-1,1\}^m)$ is given by the family of **characters** $\{\chi_S\}$ indexed by the subsets $S\subset[m]$. They are defined as follows: for each subset $S\subset[m]$:

$$\chi_S(x_1,\ldots,x_m) := \prod_{i\in S} x_i\,,$$

with the convention that $\chi_\emptyset :\equiv 1$. Notice that χ_{S_1} is clearly orthogonal to χ_{S_2} for the above scalar product when $S_1\neq S_2$. Furthermore, the number of such characters is equal to 2^m, the dimension of the space $L^2(\{-1,1\}^m)$. Therefore as in usual Fourier analysis, one can project any function $f\in L^2(\{-1,1\}^m)$ (and thus any Boolean function) on this basis. This leads us to the so-called **Fourier-Walsh** expansion of f:

$$f(\omega) = \sum_{S\subset[m]} \hat f(S)\,\chi_S(\omega)\,, \tag{4.1}$$

where the **Fourier-Walsh** coefficients (or simply the Fourier coefficients) satisfy

$$\hat f(S) := \langle f,\chi_S\rangle = \mathbb{E}\big[f(\omega)\chi_S(\omega)\big]\,. \tag{4.2}$$

Notice that $\hat f(\emptyset)$ is given by the expectation $\mathbb{E}\big[f\big]$.

The Fourier expansion of Boolean functions is particularly well-suited for the study of their noise sensitivity. Indeed, the following relation is easy to check:

$$\mathrm{Cov}\big[f(\omega(0)), f(\omega(t))\big] = \sum_{S\neq\emptyset} \hat f(S)^2 e^{-t|S|}\,. \tag{4.3}$$

Let us now introduce a notion which will be crucial later on.

Definition 4.4. *For any Boolean function $f:\{-1,1\}^m\to\{-1,1\}$, we define its* **spectral probability measure** ν_f *on the subsets of $[m]$ as follows:*

$$\nu_f(S) := \hat f(S)^2\,, \tag{4.4}$$

for any $S\subset[m]$. Note that we indeed obtain a probability measure since by Parseval, $\sum_S \hat f(S)^2 = \|f\|_2^2 = 1$ since we defined on purpose our Boolean functions with values in $\{-1,1\}$. The random subset of $[m]$, $\mathscr{S}=\mathscr{S}_f\sim\nu_f$ is called the ***spectral sample*** *of f.*

In general, it is highly non-trivial to compute or even understand precisely the spectral measure of a Boolean function f (for example if f is the crossing event of a large rectangle as in Definition 1.3, this is the focus of the works [BKS99, SchSt10, **G3**]). We refer to [**G20**] where several simple examples of Boolean functions are given where one can compute exactly the spectral measure.

Notice that one can rewrite the above correlation in terms of the spectral sample $\mathscr{S} \sim \nu_f$ as follows:

$$\mathrm{Cov}\big[f(\omega(0)), f(\omega(t))\big] = \mathbb{E}\big[e^{-t|\mathscr{S}|}\big]\,.$$

In particular, on sees that a sequence of Boolean function $\{f_n\}$ is noise sensitive if and only if the spectral measures ν_{f_n} are supported on higher and higher sets as $n \to \infty$. (In other words the spectral mass needs to diverge to infinity). As we explained in the introduction, for some applications, one needs quantitative bounds on the noise sensitivity of f_n: this corresponds to detecting at which speed the spectral mass of ν_{f_n} diverges to infinity. (See the Theorems 1.12, 1.11 and 1.12).

Let us end this section by mentioning what would have been a direct spectral approach to study the **exclusion sensitivity** of Boolean functions. As pointed out in the introduction, the characters χ_S are eigenfunctions of the **Markov semigroup** associated to the i.i.d. dynamics $\omega(t)$. The set of eigenfunctions of the Markov semigroup associated to the symmetric exclusion dynamics $t \mapsto \omega^P(t)$ is in general much harder to describe (not to mention the case of infinite volume exclusion processes in Examples 4.3 and 4.4). In the finite volume case ($|V| = m$, say), suppose $\{\phi_i\}_{1\le i \le 2^m}$ is an orthonormal family of eigenfunctions (with eigenvalues $\{\lambda_i\}$) for the Markov semigroup associated to $\omega^P(\cdot)$. Then, one has

$$\mathrm{Cov}\big[f(\omega^P(0)), f(\omega^P(t))\big] = \sum_i e^{-t\lambda_i} \langle f, \phi_i\rangle^2\,.$$

Due to the above difficulty of describing $\{\phi_i\}_i$, this expression is in general of no use.

4. — Sketch of proof of Theorem 4.1

In this last section, we wish to explain how Theorem 4.1 is proved in [**G8**]. As outlined in the above section, there is absolutely no hope to use a direct spectral approach for such medium-range dynamics. Our approach in [**G8**] consists in keeping the spectral approach which is well suited to an i.i.d dynamics, i.e. we still project our Boolean functions on the basis given by the **characters** even though the later ones are not natural from the point of view of the exclusion process. Let us analyze how the decorrelation reads on this basis:

$$\mathbb{E}\big[f(\omega^P(0)) f(\omega^P(t))\big] = \sum_{S,S'} \hat f(S)\hat f(S')\mathbb{E}\big[\chi_S(\omega^P(0))\chi_{S'}(\omega^P(t))\big] \tag{4.5}$$

It is easy to see that if $|S| \neq |S'|$, then $\mathbb{E}\big[\chi_S(\omega^P(0))\chi_{S'}(\omega^P(t))\big] = 0$. If on the other hand $|S| = |S'|$, since $\omega^P(0)$ is made of independent coin flips in $\{-1,1\}$, it is not hard to check that $\chi_S(\omega^P(0))\chi_{S'}(\omega^P(t))$ will average to 0 unless all the bits in S at time 0 will travel to the set S' by time t by following the transpositions induced by the exclusion dynamics. We call this probability $\mathbb{P}_t(S,S')$. As such, one may rewrite the decorrelation of a sequence of Boolean functions $\{f_n\}$ as follows:

$$\mathrm{Cov}\big[f_n(\omega^P(0)) f_n(\omega^P(t))\big] = \sum_{0<|S|=|S'|} \hat f_n(S)\hat f_n(S')\mathbb{P}_t(S,S') = \text{"}\Big\langle \hat f_n, \mathbb{P}_t \star \hat f_n \Big\rangle\text{"} \tag{4.6}$$

In order to prove that a sequence $\{f_n\}$ is **exclusion sensitive**, one thus needs to prove that the "vector" $\{\hat{f}_n(S)\}_{S\subset V_n}$ is asymptotically orthogonal to its convolution w.r.t $\mathbb{P}_t$: $\{\mathbb{P}_t \star \hat{f}_n(S)\}$. Unfortunately, back to our crossing events f_n (Definition 1.3), very little is known on these vectors which carry all the spectral information. A lot more is known on the vector $\{\hat{f}_n(S)^2\}_S$ which is directly related to the **spectral measure** ν_{f_n} from Definition 4.4. Indeed already theorems 1.10 and 1.11 provide informations on the typical size of the spectral sample. But in expressions such as equation (4.6), one sees that having just an information on the cardinality of $\mathscr{S} \sim \nu_{f_n}$ will not give ANY information on the exclusion sensitivity (4.6). Therefore, one needs to bring more **geometric** information on the spectral sample $\mathscr{S}_{f_n}$ into the analysis if one ever wants to obtain an exclusion sensitivity result such as Theorem 4.1. Fortunately for us, the strategy in [**G3**] in order to derive the sharp estimates in Theorem 1.12 was in some sense to study the typical **fractal geometry** of the spectral sample $\mathscr{S}_{f_n} \sim \nu_{f_n}$. It is shown in particular that $\mathscr{S}_{f_n}$ is asymptotically a random Cantor-like set of Hausdorff dimension $3/4$. In particular, it a.s. does not have any isolated points.

The key step in [**G8**] in order to prove Theorem 4.1 is the following Proposition.

Proposition 4.1. *For any medium-range exponent $\alpha > 0$ and any fixed $t > 0$: as $n \to \infty$, the probability measures ν_{f_n} and $\mathbb{P}_t \star \nu_{f_n}$ are asymptotically mutually* ***singular****. Furthermore, this remains true if $t = t_n \geq n^{-\beta(\alpha)}$.*

In other words, if one lets the spectral sample **diffuse** under the α-medium range exclusion process (even for a very little while), then it quickly looks very different from an unperturbed spectral sample! It is not hard to check (see [**G8**]) that such an asymptotic singularity (about the "vectors" $\{\hat{f}_n(S)^2\}_S$) implies that the decorrelation given by (4.6) tends to zero, which then implies Theorem 4.1.

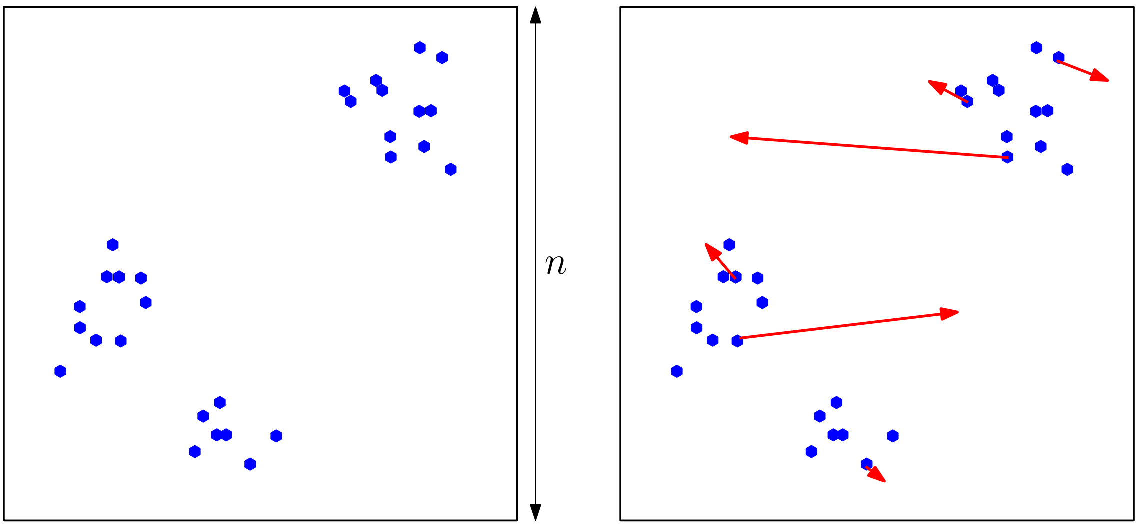


Figure 4.1: On the left, we represented what a spectral sample $\mathscr{S}_{f_n}$ of a large crossing event f_n "typically" looks like: it is large, i.e. $|\mathscr{S}_{f_n}| \approx n^{3/4}$ by [**G3**] and "Cantor-like". On the right, we illustrated how this spectral sample typically diffuses under a medium-range exclusion process with a **small** exponent α. When α gets larger, it is much harder to detect a singular behavior.

The proof of Proposition 4.1 essentially distinguishes two cases:

1. When $\alpha > 0$ is small, the exclusion dynamics is very non-local and the spectral sample $\mathscr{S}_{f_n}$ is large enough ($|\mathscr{S}_{f_n}| \approx n^{3/4}$ by [G3]) so that some of the points in $\mathscr{S}_{f_n}$ will jump macroscopically far under $\omega^P(\cdot)$. This case is easy to analyze since it is proved in [G3, G8] that it is very unlikely for the spectral sample $\mathscr{S}_{f_n} \sim \nu_{f_n}$ to have "isolated" points. Figure 4.1 illustrates how the singular behavior is established when α is small enough.

2. When α gets larger and the dynamics gets more local, it becomes more difficult to detect a singular behavior for $\mathbb{P}_t \star \nu_{f_n}$. In particular, when α is large enough, there are no isolated points anymore once the spectral sample diffuses and new ideas of events are needed. A careful "mesoscopic" analysis is provided in [G8] to show a singular behavior for arbitrary large values of α.

Remark 4.1. When "$\alpha = \infty$" which would correspond to the nearest-neighbour simple exclusion process, it becomes extremely hard to detect a singular behavior between $\mathscr{S}_{f_n}$ and $\mathbb{P}_t \star \mathscr{S}_{f_n}$. Due to this difficulty, we believe that the spectral technique used in [G8] will not lead to any result in the nearest-neighbour case. A different approach should then be used to study this interesting open problem.

Chapter **5**

Magnetization field of the critical Ising model

Based on joint works with
Federico Camia and Charles Newman

1. — Choosing the appropriate renormalization

Let us recall our definition of the renormalized magnetization field (Definition 1.5):

Definition 5.1. *For any $a > 0$, define the renormalized* **magnetization field** *to be the following random distribution on the plane:*

$$\Phi^a := a^{15/8} \sum_{x \in a\mathbb{Z}^2} \sigma_x \,,$$

where $\{\sigma_x\}_{x \in a\mathbb{Z}^2}$ is distributed according to a critical Ising model in the plane.

This definition easily extends to the magnetization field in a bounded domain Ω equipped with free *or* + *boundary conditions. Namely,*

$$\Phi^a_\Omega := a^{15/8} \sum_{x \in \Omega_a} \sigma_x \,,$$

where Ω_a is an approximation of Ω by the grid $a\mathbb{Z}^2$ (for example, the largest connected component of $\Omega \cap a\mathbb{Z}^2$).

We wish to explain in this section the reason for the renormalization in $a^{15/8}$. Let us then consider the total magnetization in the square $[0,1]^2$ defined by $m^a := \langle \Phi^a, 1_{[0,1]^2} \rangle$ in the case of the plane magnetization field $\Phi^a = \Phi^a_{\mathbb{C}}$. This random variable is $a^{15/8}$ times the unrenormalized total magnetization in $[0,1]^2$:

$$M^a := \sum_{x \in a\mathbb{Z}^2 \cap [0,1]^2} \sigma_x \,,$$

where $\{\sigma_x\}_{x \in a\mathbb{Z}^2}$ is distributed according to a critical Ising model in the plane. As in the classical CLT, the accurate renormalization is dictated by the variance of M^a which is easily expressed as follows:

$$\mathrm{Var}\big[M^a\big] = \sum_{x,y \in a\mathbb{Z}^2 \cap [0,1]^2} \mathbb{E}\big[\sigma_x \sigma_y\big] \,.$$

Now, using known results which go back to Onsager together with the RSW theorem from [DHN11], one has that $\langle \sigma_x \sigma_y \rangle \asymp \|x-y\|_2^{-1/4}$. This leads easily to $\mathrm{Var}\big[M^a\big] \asymp a^{-2}\, a^{-2}\, a^{-1/4} = a^{-15/4}$ and thus explains our renormalization in $a^{15/8}$.

Remark 5.1. Since we wish to prove a scaling limit result and not just a tightness result, an estimate up to constants such as $\langle \sigma_x \sigma_y \rangle \asymp \|x-y\|_2^{-1/4}$ is not sufficient for us. Fortunately, there is a very precise asymptotics when the sites x and y are along the diagonal which goes back to T.T. Wu (in [Wu66, MW73]). The extension to any x and y follows for example from [CHI12] or [Du11]. See the discussions on this two-point function in [**G10**, CHI12].

2. — TIGHTNESS OF $\{\Phi^a\}_a$

Notice that the above argument implies by itself the tightness of the set of random variables $\{m^a\}_{a>0}$ (defined by $m^a = \langle \Phi^a, 1_{[0,1]^2} \rangle$). Proving tightness for a family of random distributions requires some more analysis. Let us briefly explain what will be our topological setup here. We refer to Appendix A in [**G10**] for more details. See also [Du09] from which the topological setup in [**G10**] was inspired. For simplicity we will stick to the case where $\Omega = [0,1]^2$ which allows us to rely on the following orthonormal basis of the space $C_0^\infty([0,1]^2)$ endowed with the L^2 norm: for any $j,k \in \mathbb{N}^+$, let

$$\mathbf{e}_{j,k}(x,y) := 2\, \sin(j\pi x)\, \sin(k\pi y) \,. \tag{5.1}$$

Fix some $\alpha > 0$. For any $f \in C_0^\infty([0,1]^2)$ with Fourier expansion $f = \sum_{j,k>0} a_{j,k} \mathbf{e}_{j,k}$, define

$$\|f\|_{\mathcal{H}^\alpha}^2 := \sum_{j,k>0} a_{j,k}^2 (j^2+k^2)^\alpha \,.$$

The Sobolev space $\mathcal{H}_0^\alpha$ is defined as the closure of the space $C_0^\infty([0,1]^2)$ for the norm $\|\cdot\|_{\mathcal{H}^\alpha}$. It is an Hilbert space and its dual Hilbert space is the Sobolev space $\mathcal{H}^{-\alpha}$ with norm

$$\|h\|_{\mathcal{H}^{-\alpha}} := \sup_{g \in C_0^\infty([0,1]^2)\,:\, \|g\|_{\mathcal{H}^\alpha} \le 1} \langle h, g \rangle \,.$$

Furthermore, if $h \in L^2 \subset \mathcal{H}^{-\alpha}$, then h has a Fourier expansion and its $\|\cdot\|_{\mathcal{H}^{-\alpha}}$ norm can be expressed as

$$\|h\|_{\mathcal{H}^{-\alpha}}^2 = \sum_{j,k} a_{j,k}^2 \frac{1}{(j^2+k^2)^\alpha} \,. \tag{5.2}$$

For any $a > 0$, we will thus consider our magnetization field Φ^a as an element of the Polish space $\mathcal{H}^{-3}$ with operator norm $\|\cdot\|_{\mathcal{H}^{-3}}$. Since Dirac point masses do not belong to $\mathcal{H}^{-\alpha}$ for $\alpha \le 1/2$, it will be preferable to change slightly the definition of the distribution Φ^a to the following one:

Definition 5.2. *We let*

$$\Phi^a := a^{15/8} \sum_{x\in[0,1]^2\cap a\mathbb{Z}^2} \frac{\sigma_x}{a^2} 1_{S_a(x)}\,,$$

where $S_a(x)$ denotes the square centered at x of side-length a.

With this new definition, Φ^a belongs to L^2 and hence has a Fourier expansion. Using the latter expansion it is not hard to prove, as we will see below, the following Proposition:

Proposition 5.1 (Proposition A.2 in [G10]). *Consider the magnetization field Φ^a in the square domain $[0,1]^2$ with $+$ of* free *boundary conditions. Then,*

$$\limsup_{a\searrow 0} \mathbb{E}\big[\|\Phi^a\|^2_{\mathcal{H}^{-2}}\big] < \infty\,.$$

This proposition enables us to conclude that $\{\Phi^a\}_{a>0}$ is indeed **tight** in the space $\mathcal{H}^{-3}$ thanks to the classical Rellich Theorem which asserts that for any $R>0$, the ball

$$\overline{B_{\mathcal{H}^{-2}}(0,R)}$$

is compact in $\mathcal{H}^{-3}$. In particular we obtain the existence of subsequential scaling limits Φ^{a_k} to some random distribution Φ^* in the space $\mathcal{H}^{-3}$.

Proof of Proposition 5.1:
We wish to bound from above the quantity

$$\mathbb{E}\big[\|\Phi^a\|^2_{\mathcal{H}^{-2}}\big] = \sum_{j,k>0} \frac{1}{(j^2+k^2)^2} \mathbb{E}\big[\langle \Phi^a, \mathbf{e}_{j,k}\rangle^2\big]\,.$$

Hence it is enough to show that there is a constant $C>0$ such that for all $j,k>0$

$$\limsup_{a\to 0} \sup_{j,k} \mathbb{E}\big[\langle \Phi^a, \mathbf{e}_{j,k}\rangle^2\big] < C\,.$$

To see why this holds, write

$$\begin{aligned}
\mathbb{E}\big[\langle \Phi^a, \mathbf{e}_{j,k}\rangle^2\big] \leq & a^{15/4} \sum_{x\neq y\in[0,1]^2\cap a\mathbb{Z}^2} \Big| \iint_{S_a(x)\times S_a(y)} \frac{\mathbb{E}\big[\sigma_x\,\sigma_y\big]}{a^4} \mathbf{e}_{j,k}(\bar{x})\mathbf{e}_{j,k}(\bar{y}) dA(\bar{x}) dA(\bar{y}) \Big| \\
& + a^{15/4} \sum_{x\in[0,1]^2\cap a\mathbb{Z}^2} \Big(\int_{S_a(x)} \frac{1}{a^2} \mathbf{e}_{j,k}(x) dA(\bar{x}) \Big)^2\,. \\
\leq & a^{15/4} \|\mathbf{e}_{j,k}\|^2_\infty \sum_{x\neq y\in[0,1]^2\cap a\mathbb{Z}^2} |\mathbb{E}\big[\sigma_x\,\sigma_y\big]| + a^{15/4} \|\mathbf{e}_{j,k}\|^2_\infty \sum_{x\in[0,1]^2\cap a\mathbb{Z}^2} 1 \\
\leq & O(1)\,,
\end{aligned}$$

similarly as in the variance bound in section 1. □

3. — Two sketches of proofs

As explained in the introduction, two proofs of our main Theorem 1.14 are given in [G10]. (One relies on Assumption 5.1 while the second one is unconditional).

3.1. — Sketch 1: using area measures on the FK clusters

Consider a critical Ising model in $[0,1] \cap a\mathbb{Z}^2$ with free boundary conditions (+ boundary conditions would be handled similarly). Recall that one may sample the spin configuration $\{\sigma_x\}_{x\in[0,1]^2\cap a\mathbb{Z}^2}$ via the following procedure:

1. Sample a critical FK percolation ω_a in $[0,1]^2 \cap a\mathbb{Z}^2$ with free boundary conditions. See Definition 1.6.

2. Independently for each cluster (connected component) C of ω_a, sample a spin $\sigma_C \in \{\pm\}$ and for each vertex $x \in C$, declare $\sigma_x := \sigma_C$. The resulting configuration $\{\sigma_x\}_{x\in[0,1]^2\cap a\mathbb{Z}^2}$ has the desired distribution.

If $\{C_i^a\}_i$ denotes the family of all clusters of the FK configuration ω_a, one may thus rewrite the magnetization field Φ^a as the following sum of **signed area measures**:

$$\Phi^a = \sum_i \sigma_{C_i^a} \mu_{C_i^a} \,,$$

where for each cluster C_i^a, the measure $\mu_{C_i^a}$ denotes the renormalized counting measure on the cluster C_i^a, i.e.

$$\mu_{C_i^a} := \sum_{x\in C_i^a} \delta_x \, a^{15/8} \,. \tag{5.3}$$

Note that this definition of the measures $\mu_{C_i^a}$ is very similar to the **pivotal measures** (Definition 2.11) used throughout Chapter 2. Recall these later pivotal measures remained measurable in the scaling limit w.r.t. $\omega_\infty \sim \mathbb{P}_\infty$. It is thus believable that the renormalized area measures on "macroscopic" clusters should also remain measurable w.r.t a good notion of scaling limit of the FK configurations ω_a. Since ω_a naturally belongs to the Schramm-Smirnov space $\mathscr{H}$ (see Definition 2.3), one has readily subsequential scaling limits by compactness of $(\mathscr{H}, d_{\mathscr{H}})$. From the works [Sm10, CD+13], it should follow that the FK percolation ω_a has a scaling limit in $\mathscr{H}$ as $a \searrow 0$. Yet, some work is needed to prove this and it has not been done yet (this is why this first proof is a conditional one). The techniques should be somewhat similar to what was done in [CN06] for critical percolation. Finally, notice that we also need to keep track of the spins of each macroscopic cluster. There are several ways to do this. For technical reasons, the point of view in [**G10**] is to divide the FK configuration ω_a into two configurations: ω_a^+, ω_a^-, where in ω_a^+ (resp. ω_a^-), we only keep all the clusters of ω_a with a + (resp. −) spin. This way, we obtain a coupling (ω_a^-, ω_a^+) in the compact space $\mathscr{H} \times \mathscr{H}$. The first proof in [**G10**] thus relies on the following assumption:

Assumption 5.1. The FK coupling $(\omega_a^+, \omega_a^-) \in \mathscr{H} \times \mathscr{H}$ has a scaling limit as $a \searrow 0$, which we shall denote by $(\omega_\infty^+, \omega_\infty^-) \in \mathscr{H} \times \mathscr{H}$.

As discussed above, this assumption is very reasonable, based on the convergence of discrete interfaces to $SLE_{16/3}$ curves [CD+13]. However, as explained in [SchSm11], it is not always easy to go from one notion of scaling limit to another. See Section 2.2.2 in [**G10**] for a more detailed discussion on how this assumption could be proved.

Based on Assumption 5.1, the main work in [**G10**] is to prove an analog of Theorem 2.4 for the area measures μ_C (for macroscopic clusters C). The technique is similar: one approximates the true counting measures $\mu_{C_i^a}$ by using only "macroscopic" crossing informations which pass to the limit (i.e. measurable in the limit w.r.t to $(\omega_\infty^+, \omega_\infty^-)$). See Figure 2.5. In particular, as in Chapter 2, one proves a **coupling result** (as illustrated in

Figure 2.6) which decouples FK percolation conditioned on a *one-arm* event. See Lemma 2.9 in [**G10**]. Let us point out two main differences with the coupling result needed in Chapter 2.

1. First, the present situation here is simpler since we condition our FK percolations on a monotone event: the *one-arm* event. This makes things much simpler since we do not need to rely on the **separation of arms techniques** in order to deal with the possible degeneracy of four-arms events illustrated on the right of Figure 2.6.

2. On the other hand, there is an additional difficulty in the present case since one needs to handle the dependency inherent to FK percolation. Our way of handling this dependency is by using monotonicity as well as the very useful RSW Theorem from [DHN11].

Finally, similarly as in Chapter 2, the above strategy is useful only to recover the contribution of macroscopic clusters in the limiting field Φ^∞. It could well be that the main contribution would come from "microscopic" clusters in which case building such **area measures** would be useless for describing Φ^∞. Fortunately, it turns our that small clusters contribute very little in average as it is quantified by the following Proposition.

Proposition 5.2. *There is a uniform constant $C > 0$ such that uniformly in $a \to 0$,*

$$\mathbb{E}\Big[\Big(\sum_{\mathrm{diam}(C_i^a)\leq \epsilon} \sigma_{C_i^a}\, \mathrm{Area}_a(C_i^a)\Big)^2\Big] \leq C\,\epsilon^{7/4}\,,$$

where $\mathrm{Area}_a(C_i^a)$ *stands for the above renormalized area of the cluster* $\mathcal{C}_i^a$*.*

3.2. — Sketch 2: using the n-point correlation function from [CHI12]

Let Ω be a smooth bounded domain of the plane. From our tightness result obtained in Section 2, recall that there exist subsequential scaling limits $\Phi^\star = \lim \Phi^{a_n}$ for the convergence in law in the space $\mathcal{H}^{-3}$. Our second (unconditional) proof that there is a unique such subsequential scaling limit relies on the following classical fact (see for example [LT11]):

Fact 5.1. *If h is a random distribution in $\mathcal{H}^{-3}$ (for the sigma-field generated by the topology of $\|\cdot\|_{\mathcal{H}^{-3}}$), then the law of h is uniquely characterized by*

$$\phi_h(f) := \mathbb{E}\big[e^{i\,\langle h,f\rangle}\big]\,,$$

as a function of $f \in \mathcal{H}^3$.

Using the tightness property from Proposition 5.1, Theorem 1.14 thus follows from the following proposition:

Proposition 5.3. *For any $f \in \mathcal{H}^3$, the quantity*

$$\phi_{\Phi^a}(f) = \mathbb{E}\big[e^{i\,\langle \Phi^a,f\rangle}\big]$$

converges as the mesh size $a \searrow 0$.

The proof of this proposition in [G10] is divided into two main steps as follows

1. First, we show that Φ^a has "uniform exponential moments" which allows us to express its characteristic function using the series expansion

$$\phi_{\Phi^a}(f) = \mathbb{E}\big[e^{i\,\langle \Phi^a, f\rangle}\big] = 1 + \sum_{k\geq 1} \frac{i^k\,\mathbb{E}\big[\langle \Phi^a, f\rangle^k\big]}{k!}\,.$$

   Since the fact that Φ^a has exponential moments is interesting in its own and will be used at several occasions later (including in Chapter 6), we will discuss this aspect in the separate Section 6.

2. Once we have the above series expansion, it remains to compute each k^{th} moment $\mathbb{E}\big[\langle \Phi^a, f\rangle^k\big]$, i.e., to show uniqueness as $a \to 0$. For this, one uses the important results from [CHI12] which provide scaling limits of the k-point correlation functions. (Note that in the case of the plane field $\Phi^\infty_{\mathbb{C}}$, one may also rely on the results by Dubedat in [Du11]). The limit "$\mathbb{E}\big[\langle \Phi^\infty, f\rangle^k\big]$" is morally given by integrating f against these k-point functions $\phi_k(x_1, \ldots, x_k)$ except some care is needed since the results from [CHI12] hold under the condition that the points k points $x_1, \ldots, x_k$ remain at macroscopic distance (say, ϵ) from each other as $a \searrow 0$. To conclude, one thus need to show that the contribution of "local tuples" is negligible which is the purpose of the proposition below.

Proposition 5.4 (Proposition 3.9 in [G10]). *Let Ω be a domain with $+$ boundary conditions. For any $k \geq 1$, there exist constants $C_k = C_k(\Omega) < \infty$ such that, for all $0 < a < \epsilon$,*

$$\sum_{(x_1,\ldots,x_k): \inf_{i\neq j}\{|x_i - x_j|\} < \epsilon} a^{15k/8}\, \mathbb{E}\Big[\prod_1^k \sigma_{x_i}\Big] \leq C_k\, \epsilon^{7/4}\,.$$

The proof of this local control relies on the FK representation and proceeds by induction over k. See Proposition 3.9 and Lemma 3.10 in [G10].

4. — Conformal covariance

We now investigate the properties of the limiting fields we obtained. We start by the following conformal covariance structure satisfied by the magnetization fields $\Phi^\infty = \Phi^\infty_\Omega$:

Theorem 5.1 (Conformal covariance of Φ^∞). *Let $\Omega, \tilde\Omega$ be two simply connected domains of the plane (not equal to $\mathbb{C}$) and let $\phi : \Omega \to \tilde\Omega$ be a conformal map. Let $\psi = \phi^{-1}$ be the inverse conformal map from $\tilde\Omega \to \Omega$. Let Φ^∞ and $\tilde\Phi^\infty$ be the continuum magnetization fields respectively in Ω, $\tilde\Omega$. Then, the pushforward distribution $\phi * \Phi^\infty$ of the random distribution Φ^∞ has the same law as the random distribution $|\psi'|^{15/8}\tilde\Phi^\infty$, where the latter distribution is defined as*

$$\langle |\psi'|^{15/8}\tilde\Phi^\infty, \tilde f\rangle := \langle \tilde\Phi^\infty, w \mapsto |\psi'|^{15/8}(w)\tilde f(w)\rangle\,,$$

for any test function $\tilde f : \tilde\Omega \to \mathbb{C}$.

This conformal covariance property easily follows from each of the above proofs of the convergence of Φ^a to Φ^∞. Let us highlight the following interesting Corollary:

Corollary 5.2. *Let m^∞ be the scaling limit of the renormalized magnetization in the square (i.e., $m^\infty = \langle \Phi^\infty, 1_{[0,1]^2}\rangle$). For any $\lambda > 0$, let m^∞_λ be the scaling limit of the renormalized magnetization in the square $[0,\lambda]^2$. Then one has the following identity in law:*

$$m^\infty_\lambda \overset{d}{=} \lambda^{15/8}\, m^\infty\,. \tag{5.4}$$

5. — Tail behavior of the magnetization field

In this section, we wish to understand the tail behavior of Φ^∞. More precisely in the domain $\Omega = [0,1]^2$, we consider the total magnetization $m^\infty := \langle \Phi^\infty, 1_{[0,1]^2}\rangle$ and we would like to understand the asymptotics of $\mathbb{P}\big[m^\infty > x\big]$ for large values of x. Let us start with a naive analysis:

— *Heuristics* —

Suppose $x \gg 1$ is very large. From the first proof sketched above, we know that m^∞ is given by the sum of the signed (renormalized) area measures on the FK clusters of $\omega^{\mathrm{FK}}_\infty$ which we may write as follows:

$$m^\infty = \sum_{\text{clusters } C} \sigma_c \,\mathrm{area}(C)\,.$$

This expansion suggests two possible strategies that lead to high values for the total magnetization m^∞:

- Either, one biaises the spins σ_C of a large number of FK clusters and we leave the FK percolation unbiaised. Since there are few "large" clusters, one may fix some $\epsilon = \epsilon(x) > 0$ to be chosen later and decide to biais all the FK clusters whose diameter are of order ϵ. It is not hard to check that these clusters typically have renormalized area $\epsilon^{15/8}$ (notice this is consistent with Corollary 5.2). Since there are about ϵ^{-2} such clusters in the domain $[0,1]^2$, the present biasing strategy will lead to a total magnetization of order $\epsilon^{-2}\epsilon^{15/8} = \epsilon^{-1/8}$. We may such choose the scale $\epsilon = \epsilon(x) := x^{-8}$ to obtain this way a large magnetization of order x. Now, the cost of our strategy is $\mathbb{E}\big[(1/2)^{\text{number of } \epsilon\text{-clusters}}\big]$ which is of order: $(1/2)^{\epsilon^{-2}} = (1/2)^{x^{16}} = \exp(-c_1\, x^{16})$.
- A second natural strategy is to biais instead the FK percolation configuration by conditioning it on having a very large cluster C of renormalized area of order x. A similar argument based on glueing $\epsilon = \epsilon(x)$-clusters by using an ϵ-grid of RSW (from [DHN11]) also leads to a bound in $\exp(-c_2\, x^{16})$ for some different constant c_2.

These two strategies give us two different lower bounds on $\mathbb{P}\big[m^\infty > x\big]$. The optimal strategy is probably a complicated mixture of both but in any case this heuristics suggests that the tail behavior of m^∞ should behave like $e^{-cx^{16}}$ for some universal constant $c > 0$ (very likely to be stricly smaller than $c_1 \wedge c_2$).

— *Main result* —

Theorem 5.3. *There exists a universal constant $c > 0$ such that for any prescribed boundary conditions $\xi \in \{+, -, \text{free}\}$ around the square $[0,1]^2$, the (continuum) magnetization $m = m^\xi = \Phi^\xi([0,1]^2)$ in $[0,1]^2$ satisfies as $x \to \infty$:*

$$\log \mathbb{P}\big[m^\xi > x\big] \sim -c\, x^{16}\,.$$

Remark 5.2. As a Corollary, one has that Φ^∞ is not a Gaussian field. (As mentioned above, this also follows from the k-point functions obtained in [CHI12] which do not satisfy wick's formula).

Remark 5.3. Let us highlight here that the constant c does not depend on the prescribed boundary conditions ξ while the limiting magnetization random variable m^ξ does. As it will be explained below, this is the main difficulty in the proof.

— ***Sketch of proof*** —

Since it seems apriori very difficult to understand what is really the best strategy which "achieves" $\mathbb{P}\big[m^\infty > x\big]$, we rely in [**G12**] on a less explicit approach by studying the **moment generating function** of $m = m^\infty$. Namely, we prove the proposition below which then implies Theorem 5.3 by a specific **Tauberian** theorem by Kasahara (Corollary 1 in [Kas78]).

Proposition 5.5 (Proposition 2.2 in [G12]). *There exists a universal constant $b > 0$ such that for **any** boundary conditions ξ around $[0,1]^2$, one has as $t \to \infty$:*

$$\log \mathbb{E}\big[e^{t\, m^\xi}\big] \sim -b\, t^{\frac{16}{15}}$$

Sketch of proof:
The first step is to prove that the moment generating function is well-defined, i.e. that $m = m^\infty$ has **exponential moments**. Recall we already needed this property in Subsection 3.2 and this will be discussed in Section 6.

To estimate $\log \mathbb{E}\big[e^{t\,m}\big]$ for large t, we will rely on the scaling relation from Corollary 5.2: for any $\lambda > 0$, denote by m_λ^∞ the scaling limit of the renormalized magnetization field (with prescribed b.c. ξ, say) in the square $[0,\lambda]^2$. Then, we have

$$\begin{aligned}\log \mathbb{E}\big[e^{t\, m^\infty}\big] &\overset{(d)}{=} \log \mathbb{E}\big[e^{m^\infty_{\lambda_t}}\big] \\ &= \lambda_t^2 \frac{1}{\lambda_t^2} \log \mathbb{E}\big[e^{m^\infty_{\lambda_t}}\big]\,,\end{aligned} \tag{5.5}$$

where the scale λ_t needs to satisfy $\lambda_t := t^{8/15}$. As such the scaling relation from Corollary 5.2 naturally brings us to the study of a "**free energy**" for our limiting magnetization field. We state the following result as a Theorem since it is interesting on its own:

Theorem 5.4 (Existence of free energy, [G12]). *For any $L > 0$ and any boundary conditions ξ around $[0,L]^2$, let $f_L^\xi(u) := \frac{1}{L^2} \log \mathbb{E}^\xi\big[e^{u\, m_L}\big]$.*

There is a universal constant $b > 0$, which does not depend on the boundary conditions ξ, such that for any $u \in \mathbb{R}$

$$f_L^\xi(u) := \frac{1}{L^2} \log \mathbb{E}^\xi\big[e^{u\, m_L}\big] \underset{L\to\infty}{\longrightarrow} b\, |u|^{16/15}.$$

Plugging this free energy into equation (5.5) readily implies Proposition 5.5.
The proof of the Theorem 5.4 on the free energy of Φ^∞ is essentially divided as follows:

1. Using classical arguments of subadditivity (as with a discrete system), one obtains limits of $f_L^+(u)$ and $f_L^-(u)$ along dyadic scales $L = 2^k$ for boundary conditions in $\{\pm\}$.

2. The extension to a limit along $L \to \infty$ is not difficult (but a bit different form the discrete case).

3. The main difficulty compared to the classical proof in the discrete case here is to prove that the limits $f^+(u) = \lim_{L\to\infty} f_L^+(u)$ and $f^-(u)$ do coincide. (Which implies by monotonicity the independence of the free energy w.r.t boundary conditions). In the discrete, one would compare both by forcing all the spins of the boundary to be of the opposite spin and use the fact that the cost of this procedure is negligible in the exponential regime compared to the area L^2. In our continuous setting, this trick no longer applies. To overcome this, one relies on the RSW theorem from [DHN11] in **thin long tubes** along the boundary, which makes the proof significantly more technical than in the discrete case. See [**G12**].

4. Finally one concludes the proof by using the scaling law and showing that the constant b thus obtained is degenerate (equal to zero).

Remark 5.4. Let us end this section by noticing that these tail asymptotics are in very different regime from classical large deviations quantities such as $\mathbb{P}_\beta\big[\sum_{x\in\Lambda_N} \sigma_x > u\, N^2\big]$ for atypical values of u.

6. — Exponential moments for Φ^∞

In this Section we wish to sketch why Φ^∞ has exponential moments. Recall this was a crucial step in the second proof sketched in Subsection 3.2 as well in the proof of Theorem 5.3. This will also be instrumental in the study of **near-critical Ising model** in the h-direction in the next Chapter 6.

Proposition 5.6 (Proposition 3.5 and Corollary 3.8 in [G10]). *For any boundary condition ξ (either $+$, $-$ or* free *boundary conditions) around $[0,1]^2$, and for any $t \in \mathbb{R}$, if $m = m^\xi$ is the continuum magnetization of the unit square, then one has*

(i) $\mathbb{E}\big[e^{t\, m}\big] < \infty$.

(ii) Furthermore, as the mesh $a \to 0$, $\mathbb{E}\big[e^{t\, m^a}\big] \to \mathbb{E}\big[e^{t\, m}\big]$.

The proof of this proposition relies on the **GHS inequality** from [GHS70] which implies the following "convexity" property along the h-direction for the **partition function** introduced in Definition 1.4. Namely,

$$\partial_h^3 \log Z_{\beta,h} \leq 0\,,$$

which is equivalent to $\partial_h^3 \log \mathbb{E}\big[e^{h\sum \sigma_x}\big] \leq 0$. In particular, this gives us a way to bound from above the moment generating function of $\sum \sigma_x$ using ONLY the first and second moments of the magnetization at $h=0$ (the derivative of $\log Z_{\beta,h}$ at $h=0$ gives the first moment, while the second derivative gives the variance):

$$\log \mathbb{E}\big[e^{h\sum \sigma_x}\big] \leq h\,\mathbb{E}_{\beta_c,h=0}\big[\sum \sigma_x\big] + \frac{h^2}{2}\mathrm{Var}\big[\sum \sigma_x\big]\,.$$

By rescaling this estimate using $h_a := h\,a^{15/8}$ and controlling the first and second moments, it is not difficult to deduce Proposition 5.6.

7. — FOURIER TRANSFORM

We end this Chapter by the following result which shows that the random variable $m = m^\infty$ is absolutely continuous w.r.t Lebesgue measure and that its density function is very regular:

Theorem 5.5 ([G12]). *Let us consider the scaling limit $m = m^\xi$ of the magnetization in the square $[0,1]^2$ with prescribed boundary conditions $\xi \in \{+,-,\mathrm{free}\}$. There is a constant $\tilde{c} > 0$ such that for all $t \in \mathbb{R}$ one has*

$$|\mathbb{E}^\xi\big[e^{i\,t\,m}\big]| \leq e^{-\tilde{c}\,|t|^{\frac{16}{15}}}\,.$$

In particular, the density function $f = f^\xi$ of the random variable $m = m^\xi$ can be extended to an entire function on the whole complex plane $\mathbb{C}$.

Brief sketch: In our study of $|\mathbb{E}^\xi\big[e^{i\,t\,m}\big]|$, we rely on the FK representation and we prove that with very high probability (of order $1 - e^{-c|t|^{16/15}}$), one can find $O(1/\epsilon^2)$ mesoscopic squares of well-chosen size $\epsilon = \epsilon_t$ which contain an FK cluster of "mass" about $1/t$.

CHAPTER 6

Near-critical Ising model

Based on joint works with Federico Camia,
Hugo Duminil-Copin, Gábor Pete and Charles Newman

1. — Self-organized near-criticality

As we have seen in the introduction, we discovered in [G9] that the correlation length in the FK-Ising model, $L(p) = L^{\xi}_{p,\epsilon}(p)$ (see Definition 1.7) does not satisfy Kesten's scaling relation

$$L(p)^2 \alpha_4^{\mathrm{FK}}(L(p)) \asymp \frac{1}{|p - p_c|} \,. \tag{6.1}$$

This means in particular that the intuition based on picture 1.6 which leads to this scaling relation cannot possibly hold with FK-Ising percolation. Indeed, we have seen that the "true" correlation length for the FK-Ising model (in $|p - p_c(2)|^{-1}$) is much smaller than what Kesten's relation would give (i.e. in $|p - p_c(2)|^{-24/13}$). Hence when one increases p slightly above $p_c(2)$, the system needs to add new edges in a more "efficient" manner (compared to standard $q = 1$ percolation) so that the infinite cluster emerges more quickly. In [G9], we attempted to explain what is going on along the following lines:

1. First by looking at a natural **monotone coupling** of FK-Ising percolation configurations for $p \in [0, 1]$ and studying its properties.

2. Then, by mimicking Kesten's proof and looking at the appropriate notion of **influence** for FK-Ising percolation.

1.1. — Monotone couplings of FK percolation

We wish to study the FK percolation model model through its phase transition by constructing a monotone coupling of random-cluster models with fixed cluster-weight $q \geq 1$.

In the case of standard bond percolation ($q = 1$), the standard monotone coupling of percolation configurations is both straightforward to define and very useful. See for example the beginning of Subsection 1.3 in the introduction (Chapter 1). For other values of q ($q > 1$), due to the dependency structure between edges, it is harder to construct **explicit** monotone couplings. Note in contrast that the existence of such couplings follows easily from a generalized Strassen's theorem and comparison inequalities between $\mathbb{P}_{p,q}$ and $\mathbb{P}_{p',q}$ when $q \geq 1$ and $p \leq p'$. See [Gri06].

Fortunately, Grimmett constructed in [Gri95] a somewhat explicit **monotone coupling** of FK percolation configurations $(\omega_p)_{p\in[0,1]}$. His coupling is not quite explicit yet since it is defined as the invariant measure of a simple (non-local) Markov process on $[0, 1]^{E(G)}$

(See [Gri95, **G9**] for details). Yet it is already possible to extract interesting informations on this coupling. In particular, one can see that this monotone coupling differs in several essential ways from the standard monotone coupling ($q = 1$):

1. First, the **edge-intensity** has a singularity near $p_c = p_c(2)$. This means that more edges appear in the monotone coupling near the critical point. We initially believed that this could explain the braking of Kesten's scaling relation, but the blow-up of the edge-intensity is only logarithmic, i.e. $\frac{d}{dp}\mathbb{P}_p\big[e \text{ is open }\big] \asymp \log|p-p_c|^{-1}$, which cannot explain by itself a difference of exponents (1 instead of 24/13).

2. We show in [**G9**] that the monotone stochastic process $p \in [0,1] \mapsto \omega_p^{\mathrm{FK}}$ given by Grimmett's coupling is an inhomogeneous **Markov** process in the space of percolation configurations and that as p increases, **"clouds"** of several edges appear simultaneously ! (This holds for any $q > 1$).

3. Finally, the location of these clouds of edges highly depend on the current configuration ω_p which is a hint of an interesting **self-organized mechanism**.

These properties which are established in [**G9**] (besides item 1 which is harder and corresponds to the so-called **specific heat**) are not difficult to prove but show a drastic difference with the case $q = 1$. Yet, a much more detailed understanding of these emerging clouds would be needed in order to understand the near-critical geometry of FK-percolation around $p_c(q)$. This seems to be an ambitious program. For example, we couldn't even prove in [**G9**] that these clouds should be of "macroscopic" diameter near p_c and should only be "local" away from the critical point. See Questions 1 and 2 in Section 3 of [**G9**].

See also the end of Section 3 in [**G9**] for a heuristics justification of the existence of these clouds based on the Markovian property of $p \mapsto \omega_p^{\mathrm{FK}}$ together with a **storing mechanism** which takes care of the factor $q^{\sharp\text{ clusters}}$ in the partition function of FK percolation.

1.2. — Influences and Russo's formula for FK percolation

An alternative approach to a "geometric" understanding of the near-critical FK percolation model is through the study of influences. As a continuation of the work by Kesten on near-critical percolation [Ke87], Russo's formula should be replaced by the following generalization. Fix $q \geq 1$, $\epsilon > 0$ and an increasing event A. Then for any $p \in [\epsilon, 1-\epsilon]$

$$\frac{d}{dp}\phi^{\xi}_{G,p,q}(A) \asymp \sum_{e\in E} I^p_A(e), \tag{6.2}$$

where the constants in $\asymp$ depend on q and ϵ only (see [Gri06]). Above, $I^p_A(e)$ denotes the **(conditional) influence** on A of the edge $e \in E$ defined by

$$I^p_A(e) := \phi^{\xi}_{G,p,q}(A|e \text{ is open}) - \phi^{\xi}_{G,p,q}(A|e \text{ is closed}). \tag{6.3}$$

Let us analyze the consequences of this generalized Russo's formula. To avoid boundary issues, let us consider the case of the torus $\mathcal{T}_n := \mathbb{Z}^2/n\mathbb{Z}^2$, and let A_n be the event that there is an open circuit with non-trivial homotopy in $\mathcal{T}_n$. It is easy to check (by self-duality) that $\phi_{p_c,2}(A_n) \leq 1/2$. On the other hand, our main result in [**G9**] (i.e. Theorem 1.15 in the introduction) easily implies that there exists a constant $\lambda > 0$ such that if $p_n := p_c(2) + \lambda\frac{\log n}{n}$, then

$$\phi_{p_n,2}(A_n) \geq 3/4\,.$$

Integrating (6.2), this says that

$$\int_{p_c}^{p_c+\lambda\frac{\log n}{n}} I^p_{A_n}(e)dp \geq \Omega(1)\frac{1}{n^2}\,,$$

where e is any edge in $\mathcal{T}_n$. In particular, this implies that near $p_c(2)$, influences should be of order $1/n$ (See conjectures 1 and 2 in [G9]) which is much larger than the probability for an edge e to be pivotal, i.e. $\alpha_4^{\mathrm{FK}}(n) \approx n^{-35/24}$. Going back to the definition of the (conditional) influence in equation (6.3), this means that the FK percolation near p_c and conditioned on the edge e to be open is drastically different from the unconditional FK percolation.

2. — Study of the correlation length using Smirnov's observable

The previous section highlights an interesting self-organized mechanism which makes the correlation length of FK-Ising percolation much smaller than what the typical amount of pivotal points would suggest. Once we realized that the quantity $\alpha_4^{\mathrm{FK}}(R)$ would not easily lead us to an analysis of the near-critical window (due to the lack of understanding of the above mechanism), we had to rely on a completely different approach in [G9] in order to prove our Theorem 1.15. The idea is to rely on the powerful **fermionic** or **Smirnov**'s observable which we now briefly introduce.

The fermionic observable. Consider a discrete domain (Λ, a, b) with two marked points on $\partial\Lambda$. (For example the approximation of a fixed simply connected domain Ω by $\delta\mathbb{Z}^2$). If one equips (Λ, a, b) with Dobrushin's boundary conditions (i.e. wired b.c along the arc (ab) and free b.c. on (ba)), then to each FK percolation configuration sampled with these b.c., one can associate an interface γ which runs on the so-called **medial lattice** from a to b. See Figure 6.1. The observable introduced by Smirnov in [Sm10] is a function $F : E_\diamond \to \mathbb{C}$, where $E_\diamond$ is the set of edges of the medial graph. Roughly speaking, F is a modification of the probability that the exploration path passes through a given edge. More precisely, we have the following Definition:

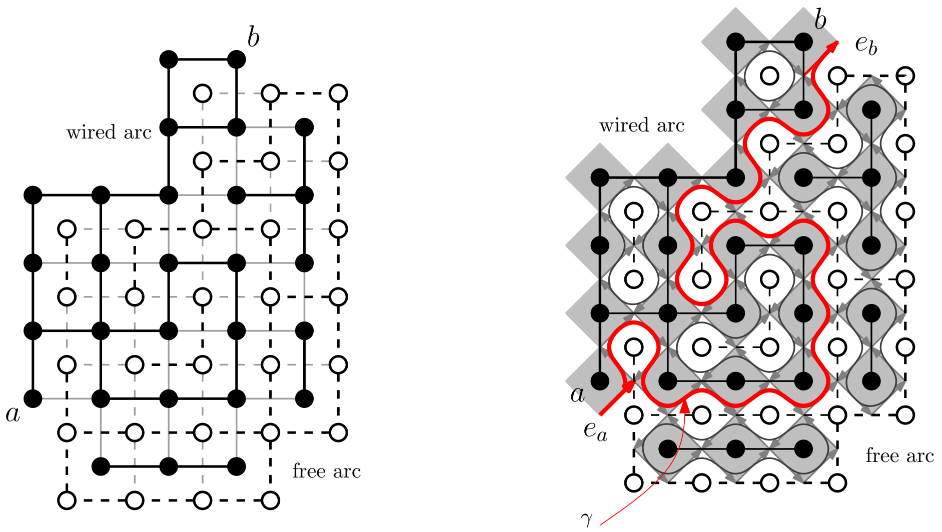


Figure 6.1: Construction of the **medial lattice** represented on the right.

Definition 6.1 (Fermionic observable introduced in [Sm10]). *We define the observable $F = F_p$ for any edge $e \in E_\diamond$ as follows*

$$F(e) = F_p(e) := \mathbb{E}^{a,b}_{p,2}\big[e^{\frac{i}{2} W_\gamma(e,e_b)} \mathbb{1}_{e\in\gamma}\big], \tag{6.4}$$

where $W_\gamma(e,e_b)$ is the winding *of the exploration path γ from the edge e to the exiting edge e_b near b. See [Sm10, DS12, BD12a] for useful references on this observable.*

Remark 6.1. Due to the winding term which averages to zero, when the edge e is in the "bulk" of the domain, one has $|F_{p_c}(e)| \ll \mathbb{P}_{p_c,2}\big[e \in \gamma\big]$. On the other hand, if e is along the boundary, then we have for any $p \in [0,1]$,

$$|F_p(e)| = \mathbb{P}_{p,2}\big[e \in \gamma\big]\,.$$

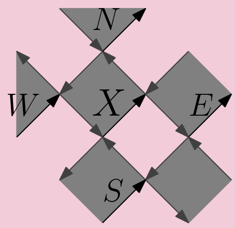
In particular, one can easily understand crossing properties of domains with Dobrushin's b.c. by estimating the Fermionic observable F. Note that this fruitful observation has already been used at many occasions since [Sm10].

One of the key results proved in [Sm10] is the following harmonicity result:

Theorem 6.1 (Exact harmonicity at criticality, Smirnov, [Sm10]).

For $q = 2$ and $p = p_c(2) = \sqrt{2}/(1+\sqrt{2})$, the fermionic observable F_{p_c} is **harmonic** *once restricted to a proper sub-lattice (North-East pointing edges), i.e.*

$$\Delta F_{p_c}(e_X) = 0$$



This Theorem was a crucial step in the proof of the conformal invariance of FK-Ising percolation in [Sm10]. Away from the critical point, the exact harmonicity of F is replaced by the following **massive harmonicity**:

Theorem 6.2 (Beffara and Duminil-Copin, [BD12a]). *When $p < p_c$, the observable F_p is now* **massive harmonic***: namely*

$$\Delta F_p(e_X) = m(p)\, F_p(e_X)\,,$$

in the "bulk" where the **mass** *$m(p)$ satisfies $m(p) \asymp |p - p_c|^2$. On the boundary, the defect of harmonicity is quite different, see [BD12a, **G9**].*

It is a classical fact that one may interpret such massive harmonic functions using **massive random walks** $(X_n^{(p)}, m_n^{(p)})$, where $X_n^{(p)}$ is a simple RW on the NE pointing edges inside the domain which is reflected on the free arc according to some explicit kernel (see [BD12a]) and which is killed on the wired arc of the Dobrushin domain. The mass $m_n^{(p)}$ decays by a factor $1/(1+m(p))$ at each step in the bulk and may behave differently (even increase) on the free arc. If τ denotes the stopping time when the Random Walk hits the wired arc, one has (see [BD12a, **G9**])

$$F_p(e) = \mathbb{E}_p^e\big[F_p(X_\tau) m_\tau\big] \,. \tag{6.5}$$

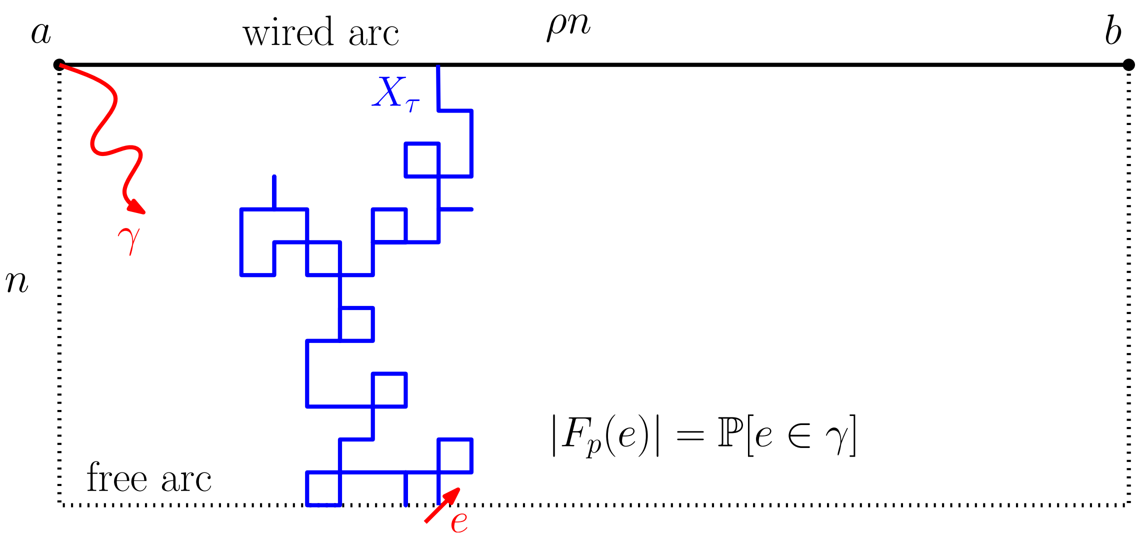


Figure 6.2: Recall the observable on the free arc gives the probability that the exploration path goes through e.

One can use this observable in a domain R_n with Dobrushin b.c. as in Figure 6.2. Recall that for any edge e on the free arc of R_n, we have

$$\mathbb{P}\big[e \in \gamma\big] = |\mathbb{E}_p^e\big[F_p(X_\tau) m_\tau^{(p)}\big]| \leq \mathbb{E}_p^e\big[m_\tau^{(p)}\big] \,.$$

For simplicity, let us neglect the possible accumulation of mass along the free arc of R_n and let us assume that $\mathbb{E}_p^e\big[m_\tau^{(p)}\big] \lesssim \mathbb{E}_e\Big[\Big(\frac{1}{1+m(p)}\Big)^\tau\Big]$ (This can be easily justified in the near-critical window we are interested in, see [**G9**]). Since $m(p) \asymp (p - p_c)^2$, and since the "reflected" RW on the free arc takes about n^2 steps to reach the wired arc of R_n, one obtains on a very heuristical level that

$$\mathbb{E}_p^e\big[m_\tau^{(p)}\big] \lesssim e^{-n^2(p-p_c)^2} \,,$$

which explains why something changes near $n \approx \frac{1}{|p-p_c|}$. Let us now detail a little bit more how the proof of Theorem 1.15 goes:

1. First of all, the "analysis" above holds for $p < p_c$. In order to prove an **upper bound** on the correlation length $L^\xi(p')_{\rho,\epsilon}$ with $\xi =$ Dobrushin b.c. but reversed w.r.t to Figure 6.2 and $p' > p_c$, we take $p = (p')^*$, the dual value of p' (see [DS12] for the duality in FK percolation) and we wish to show that it is very likely for R_n to be traversed from left to right. By duality, this is equivalent to showing that it is very unlikely for the dual percolation to traverse from top to bottom with this time the same Dobrushin b.c. as in Figure 6.2. Now, if $p' > p_c + \lambda \frac{\log n}{n}$, and λ is large enough, then $p = (p')^* < p_c - \tilde\lambda \frac{\log n}{n}$ and it can be shown ([**G9**]) that for any e on the bottom arc, $|F_p(e)| \leq 1/n^4$. Summing over all such edges implies the result.

2. To obtain a lower bound on the correlation length, one proves that if p' (and thus p) is close enough to p_c (in $1/n$), then $|F_p(e)|$ is close to $|F_{p_c}(e)|$ which is known to be of order $1/\sqrt{n}$. Using readily the second moment argument from [DHN11] enables us to conclude.

3. The above arguments give upper and lower bound as in Theorem 1.15, but for a **very particular boundary condition** ξ around R_n! Since one needs (for potential applications) correlation lengths estimates for the worse possible b.c. a non-trivial work is still needed. This is probably the main technical contribution of [**G9**], i.e. to deal with boundary conditions that are not suitable to a direct study using Dobrushin b.c., interfaces, fermionic observables and so on. See [**G9**].

3. — Heat-bath dynamics on FK percolation

Similarly to dynamical percolation, there is a natural dynamics with the random-cluster model $\mathrm{FK}(p,q)$ as stationary distribution, called the **heat-bath dynamics** or **Sweeny algorithm**: edges have independent exponential clocks, and when the clock of $e = \langle x, y\rangle$ rings, the state of e is updated according to the $\mathrm{FK}(p,q)$ measure conditioned on the rest of the configuration ω, of which the only relevant information is whether x and y are connected in $\omega \setminus \{e\}$. The conditional law can be written as follows:

$$\mathbb{P}_{p,q}\big[e \text{ is open } \big| \, \omega \text{ on } G\setminus\{e\}\big] = \begin{cases} p & , \ \text{if } x \overset{\omega}{\longleftrightarrow} y \text{ in } G\setminus\{e\} \\ \frac{p}{p+(1-p)q} & , \ \text{otherwise}. \end{cases}$$

This dynamics has been studied both for theoretical reasons (see [Gri95, Gri06]) and practical ones (see [DGS07] for a good account of recent works). The extension to infinite volume dynamics is non-trivial (due to the non-local aspect of the dynamics). See [Gri95].

As we have seen in the first section, pivotal points are somewhat misleading if one wants to understand the near-critical behavior of FK-Ising percolation. Nevertheless, it turns out that for the above heat-bath dynamics, they seem to be the accurate quantity to look at. For example, they do dictate (as in the case $q = 1$) at which speed the heat-bath dynamics starts decorrelating (work in progress [**G16**]). Let us point out though that we are far from being able to prove a total decorrelation on this time-scale as in [**G3**] for $q = 1$ or even a noise sensitivity result as in [BKS99]. It turns out that one can also use the quantity $\alpha_4^{\mathrm{FK}}(R)$ from [DG] in order to give an upper bound on the possible set of exceptional times where an infinite cluster appears along the above heat-bath dynamics.

Let us recall how this works in the case $q = 1$: run dynamical percolation on the triangular lattice $\mathbb{T}$ for a time one. We wish to find an upper bound on the number of ϵ intervals which contain an exceptional time where the origin 0 is connected to ∞. For this, divide $[0,1]$ into $1/\epsilon$ intervals of length ϵ. The probability that each of these intervals contains an infinite cluster is bounded from above by the probability that the origin is connected to infinity for a slightly supercritical percolation $p = p_c + \epsilon$. As such the average number of ϵ-intervals is bounded by $\epsilon^{-1}\theta(p_c+\epsilon) = \epsilon^{-1}\epsilon^{5/36+o(1)} = \epsilon^{-31/36+o(1)}$.

Back to the case of FK-Ising percolation, one may thus expect an upper bound of the form $\epsilon^{-1}\theta^{\mathrm{FK}}(p_c+\epsilon)$. But this is not at all the case! The dynamics $\omega^{\mathrm{FK}}(t)$ on a small interval of length ϵ is no longer dominated by a slightly supercritical FK-Ising percolation but rather by a critical configuration which is thickened by a **Poissonian** cloud of new edges. This way the naive analysis performed in Section 1 which failed for the near-critical window is now accurate in this setting ! This gives rigorously a **dynamical correlation length** of order $L^{\mathrm{dyn}}(p) = |p-p_c|^{-24/13+o(1)}$ (based on [DG, **G16**]). Using this correlation length together with the one-arm exponent $\alpha_1^{\mathrm{FK}}(R) \asymp R^{-1/8}$ (as in the computation leading to $\theta_{\mathbb{T}}(p)$) one obtains in [**G9**] a rigorous upper bound on the set of exceptional times for the critical beta-bath dynamics ($q = 2$) equal to $10/13$.

This analysis is rigorous only for FK-Ising percolation ($q = 2$) but it can be extrapolated using the conjectural values of the critical exponents for $q \neq 1$. This leads us to the following interesting phenomenon mentioned in the introduction:

Conjecture 1. *There exist "natural" critical two-dimensional systems (i.e., scale-invariant and so on) with the property that they have pivotal points at all scales (hence are expected to be noise sensitive), but for which there are no exceptional times of infinite clusters along the natural heat-bath dynamics (hence are not dynamically sensitive).*

This conjecture can be made more precise in our context:

Conjecture 2 ([G16]). *Let*

$$q^* := 4\cos^2(\frac{\pi}{4}\sqrt{14}) \approx 3.83.$$

- *For* $q \in (q^*, 4]$, *critical* FK_q *percolation is NOT dynamically sensitive.*
- *If* $q \in [1, q^*]$, *one has a.s.*

$$\dim(\mathcal{E}_q) \le \frac{1 - 8u(q) + 2u(q)^2}{3u(q)^2 - 8u(q)}, \tag{6.6}$$

where, $u(q) := \frac{2}{\pi}\arccos(\frac{\sqrt{q}}{2}) = 2 - \frac{8}{\kappa(q)}$.

We may thus conclude that Kesten's scaling relation $L^2(p)\alpha_4(L(p)) \asymp |p - p_c|^{-1}$ is not meaningful for the near-critical regime but is very accurate in the dynamical setting: it gives the correct notion of **dynamical correlation length**.

4. — Near-critical Ising model with vanishing exterior magnetic field

All the results so far in this Chapter concerned near-critical Ising model when one varies the temperature (i.e. FK-Ising percolation when one varies p near p_c). We focus now on the Ising model at $\beta = \beta_c$ with some small external magnetic field $h > 0$. We mentioned two results in the introduction which are of different flavors (Theorems 1.16 and 1.17). We will only describe here the technology behind the first result. Let us recall the statement of our main result in [**G11**]:

Theorem 6.3. *Consider the Ising model on* $\mathbb{Z}^2$ *at* β_c *with a positive external magnetic field* $h > 0$, *then*

$$\langle \sigma_0 \rangle_{\beta_c, h} \asymp h^{\frac{1}{15}}.$$

We highlight below how the proof works.

The Lower bound. The proof is rather classical: one introduces a correlation length $L(h) \asymp h^{-8/15}$ and we prove that clusters of diameter $\asymp L(h)$ are likely to be connected to the **ghost** vertex in the FK representation of the Ising model with exterior magnetic field (see also for example [Gri67]). This property is the main part of the proof (one needs to show that such clusters are unlikely to be too thin). Then, by monotonicty, the origin is connected to these large clusters with probability at least $\alpha_1^{\mathrm{FK}}(L(p))$, which gives the desired lower bound.

The Upper bound. The upper bound is usually what is highly non-trivial to prove in such near-critical results, for example in the proof that $\theta_{\mathbb{T}}(p) = (p - p_c)^{5/36+o(1)}$ ([Ke87, SW01]). Surprisingly, by using the special "convexity" properties satisfied by the Ising model, in particular the GHS inequality, one manages to control the near-critical effect induced by $h > 0$ in a rather simple manner. Here is roughly how it goes. First one reduces the problem to a finite domain $\Lambda_L := [-L, L]^2$ with $+$ boundary conditions as follows. First, by translation invariance, one has for any $L \geq 1$,

$$\langle \sigma_0 \rangle_{\beta_c,h} = \frac{1}{|\Lambda_L|} \langle M_L^{\mathbb{Z}^2} \rangle_{\beta_c,h} \,,$$

where $M_L := \sum_{x \in \Lambda_L} \sigma_x$ is the total magnetization in Λ_L. Then, by monotonicity one has, for any $L \geq 1$,

$$\langle \sigma_0 \rangle_{\beta_c,h} \leq \frac{1}{|\Lambda_L|} \langle M_L \rangle_{\beta_c,h,+} \tag{6.7}$$

where $\langle M_L \rangle_{\beta_c,h,+}$ is the average total magnetization in Λ_L with $+$ boundary conditions. The main idea in the proof of the upper bound is to rewrite the expected magnetization $\langle M_L \rangle_{\beta_c,h,+}$ as follows:

$$\langle M_L \rangle_{\beta_c,h,+} = \frac{\langle M_L \, e^{hM_L} \rangle_{\beta_c,0,+}}{\langle e^{hM_L} \rangle_{\beta_c,0,+}} = \frac{\frac{\partial}{\partial h} \langle e^{hM_L} \rangle_{\beta_c,0,+}}{\langle e^{hM_L} \rangle_{\beta_c,0,+}} \,,$$

and then to apply the GHS inequality. Indeed the latter (see [GHS70]) says that, for $+$ boundary conditions,

$$\begin{aligned}
& \frac{\partial^3}{\partial h^3} \log\Big(\sum_\sigma e^{-\beta_c E_L(\sigma) + hM_L(\sigma)} \Big) && \leq 0 \\
\Leftrightarrow \qquad & \frac{\partial^3}{\partial h^3} \log\Big(\frac{\sum e^{-\beta_c E_L + hM_L}}{\sum e^{-\beta_c E_L}} \Big) && \leq 0 \\
\Leftrightarrow \qquad & \frac{\partial^2}{\partial h^2} \Big(\frac{\frac{\partial}{\partial h} \langle e^{hM_L} \rangle_{\beta_c,0,+}}{\langle e^{hM_L} \rangle_{\beta_c,0,+}} \Big) && \leq 0 \,.
\end{aligned}$$

Let $F(h) = F_L(h) := \frac{\frac{\partial}{\partial h} \langle e^{hM_L} \rangle_{\beta_c,0,+}}{\langle e^{hM_L} \rangle_{\beta_c,0,+}} = \langle M_L \rangle_{\beta_c,h,+}$. Then one has for any $h \geq 0$:

$$\begin{aligned}
F(h) &\leq F(0) + h\, F'(0) \\
&= \langle M_L \rangle_{\beta_c,0,+} + h\big(\langle M_L^2 \rangle_{\beta_c,0,+} - \langle M_L \rangle_{\beta_c,0,+}^2 \big) \,.
\end{aligned} \tag{6.8}$$

We learned afterwards that an inequality similar to (6.8) was already used in [FFS92] (see Eq. (14.230), p. 345) to obtain a lower bound for the spontaneous magnetization, leading to another inequality involving critical exponents.

Now, using the estimates on the first and second moment of the total magnetization M_L at $\beta = \beta_c$ obtained in [**G10**] (see Proposition B.2 in [**G10**] whose proof relies essentially on the RSW theorem for Ising-FK percolation proved in [DHN11]), one concludes the proof of the upper bound by optimizing in $L = L(h)$. Let us point our that the optimal scale thus obtained also behaves like $L(h) \asymp h^{-8/15}$ as in the lower bound. □

CHAPTER 7

Coalescing flows of Brownian motion: a new perspective

Based on joint works with
Nathanael Berestycki and Arnab Sen

1. — THE SPACE OF COALESCING FLOWS

Inspired by the Schramm-Smirnov space $\mathscr{H}$, we wish to represent a coalescing flow ξ by the set of all "tubes" which are traversed by at least one particle. In principle, one could even keep the setup used for percolation, i.e. the space $\mathscr{H}$ itself, but the **quads** which are not aligned along the time axis would not be natural in this case and would require some extra analysis. This is why we adapt slightly the setup from [SchSm11] (see also the presentation in Chapter 2) and we replace the space of quads $\mathcal{Q}$ by the following space of **tubes** $\mathcal{T}$. Since our aim is to analyze cowlescing flows on fractal structures which may be d-dimensional, we define our tubes in dimension $d+1$, for any $d \geq 1$.

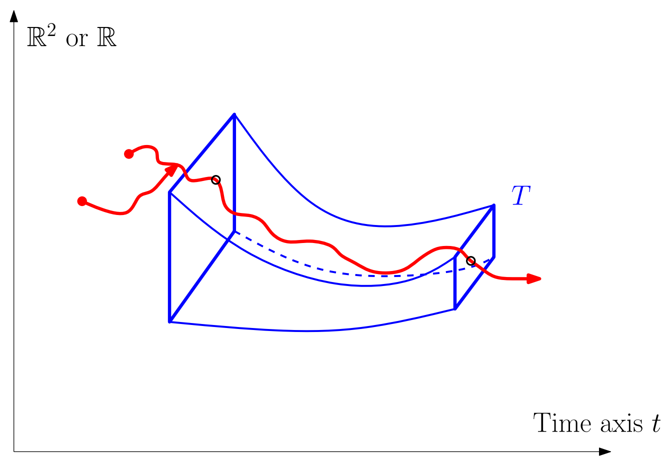


Figure 7.1:

Definition 7.1 (The space of tubes $\mathcal{T}$, [G15]). *A* **tube** T *is a triplet* $([T], \partial_0 T, \partial_1 T) := \big(\varphi([0,1]^{d+1}), \varphi([0,1]^d \times \{0\}), \varphi([0,1]^d \times \{1\})\big)$ *where* $\varphi : [0,1]^d \times [0,1] \to \mathbb{R}^d \times \mathbb{R}$ *is a* *homeomorphism such that* $\varphi([0,1]^d \times \{0\})$ *and* $\varphi([0,1]^d \times \{1\})$ *are subsets of* $\mathbb{R}^d \times \{t_0\}$ *and* $\mathbb{R}^d \times \{t_1\}$ *respectively for some* $t_0 < t_1$. *Furthermore, we require that* $[T] = \phi([0,1]^{d+1})$ *is included in* $\mathbb{R}^d \times [t_0, t_1]$. *Informally,* T *is a topological cube along with a distinct pair of opposite faces which are both orthogonal in* $\mathbb{R}^{d+1}$ *to the time axis. We call* t_0 *the* start time *of* T *and* t_1 *the* end time *of* T. *The sets* $\partial_0 T$ *and* $\partial_1 T$ *are called the* **left face** *and the* **right face** *of* T *respectively. See figure 7.1.*

The space of all tubes is denoted by $\mathcal{T}$ *and is equipped with the following metric:*

$$d_{\mathcal{T}}(T_1, T_2) := d_{\text{Haus}}([T_1], [T_2]) + d_{\text{Haus}}(\partial_0 T_1, \partial_0 T_2) + d_{\text{Haus}}(\partial_1 T_1, \partial_1 T_2),$$

where d_{Haus} *is the usual Hausdorff metric on the compact subsets of* $\mathbb{R}^{d+1}$. *It is easy to see that the space* $(\mathcal{T}, d_{\mathcal{T}})$ *is separable.*

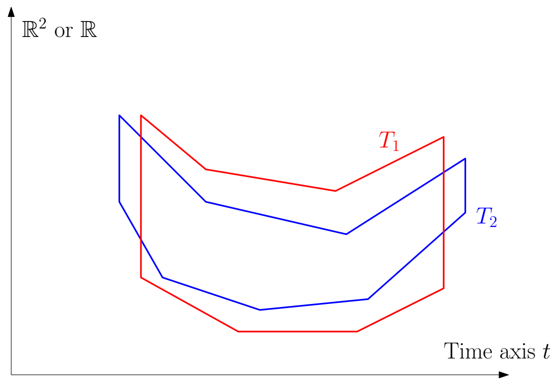


Figure 7.2:

Given a discrete coalescing flow ξ and a tube T, we say that the tube T is **crossed** or **traversed** by ξ if one can find a particle in ξ which enters the tube through $\partial_0 T$ and remains inside $[T]$ until it leaves the tubes through $\partial_1 T$. From this point of view of crossings, there is a natural partial order on $\mathcal{T}$: we write $T_1 \leq T_2$ if any crossing of T_2 contains a crossing of T_1. See Figure 7.2. One may thus proceed exactly as with the Schramm-Smirnov space and define a subset $S \subset \mathcal{T}$ to be **hereditary** if whenever $T \in S$ and $T' \in \mathcal{Q}$ satisfies $T' \leq T$, we also have $T' \in S$. This leads us to the analog of the space $\mathscr{H}$:

Definition 7.2 (The space of coalescing flows $\mathscr{C}$). *Let* $\mathscr{C}$ *be the space of all* **closed** *hereditary subsets of* $\mathcal{T}$. *We call* $\mathscr{C}$ *the* **Schramm–Smirnov space of coalescing flows** *or simply the* **space of coalescing flows**.

The discrete coalescing flows ξ_η or ξ_η^G as defined in the Introduction are naturally associated to random elements in $\mathscr{C}$ (with $d = 1$ and $d = 2$). With a slight abuse of notation, we will still denote these by ξ_η and ξ_η^G and we will denote by $\mathbb{P}_\eta$ and $\mathbb{P}_\eta^G$ the probability measures on $\mathscr{C}$ induced by these coalescing flows. We still need to precise what the topology on $\mathscr{C}$ is. Exactly as with the space $\mathscr{H}$, we define the following subsets of $\mathscr{C}$:

(i) For any tube $T \in \mathcal{T}$, let $\boxminus_T := \{\xi \in \mathscr{C} : T \in \xi\}$.

(ii) For any open $U \subset \mathscr{C}$, let $\boxdot_U := \{\xi \in \mathscr{C} : \xi \cap U = \emptyset\}$.

And we define the following analog of the **quad-crossing topology** from Definition 2.4:

Definition 7.3 (The tube-topology [G15]). *We define the* **tube-topology** *to be the minimal topology on $\mathscr{C}$ that contains every $\boxminus_T^c$ and $\boxdot_U^c$ as open sets.*

Finally, Theorem 2.1 for $\mathscr{H}$ is replaced here by:

Theorem 7.1. *The space of coalescing flows $\mathscr{C}$ equipped with the tube-topology has the following properties:*

1. *$\mathscr{C}$ is separable, metrizable and hence a Polish space. We will fix a metric $d_{\mathscr{C}}$ compatible with the tube-topology.*
2. *$\mathscr{C}$ is compact.*
3. *For any dense set $\mathcal{S}$ of $\mathcal{T}$, the σ-field generated by $\{\boxminus_T : T \in \mathcal{S}\}$ is the Borel σ-field on $\mathscr{C}$.*

2. — Construction of the Brownian coalescing flow in $\mathscr{C}$

In this section, we wish to construct a universal scaling limit for coalescing flows on the real-line $\mathbb{R}$. We will denote it by $\xi_\infty \sim \mathbb{P}_\infty$. What we wish to build is the analog of the continuous percolation $\omega_\infty \sim \mathbb{P}_\infty$ from Theorem 2.2 which was a very non-trivial result since it relied on the Cardy's formula proved in [Sm01] as well as on results from [CN06] which handle joint laws for multiple quads. The approach here will be easier and more hands-on since it is based essentially on the Brownian motion. Here is briefly how one proceeds in [**G15**]:

(A) First, for any countable dense set $\mathcal{D} = \{(x_i, t_i)\}_{i\in\mathbb{N}}$ in $\mathbb{R}^{1+1}$, we consider for each $n \geq 1$ the coalescing flow of Brownian motions $\xi_n^{\mathcal{D}}$ which start from the space-time points $\{(x_1, t_1), \ldots, (x_n, t_n)\}$. Clearly, the flows $\xi_n^{\mathcal{D}}$ can be coupled in an "increasing" manner in the space $\mathscr{C}$ and it is not hard to show that $\xi_n^{\mathcal{D}}$ converges in law as $n \to \infty$ to a limiting object $\xi_\infty^{\mathcal{D}} \sim \mathbb{P}_\infty^{\mathcal{D}}$.

(B) In order to obtain a natural universal flow, one needs to check that the limiting flow obtained above does not depend on the choice of the countable set $\mathcal{D}$, i.e. that $\mathbb{P}_\infty^{\mathcal{D}} = \mathbb{P}_\infty^{\mathcal{D}'}$. To show this, we introduce a "nice" dense family of tubes $\hat{\mathcal{T}}$ (which is adapted to the coalescing flow one considers, for the present $\mathbb{R}$-flow, these are "rectangular" tubes, see [**G15**]) and we prove that for all $T_1, \ldots, T_k$ in $\hat{\mathcal{T}}$, one has

$$\lim_{n\to\infty} \mathbb{P}\big[\xi_n^{\mathcal{D}} \in \boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k}\big] = \lim_{n\to\infty} \mathbb{P}\big[\xi_n^{\mathcal{D}'} \in \boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k}\big]$$

As such one obtains a universal measure $\mathbb{P}_\infty$ on the space $\mathscr{C}$. This step is not difficult. The basic idea is that for each $n \geq 1$, one can choose N large enough so that among $\{(x_1', t_1', \ldots, (x_N', t_N')\}$ in $\mathcal{D}'$, one can find n space-time points which approximate well $\{(x_1, t_1), \ldots, (x_n, t_n)\}$ and then we couple the Brownian motion from $\mathcal{D}$ and $\mathcal{D}'$ so that they quickly coalesce together. To avoid unnecessary technicalities near the boundaries of the tubes, we only need to analyse what happens for tubes in the above well-chosen dense family $\hat{\mathcal{T}}$. This gives us intrinsic quantities

$$p(T_1, \dots, T_k) := \lim_{n\to\infty} \mathbb{P}\big[\xi_n^{\mathcal{D}} \in \boxminus_{T_1} \cap \dots \cap \boxminus_{T_k}\big]\,,$$

for any k-tuple of tubes in $\hat{\mathcal{T}}$.

(C) At this point, it is tempting to characterize our limiting measure as follows:

$$\mathbb{P}_\infty\big[\boxminus_{T_1} \cap \dots \cap \boxminus_{T_k}\big] := p(T_1, \dots, T_k)\,, \tag{7.1}$$

for any $T_1, \dots, T_k \in \hat{\mathcal{T}}$. By Theorem 7.1, this would indeed characterise the measure $\mathbb{P}_\infty$ on $\mathscr{C}$. But it turns out that the equality (7.1) is substantially harder to show than Items (A) and (B). Indeed, since $\boxminus_{T_1} \cap \dots \cap \boxminus_{T_k}$ is a closed set, it is obvious that

$$\mathbb{P}_\infty\big[\boxminus_{T_1} \cap \dots \cap \boxminus_{T_k}\big] \geq p(T_1, \dots, T_k)\,.$$

Proving the other inequality requires a non-trivial amount of work. This is the content of the Lemma below which is an analog of Theorem 2.3. (It is straightforward to check that it is enough to focus on a single tube). We state it in a different way as in [**G15**] in order to highlight more the similarity with Theorem 2.3 for critical percolation.

Lemma 7.1 (Theorem 3.4 in [G15]). *For any tube T in the nice family of rectangular tubes $\hat{\mathcal{T}}$, one has*

$$\mathbb{P}_\infty\big[\partial \boxminus_T\big] = 0\,.$$

The proof in [SchSm11] of Theorem 2.3 relies mostly on the three-arms events to show that crossings are unlikely to come too close to the boundary. In our present case of coalescing flows, the situation is less pleasant than with critical percolation since what happens close to $\partial_0 T$ is very different from what happens near $\partial_1 T$. For a given tube $T \in \hat{\mathcal{T}}$, we need to prove that one can find a tube T^δ which is "strictly" larger than T but which is such that $\mathbb{P}_\infty\big[\boxminus_{T^\delta}\big] \approx \mathbb{P}_\infty\big[\boxminus_T\big]$. See Figure 7.3 where we illustrate how the proof goes in [**G15**]. The main ingredient is a **coming down from infinity argument** which is easy to establish in the case of the real-line (this was already present in Arratia's work [Arr79]).

Let us summarize the above results into the following theorem.

Theorem 7.2. *There is a natural random coalescing flow on the real-line $\mathbb{R}$, $\xi_\infty \sim \mathbb{P}_\infty$, whose law $\mathbb{P}_\infty$ is characterized by*

$$\mathbb{P}_\infty\big[\boxminus_{T_1} \cap \dots \cap \boxminus_{T_k}\big] := \sup_{n\geq 1; z_1,\dots,z_n \in \mathbb{R}^2} \mathbb{P}\big[\xi(z_1, \dots, z_n) \in \boxminus_{T_1} \cap \dots \cap \boxminus_{T_k}\big]\,,$$

*for any k-tuple of "nice" tubes $T_1, \dots, T_k \in \hat{\mathcal{T}}$ and where $\xi(z_1, \dots, z_n)$ denotes a system of n coalescing Brownian motions starting from $(z_1, \dots, z_n)$. (See Definition 2.14 in [**G15**] for the choice of the dense family of tubes $\hat{\mathcal{T}}$).*

Remark 7.1. Note that we don't have such a characterization of the scaling limit of critical percolation $\omega_\infty \sim \mathbb{P}_\infty$ since the latter one is only characterized as the unique scaling limit of $\omega_\eta \sim \mathbb{P}_\eta$. (See Theorem 2.2). We proceed differently here by first building an object in the continuum and then proving an **invariance principle**. This leads us to the next section.

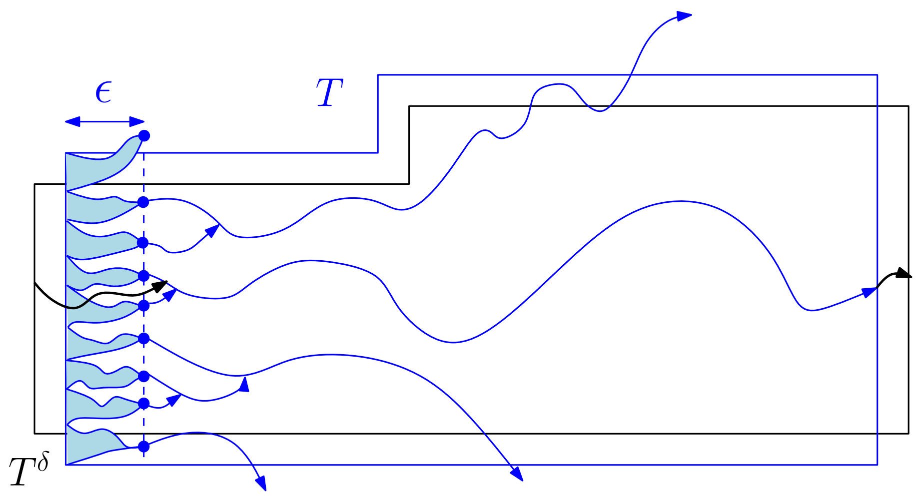


Figure 7.3: For a given "rectangular" tube T in $\hat{\mathcal{T}}$, we choose a slightly larger tube T^δ. Then, we prove that if T is traversed by a particle, it is very likely that the tube T^δ is traversed as well. For this we use the fact that at a time ϵ after $\partial_0 T$, there are finitely many particle still alive in T. This is a **coming down from infinity** type of result which is easy in this one-dimensional case. These particles are dense enough that they are likely to coalesce with a particle coming from $\partial_0 T^\delta$. Some surgery is still needed near the boundary of T. For more details, see the proof in Section 3.2 in [**G15**].

3. — Invariance principle for coalescing random walks on $\mathbb{Z}$

Recall from the introduction that we are interested in a system $\xi_\eta \in \mathscr{C}$ of independent coalescing random walks started from every space-time point (x,t) of the rescaled lattice $\mathbb{L}_\eta := \sigma^{-1}\eta\mathbb{Z} \times \eta^2\mathbb{Z}$. We assumed furthermore that the step distribution μ (of the non-rescaled step distribution) satisfies

$$\mathbb{E}[\mu] = 0, \quad \mathbb{E}[\mu^2] = \sigma^2 < \infty \quad \text{and } \mu \textbf{ is aperiodic}. \tag{7.2}$$

We recall below our main result about coalescing flows on $\mathbb{R}$:

Theorem 7.3 (Theorem 4.1 in [G15]). *Assume (7.2). Then as $\eta \to 0$,*

$$\mathbb{P}_\eta \to \mathbb{P}_\infty \,,$$

weakly in $\mathscr{C}$, where $\mathbb{P}_\infty$ is the law of the coalescing Brownian flow on $\mathscr{C}$, as defined in Theorem 7.2.

Short sketch:
Let us briefly explain how this invariance principle is proved. Since we know by Lemma 7.1 that for any k-tuple of nice tubes $T_1, \dots, T_k \in \hat{\mathcal{T}}$:

$$\mathbb{P}_\infty\big[\partial(\boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k})\big] = 0 \,,$$

it is enough to prove that for these k-tuples of tubes in $\hat{\mathcal{T}}$:

$$\lim_{\eta\to 0} \mathbb{P}\big[\xi_\eta \in \boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k}\big] = \mathbb{P}\big[\xi_\infty \in \boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k}\big] \,, \tag{7.3}$$

where the latter quantity was characterized in Theorem 7.2. Let us start with the (easier) upper bound:

Upper bound in (7.5): the upper bound is not difficult: indeed by Theorem 7.2, for any $\epsilon > 0$, there exist $z_1, \ldots, z_n \in \mathbb{R}^2$ such that

$$\mathbb{P}\big[\xi(z_1, \ldots, z_n) \in \boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k}\big] \geq \mathbb{P}\big[\xi_\infty \in \boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k}\big] - \epsilon\,.$$

Now by following the trajectories in ξ_η which start from $z_1^\eta, \ldots, z_n^\eta$ and using an easy generalization of Donsker's Theorem for finite systems of coalescing Brownian motions (see [NRS05]), one obtains that

$$\lim_{\eta\to 0} \mathbb{P}\big[\xi_\eta(z_1^\eta, \ldots, z_n^\eta) \in \boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k}\big] = \mathbb{P}\big[\xi(z_1, \ldots, z_n) \in \boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k}\big]\,, \tag{7.4}$$

which implies the upper bound.

Lower bound in (7.5): this side requires more work. Let us focus for simplicity on the case of a single tube. (We will say a few words on the case of several "interacting" tubes). Namely we wish to show that for any tube $T \in \hat{\mathcal{T}}$:

$$\lim_{\eta\to 0} \mathbb{P}\big[\xi_\eta \in \boxminus_T\big] \leq \mathbb{P}\big[\xi_\infty \in \boxminus_T\big]\,. \tag{7.5}$$

To main ingredient to prove this is a **uniform coming down from infinity** property similar as what is represented in Figure 7.3 but which holds for discrete coalescing flows **uniformly** in the mesh $\eta > 0$. It says that, after a positive amount of time $\delta > 0$ the number of rescaled coalescing random walks which are killed upon exiting a bounded region of space stays finite as $\eta \to 0$ (i.e., is a tight family of random variables). Here is the precise statement we prove in [**G15**]:

Proposition 7.1 (Proposition 4.4 in [G15]). *Let $K > 0$ be fixed. Consider coalescing random walks on $\mathbb{Z} \times \mathbb{Z}$ with increments distributed as μ satisfying (7.2), starting from each $x \in [-Kn, Kn] \cap \mathbb{Z}$ at time 0, and that are killed upon leaving the interval $[-Kn, Kn]$. For $\delta > 0$, let U_n be the number of distinct coalescing random walks at time δn^2. Then there exists a constant $C = C(\mu) < \infty$ independent of δ and n such that for all $k, n \in \mathbb{Z}_+$,*

$$\mathbb{P}(U_n \geq k) \leq \frac{C}{\delta k}.$$

Similarly as in Figure 7.3, this uniform coming down from infinity property allows us to bound $\mathbb{P}\big[\xi_\eta \in \boxminus_T\big]$ from above by the probability that a slightly smaller tube T^δ (careful, in Figure 7.3, T^δ is slightly larger) is traversed by a **tight** number of coalescing random walks. Using again the generalized Donsker's Theorem above which lead to the identity (7.4), one finds that as $\eta \to 0$, this probability is smaller than

$$\sup_{n\geq 1; z_1, \ldots, z_n \in \mathbb{R}^2} \mathbb{P}\big[\xi(z_1, \ldots, z_n) \in \boxminus_{T^\delta}\big]\,,$$

which is equal to $\mathbb{P}_\infty\big[\boxminus_{T^\delta}\big]$. Therefore, the above analysis based on the coming down from infinity result gives us the upper bound:

$$\limsup_{\eta\to 0} \mathbb{P}\big[\xi_\eta \in \boxminus_T\big] \leq \mathbb{P}_\infty\big[\boxminus_{T^\delta}\big]\,, \tag{7.6}$$

for any strictly smaller tube T^δ. We conclude by noticing that $\bigcap_{\delta>0} \boxminus_{T_\delta} = \boxminus_T$ which readily implies that $\lim_{\delta\to 0} \mathbb{P}_\infty\left[\boxminus_{T^\delta}\right] = \mathbb{P}_\infty\left[\boxminus_T\right]$ and thus concludes our proof of the lower bound (7.5).

Let us end this rough sketch with a comment on what needs to be done in the case of multiple tubes $T_1, \ldots, T_k \in \hat{\mathcal{T}}$: we would still need to prove the lower bound:

$$\limsup_{\eta\to 0} \mathbb{P}\left[\xi_\eta \in \boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k}\right] \leq \mathbb{P}\left[\xi_\infty \in \boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k}\right]. \tag{7.7}$$

The idea, as with a single tube T, is to rely on the coming down from infinity property after EACH of the sides $\partial_0 T_1, \ldots, \partial_0 T_k$. There is a difficulty in implementing this due to the fact that particles from other tubes may interact with the coming down from infinity of the tube T_i. To overcome this, we use a certain coupling in [**G15**]. See Figure 7.4 for an illustration of this.

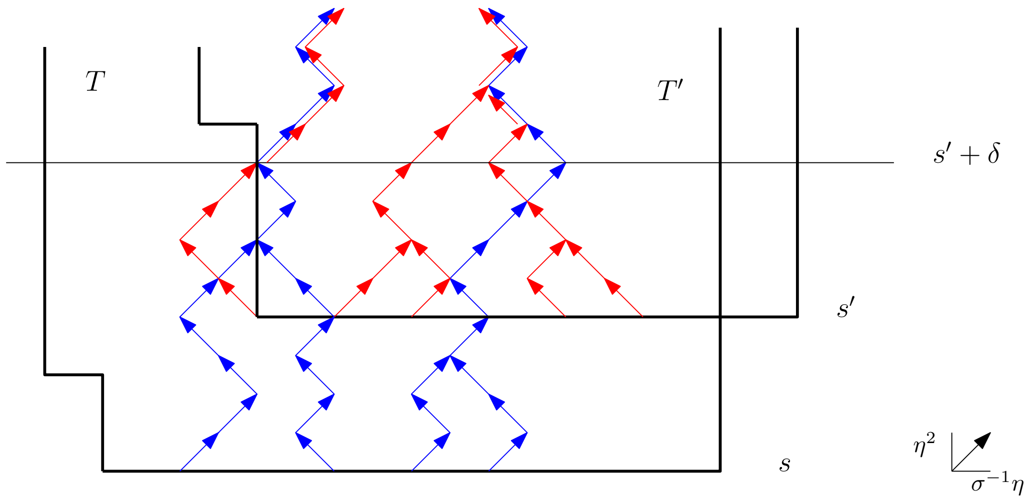


Figure 7.4:

Connection to the Brownian web of Fontes et al. As we pointed out in the Introduction, we obtain in Theorem 7.3 an invariance principle with an optimal finite second moment condition. This seems to be almost in contradiction with previous results from [BM+06] where it was shown that for the same coalescing system of random walks, a $3-\epsilon$ finite moment is needed in order to obtain a tightness criterion under [FINR04]'s setup, while a $3+\epsilon$ is sufficient. Understanding the reason for this discrepancy is instructive for both topological setups. The situation is similar to what happens with the various setups that were introduced for the scaling limit of critical percolation (see the historical list in Subsection 1.1 in Chapter 2 which was borrowed from [SchSm11]). As it is well explained in [SchSm11], it can be sometimes difficult to go from one notion of scaling limit to another. In the appendix of [**G15**], we prove that there is a **continuous** map from the metric space used in [FINR04] towards $(\mathscr{C}, d_{\mathscr{C}})$ which is such that the pushforward measure of the **Brownian web measure** in [FINR04] is our measure $\mathbb{P}_\infty$ on $\mathscr{C}$. Hence it appears that the tube topology is weaker (or coarser) than the path topology from [FINR04] suggesting that there should exist examples of events which are measurable in the path topology but whose images under the above continuous map are not measurable in the tube topology.

Another approach which does not rely on the non-crossing property. Finally, let us mention that Le Jan and Raimond, in a series of papers, [LJR04a, LJR04b], adopted yet a different point of view on coalescing flows. They viewed Arratia's flow as a random flow of probability kernels $K_{s,t}$, where $K_{s,t}(x, dy)$ represents the (random) probability that a particle starting from x at time s ends up near y at time t. While this approach is in principle very general (and in particular, does not rely on the non crossing property), it is not well-suited to the questions of taking a scaling limit of some discrete flow to its continuous counterpart. This is because the discrete and the continuous objects do not naturally live on the same space, and so the question of scaling limits can only be approached through the finite-dimensional distributions of the flow. See for instance [LJL04, LJ06].

4. — Coalescing flow on the Sierpinski gasket and invariance principle

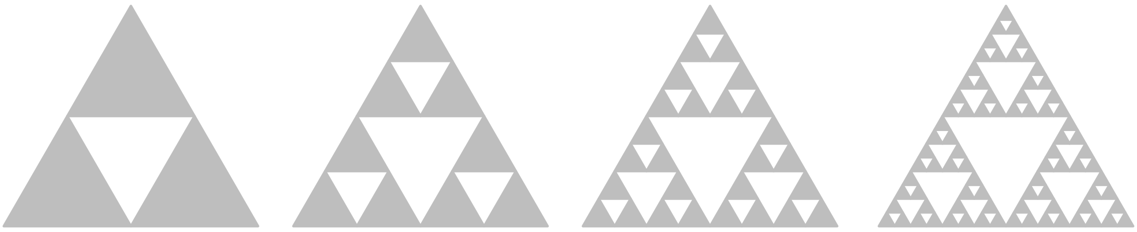


Figure 7.5: Illustration of the Sierpinski graphs G_1, G_2, G_3, G_4.

Let us now turn to coalescing flows on the Sierpinski gasket G: we wish to understand the scaling limit of the rescaled coalescing flows ξ_η^G defined in the introduction on the Sierpinski graphs G_n (with $\eta := 2^{-n}$) pictured in Figure 7.5. As in the one-dimensional case, we proceed in two steps:

1. First we build a natural coalescing flow ξ_∞^G on the (continuous) Sierpinski gasket G. See Theorem 7.4.

2. Then, we prove an **invariance principle** showing that the discrete flows ξ_η^G converge weakly to ξ_∞^G.

We only briefly explain below how to adapt the proofs used in the one-dimensional case.

Construction of the Sierpinski Brownian flow. Exactly as in the one dimensional setting, one can build scaling limits $\mathbb{P}_\infty^{G;\mathcal{D}}$ for any countable dense sets of $G \times \mathbb{R}_+$. The same argument implies that the construction does not depend on the choice of $\mathcal{D}$. As such, we obtain a natural measure $\xi_\infty^G \sim \mathbb{P}_\infty^G$. Already in dimension one, some non-trivial work was needed to characterize the limiting measure $\mathbb{P}_\infty$ (Lemma 7.1 which implied Theorem 7.2). This work is a bit more tedious in the case of the Sierpinski gasket G:

1. First we need a **coming down from infinity** result in the case of Brownian motions on the fractal G. This property was already established in [EMS09].

2. Then, it is more technical in this case to deal with the boundary issues that arise when one compares the probability of the events $\boxminus_T$ and $\boxminus_{T^\delta}$. (This difficulty of boundary issues is mentioned in Figure 7.3). To reduce the number of possible boundary configurations to analyze, we introduce in [**G15**] a well-chosen dense family of tubes $\mathcal{T}_\triangle$ (see Definition 5.3 in [**G15**]). The tubes in $\mathcal{T}_\triangle$ (which are in $\mathbb{R}^{2+1}$ as in Figure 7.1) are adapted to the geometry of G.

Following this rough sketch of proof, the following result is proved in Section 5 of [**G15**]:

Theorem 7.4. *There is a natural* **Sierpinski Brownian flow** *on the Sierpinski gasket G, $\xi^G_\infty \sim \mathbb{P}^G_\infty$, whose law $\mathbb{P}^G_\infty$ is characterized by*

$$\mathbb{P}^G_\infty\big[\boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k}\big] := \sup_{n\geq 1; z_1,\ldots,z_n \in \mathbb{R}^2} \mathbb{P}\big[\xi^G(z_1,\ldots,z_n) \in \boxminus_{T_1} \cap \ldots \cap \boxminus_{T_k}\big]\,,$$

*for any k-tuple of "nice" tubes $T_1,\ldots,T_k \in \mathcal{T}_\triangle$, the family introduced in Definition 5.3 in [**G15**].*

Invariance principle for the Sierpinski coalescing random walks ξ^G_η. Recall that the proof in the one-dimensional setting was divided into the upper and lower bounds in (7.3). The (easy) upper bound is proved along the same lines on the gasket G, except the generalization of Donsker's Theorem, i.e. the identity (7.4) which on the Sierpinsky gasket requires some extra analysis (see Proposition 6.6 in [**G15**]).

The (harder) lower bound requires the following **uniform coming down from infinity** statement for coalescing random walks on the Sierpinski graphs G_n (uniform in the mesh $\eta = 2^{-n}$). Suppose the coalescing random walks start at time 0 on G_n, and jump every 5^{-n} units of time. Given a bounded region $T \subset \mathbb{R}^2$, suppose that the random walk particles are killed as soon as they touch $\mathbb{R}^2 \setminus T$. Let $N(t)$ denotes the number of particles left at time t. The following is the analogue of Proposition 7.1 but for the case of the gasket:

Proposition 7.2 (Uniform coming down from infinity on G, [G15]). *For every $\delta > 0$ and $\epsilon > 0$, there exists $k \geq 0$ depending only on T, such that $\mathbb{P}\big[N(\delta) > k\big] \leq \epsilon$ for all sufficiently large n.*

Its proofs relies on uniform heat-kernel estimates of the random walks on the gasket due to Jones [Jon96] and is inspired by an argument in [EMS09] for the fact that coalescing Brownian particles come down from infinity as well as [ABL12]. See [**G15**].

5. — The Sierpinski Brownian web is a black noise

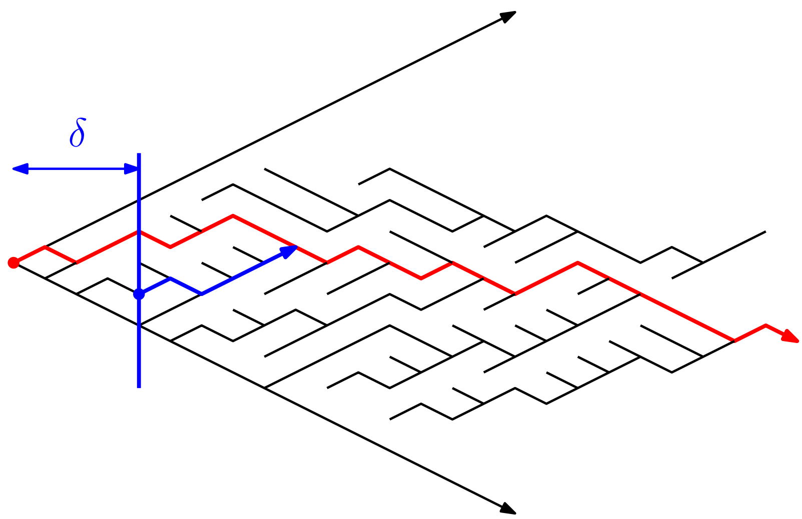
Let us say a few words on the work in preparation [**G17**] whose aim is to prove that the **Sierpinski Brownian flow** introduced in Theorem 7.4 is a **black noise**. See for example [Tsi04] for a very readable survey on the subject. Very roughly, a stochastic process giving rise to a black noise is a random process such that:

(i) its source of randomness is well structured into independent "blocks". This is the **noise property** or factorization property.

(ii) it is highly sensitive to small random perturbations (this is the **"black"** part). This corresponds to the so-called **noise sensitivity** property which was introduced in [BKS99].

Sometimes, both properties are hard to establish. For example, in the case of the scaling limit of critical percolation $\omega_\infty \sim \mathbb{P}_\infty$, the noise property (item (i)) is the main Theorem in [SchSm11] and the (macroscopic) noise sensitivity property follows from a combination of [BKS99] and [Tsi04] ([**G3**] gives a different proof).

In [**G17**], we wish to prove the following result:



Theorem 7.5 (Work in progress, [G17]).

1. *The coalescing flows ξ_η^G are* **noise sensitive**. *(Meaning that if one resamples a small fraction ϵ of the i.i.d bits that define the coalescing flow ξ_η^G, the resulting flow $\xi_\eta^{G,\epsilon}$ is almost independent of the initial one).*

2. *The Sierpinski Brownian flow ξ_∞^G is a* **black noise**.

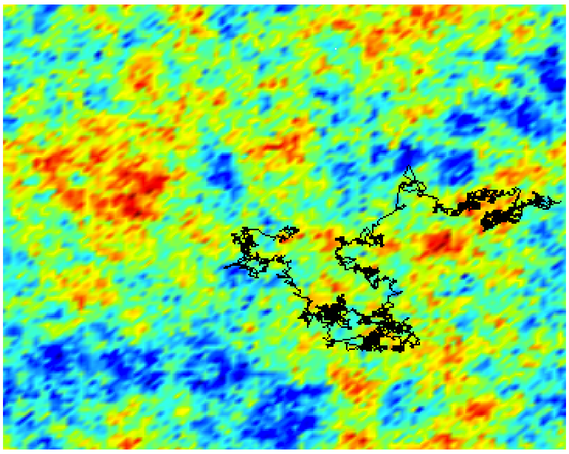
The approach we follow in [**G15**] is very different from the existing proofs that the **Brownian web** from [FINR04] is a blacknoise ([LJR04b, Tsi04]). We first establish the noise sensitivity of discrete systems (Item 1) from which the black noise property essentially follows. In order to prove the noise sensitivity of ξ_η^G, several different techniques are available to us. See for example our survey [**G18**]. The approaches such as [BKS99] or [**G3**] require a good understanding of **influences**. Unfortunately, notice that it is non-trivial in our case to estimate the probability for a point in $G_n \times 5^{-n}\mathbb{N}_+$ to be "pivotal" for the coalescing flow. (It involves a kind of **intersection exponent** for Brownian motions on the Sierpinski gasket G). We are thus left with the technique invented by Schramm and Steif [SchSt10]. They discovered that if a **Boolean function** can be evaluated using a **randomized algorithm** which reveals the value of "few" inputs (in a certain quantitative sense), then the Boolean function needs to be of high frequency (= noise sensitive). They point out in [SchSt10] that if the algorithm only approximates well the output of the Boolean functions, then their result still holds (depending of course on the quality of the approximation). This is what we use here: in order to compute where the particle initially at 0 is at time one, we sample a random point in the δ-neighborhood of zero and we follow the trajectory of this particle from time δ to time 1 as illustrated in the Figure above (in $d = 1$ for simplicity). We conclude by noticing that all the bits are unlikely to be used by the algorithm (this would not be the case without this initial randomization) and the algorithm is likely to give the correct answer since both particles will quickly coalesce.

Chapter 8

Liouville Brownian motion in $2d$ quantum gravity

Based on joint works with
Rémi Rhodes and Vincent Vargas

As explained in the introduction, we introduced in [G13] a Feller process called the **Liouville Brownian motion** on $\mathbb{S}^2$ (or $\mathbb{R}^2$). It can be seen as a Brownian motion evolving in a random geometry given formally by the exponential of a Gaussian Free Field $e^{\gamma X}$. We argued that this process is the right diffusion to consider regarding $2d$-**Liouville quantum gravity**. We will explain in this Chapter how this process is constructed when started from a fixed point $x \in \mathbb{S}^2$ (Section 1) and then will give some ideas on how it is extended in [G13] to a Feller process. Finally we will discuss the companion paper [G14] which discusses the Dirichlet form associated to the Liouville Brownian motion as well as heat-kernels.



1. — Starting from a fixed point $x \in \mathbb{S}^2$

Let us then fix $x \in \mathbb{S}^2$ and sample a Gaussian Free Field X on the sphere $\mathbb{S}^2$ with vanishing mean (see equation 1.18). Recall that we will use a regularization as in Kahane's work [Kah85], namely for each $n \geq 1$:

$$X_n = \sum_{k=1}^{n} Y_k \,,$$

where $X = \sum_{k\geq 1} Y_k$ is decomposed into a sum of independent real-Gaussian processes on $\mathbb{S}^2$ whose covariance kernels K_k are positive and positive-definite.

Remark 8.1. We are cheating a bit here since this decomposition is more suitable for example to a **massive** GFF X on $\mathbb{R}^2$. See [RV13] for the kernels that may be decomposed like that (called kernels of σ-positive type since Kahane) and [**G13**] for the Liouville Brownian motion on the sphere.

Using this approximation of the Gaussian Free Field, we defined in the introduction the following approximated SDE:

Definition 8.1. *Let $(\bar{B}_t)_{t\geq 0}$ be a fixed* **driving Brownian motion**. *For each $n \geq 1$ and each starting point $x \in \mathbb{S}^2$, define the n*-**regularized Liouville Brownian motion** *as follows:*

$$\begin{cases} \mathcal{B}^{n,x}_{t=0} = x \\ d\mathcal{B}^{n,x}_t = e^{-\frac{\gamma}{2} X_n(\mathcal{B}^{n,x}_t) + \frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}^{n,x}_t)^2]}\, d\bar{B}_t. \end{cases} \tag{8.1}$$

By using the Dambis-Schwarz Theorem, it is easy to check that one may also define the n-regularized Liouville Brownian motion as follows.

Definition 8.2. *For any $n \geq 1$ and any fixed $x \in \mathbb{R}^2$,*

$$\mathcal{B}^{n,x}_t = x + B_{\langle \mathcal{B}^{n,x}\rangle_t}\,, \tag{8.2}$$

where $(B_r)_{r\geq 0}$ is a two-dimensional Brownian motion independent of the GFF X and where the ***quadratic variation*** *$\langle \mathcal{B}^{n,x}\rangle$ of $\mathcal{B}^{n,x}$ satisfies:*

$$\langle \mathcal{B}^{n,x}\rangle_t := \inf\{s \geq 0 \, : \, \int_0^s e^{\gamma X_n(x+B_u) - \frac{\gamma^2}{2}\mathbb{E}\left[X_n(x+B_u)^2\right]}\, du \geq t\}\,. \tag{8.3}$$

Note that in the two above equivalent definitions of the n-regularized LBM, the "driving" Brownian motions $\bar{B}_t$ and B_t are both independent of the Gaussian Free field X. Nevertheless, it is not correct that $(X, \bar{B}_t, B_t)$ are mutually independent. There is some dependency between (B_t) and $(\bar{B}_t)$ which depends on the field X and of the value of $n \geq 1$. It turns out that as the regularization $n \to \infty$, these two Brownian motions become asymptotically independent. Let us now state our main result when our process starts from a fixed point $x \in \mathbb{S}^2$ (i.e. independent of the Free Field X):

Theorem 8.1 ([G13]). *Let $x \in \mathbb{S}^2$ be fixed. Then, a.s. in the Free Field X, the coupling of processes $(\mathcal{B}^{n,x}, \bar{B})$ converges in law as the regularization $n \to \infty$ (for the topology of uniform convergence on compacts sets) to a couple $(\mathcal{B}^x, \bar{B})$ where $\mathcal{B}^x$ is the* **Liouville Brownian motion** *starting from x. Furthermore, the limiting couple $(\mathcal{B}^x, \bar{B})$ satisfies the following properties (still a.s. in the realization of X):*

(i) $t \mapsto \mathcal{B}_t^x$ *is continuous.*

(ii) $t \mapsto \langle \mathcal{B}^x \rangle_t$ *is strictly increasing. I.e, the Liouville Brownian motion never gets stuck on points where the Free Field* X *is very "large".*

(iii) $\mathcal{B}^x$ *is independent of the initial driving force* $\bar{B}$*.*

Let us start by the constructing a process $\mathcal{B}^x$ which will be the limit in law as $n \to \infty$ of the processes $\mathcal{B}^{n,x}$. Following Definition 8.2 and since we are looking for a convergence in law, we may sample once and for all a Brownian motion $(B_u)_{u \geq 0}$ independent of the Free Field X. As such, for any $n \geq 1$, $\mathcal{B}^{n,x}$ is just a time-changed (given by (8.3)) of the fixed Brownian motion (B_u) and we just need to prove that the induced reparametrization converges as $n \to \infty$. In order to analyze this time-change, let us introduce the following useful additive functional:

Definition 8.3. *Let* F^n *be the following random function on* $\mathbb{R}^2 \times \mathbb{R}_+$*:*

$$F^n(x,s) := \int_0^s e^{\gamma X_n(x+B_u) - \frac{\gamma^2}{2}\mathbb{E}\left[X_n(x+B_u)^2\right]} \, du. \tag{8.4}$$

This functional will allow us to control the behavior of the quadratic variation $\langle \mathcal{B}^{n,x} \rangle_t$ thanks to the relation (obtained from (8.3)):

$$\langle \mathcal{B}^n \rangle_t = F_n(x, \bullet)^{-1}(t) \tag{8.5}$$

Let us rewrite F_n as follows:

$$F^n(x,s) := \int_{\mathbb{R}^2} e^{\gamma X_n(z) - \frac{\gamma^2}{2}\mathbb{E}\left[X_n(z)^2\right]} \, d\nu_s(z) \,, \tag{8.6}$$

where ν_s denotes the occupation time measure of the Brownian motion B up to time s. Note that this expression is very similar to the martingale $M_n(A)$ defined in the introduction in equation (1.19). In fact, already in [Kah85], Kahane considered his mutiplicative chaos integrated against general Radon measures σ. He proved the following theorem:

Theorem 8.2 ([Kah85]). *Let* X *be a Gaussian field in* $\mathbb{R}^n$ *whose kernel* K *is of* σ*-positive type and such that* $K(x,y) \sim \log \frac{1}{\|x-y\|_2}$ *as* $x \to y$*. Then for any* $0 \leq \gamma^2 < 2\alpha$ *and any finite Radon measure* σ *in the class* R_α *(see [Kah85] or the appendix in [**G13**]), the sequence*

$$M_n(\sigma) := \int_{\mathbb{R}^n} e^{\gamma X_n(z) - \frac{\gamma^2}{2}\mathbb{E}\left[X_n(z)^2\right]} d\sigma(z) \,,$$

is a **uniformly integrable** *martingale.*

Since it is well-known that the occupation time measure of Brownian motion a.s. belongs to the class R_α for any $\alpha < 2$ (see for example [**G13**]), we have that a.s. in the GFF X and a.s. in the trajectory B, $s \mapsto F_n(x,s)$ has a non-degenerate limit $s \mapsto F(x,s)$ as

$n \to \infty$. Furthermore the process thus-obtained is obviously non-decreasing. In order to prove items (i) and (ii) in Theorem 8.1, we still have to prove the following a.s. properties for the additive functional $F(x,s)$:

1. We need to show that $s \mapsto F(x,s)$ is strictly increasing. This will ensure using (8.5) that the quadratic variation $t \mapsto \langle \mathcal{B}^n \rangle_t$ is a.s. continuous which gives us item (i) in Theorem 8.1.

2. We also need to show that $s \mapsto F(x,s)$ is continuous, which will ensure, again using (8.5) item (ii) in Theorem 8.1.

Item 1 is easier to prove than item 2. It is equivalent to the fact the measure $F(x,ds)$ a.s. has full support. Now, for any non empty interval $I=[s,t]$, the event $\{F(x,I)>0\}$ is a tail event in the sigma-algebra generated by the random processes $(Y_n)_n$ used to construct the Gaussien field of X. One can thus conclude using a 0-1 law (see [**G13**] for details).

For item 2, we need to prove that a.s. in X and B, there are no Dirac point masses in the measure $F(x,ds)$. This is done in [**G13**] by studying the moments of F. Namely we prove:

Proposition 8.1 (Appendix B in [G13]). *Assume $\gamma < \gamma_c = 2$. Then for any $1 \leq q < \min(2, \frac{4}{\gamma^2})$, one has*

$$\mathbb{E}^{X,B}\big[F(x,[t,t+s])^q\big] \asymp s^{\xi(q)}\,,$$

with $\xi(q) = (1+\frac{\gamma^2}{4})q - \frac{\gamma^2}{4}q^2$. ($\mathbb{E}^{X,B}$ means here that one averages w.r.t the Brownian motion B as well as the Free Field X).

Choosing $q>1$ such that $\xi(q)>1$ one obtains for any $\delta>0$,

$$\sum_{k=0}^{n-1} \mathbb{P}\big[F(x,[\frac{k}{n},\frac{k+1}{n}] \geq \delta)\big] \leq \delta^{-q} n \mathbb{E}\big[F(x,1/n)^q\big] \leq O(1) n^{1-\xi(q)}\,,$$

which implies by union bound that there are a.s. no Dirac point masses.

Finally, let us explain how to derive item (iii) in Theorem 8.1, i.e. the asymptotic independence between the **range** of $\mathcal{B}^{n,x}$ and its driving Brownian motion $\bar{B}$. By Knight's theorem, it is enough to prove the following Lemma. (See [**G13**] for details).

Lemma 8.1. *For any $t>0$, a.s. in X and $\bar{B}$,*

$$\langle \mathcal{B}^n, \bar{B} \rangle_t \longrightarrow 0\,,$$

as $n \to \infty$, where $\mathcal{B}^{n,x}$ was defined in Definition 8.1.

To prove this, notice using (8.1) that

$$\begin{aligned}
\langle \mathcal{B}^{n,x}, \bar{B}\rangle_t &= \int_0^t e^{-\frac{\gamma}{2}X_n(\mathcal{B}^{n,x}_r)+\frac{\gamma^2}{4}\mathbb{E}[X_n(\mathcal{B}^{n,x}_r)^2]}\,dr \\
&\overset{law}{=} \int_0^t e^{-\frac{\gamma}{2}X_n(x+B_{\langle \mathcal{B}^{n,x}\rangle_r})+\frac{\gamma^2}{4}\mathbb{E}[X_n(x+B_{\langle \mathcal{B}^{n,x}\rangle_r})^2]}\,dr \\
&= \int_0^{\langle \mathcal{B}^{n,x}\rangle_t} e^{\frac{\gamma}{2}X_n(x+B_u)-\frac{\gamma^2}{4}\mathbb{E}[X_n(x+B_u)^2]}\,du \\
&= e^{-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]}\int_0^{\langle \mathcal{B}^{n,x}\rangle_t} e^{\frac{\gamma}{2}X_n(x)-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]}\,du\,.
\end{aligned}$$

We used in the last equality the invariance by rotation of the GFF X_n on $\mathbb{S}^2$. Note that by Kahane's Theorem, the integral on the right converges as $n\to\infty$. (This is the same argument as for the convergence of F_n with $\gamma/2$ instead of γ). Since $e^{-\frac{\gamma^2}{8}\mathbb{E}[X_n(x+B_u)^2]}$ goes to zero as $n\to\infty$, this proves the Lemma.

2. — Starting simultaneously from all points in $\mathbb{S}^2$

The main obstacle to extend the above construction simultaneously to all starting points $x\in\mathbb{S}^2$ is the possibility that the above functional $F_n(x,s)$ might typically (under the law of B) blow up on the **"thick points"** of the GFF X. This would force the Liouville Brownian motion to "seat" on these thick points and would prevent us from relying in [G13] on coupling techniques in order to define the semi-group P_t^X. As such, the key proposition may be stated as follows:

Proposition 8.2. *For any $T>0$ one has, a.s. in the Free Field X:*

$$\sup_{n\geq 1}\left\{\sup_{y\in\mathbb{S}^2}\mathbb{E}^B\big[F^n(y,T)\big]\right\}<\infty \tag{8.7}$$

To prove this, we observe that

$$\begin{aligned}
\mathbb{E}^B\big[F^n(y,T)\big] &= \int_{\mathbb{S}^2}\left(\int_0^T e^{-\frac{|w-y|^2}{2s}}\frac{ds}{2\pi s}\right)e^{\gamma X_n(w)-\frac{\gamma^2}{2}\mathbb{E}\left[X_n^2(w)\right]}dw \\
&= \int_{\mathbb{S}^2}\left(\int_0^T e^{-\frac{|w-y|^2}{2s}}\frac{ds}{2\pi s}\right)M_n(dw) \\
&\asymp \int_{\mathbb{S}^2}\log\frac{1}{|w-y|}M_n(dw)
\end{aligned}$$

Since these quantities are once again martingales, by Doob's inequality, it is enough to show that a.s. in the field X,

$$\sup_{y\in\mathbb{S}^2}\left\{\int_{\mathbb{S}^2}\log\frac{1}{|w-y|}M(dw)\right\}<\infty\,,$$

where $M=M_\gamma$ is the **Liouville measure**. In other words, we need to show that one can integrate the Liouville measure M_γ against all logarithmic singularities on the sphere. To overcome these logarithmic singularities, it is enough to obtain the following bound on the moments of the Liouville measure M.

Proposition 8.3. *For all $\gamma < \gamma_c = 2$ and any $\epsilon > 0$, let $\alpha := 2(1 - \frac{\gamma}{2})^2 > 0$. Then, a.s. in X there is a constant $C = C(X) < \infty$ s.t. for any $r > 0$:*

$$\sup_{y \in \mathbb{S}^2} M(B(y,r)) \leq C\, r^{\alpha - \epsilon},$$

where $B(y,r)$ is the ball around y of radius r.

The proof of this result in some sense relies on the $\star$-scale invariance of the GFF (see [RSV]) and is not difficult, see [**G13**].

To conclude the sketch of proof of the main result (Theorem 1.20 stated in the introduction) we still need to discuss the Markov, Feller and invariant measures of the semi-group. For the Feller/Markov property, the main observation is that Proposition 8.2 enables us (using couplings) to prove that the Liouville Brownian motion $\mathcal{B}_t^x$ converges in law towards $\mathcal{B}_t^{x_0}$ when $x \to x_0$. The coupling works as follows: one couples the "ranges" B^x and B^{x_0} in the usual manner so that they coalesce before leaving the ball $B(x_0, \|x - x_0\|^{1/2})$ with high probability. Then Proposition 8.2 says that the time-changed process $\mathcal{B}^x$ and $\mathcal{B}^{x_0}$ need to move at some positive speed with high probability. There is still a bit of work needed since the ranges B^x and B^{x_0} are built so that they coalesce at the same "euclidean" time but the processes $\mathcal{B}^x$ and $\mathcal{B}^{x_0}$ won't quite coalesce at the same "quantum" time. From then one, they will follow the same range but one will be a bit delayed w.r.t to the other. This delay can be controlled as well which implies the desired regularity properties of the flow.

Finally, it is straightforward (by construction) that the n-regularized Liouville measure M_n is reversible for the semi-group P_n^X associated to the n-regularized Liouville Brownian motion introduced in (8.1). Some analysis is required (especially away from the L^2 case when $\gamma \geq \sqrt{2}$) in order to pass to the limit $n \to \infty$. $\square$

3. — Liouville Dirichlet form and its associated metric

One may associate to our Liouville semigroup $(P_t^X)_{t \geq 0}$ the following Dirichlet form:

$$\Sigma(f,f) = \lim_{t \to 0} \frac{1}{t} \int \big(f(x) - P_t^X f(x)\big) f(x) M(dx) \tag{8.8}$$

whose domain $\mathcal{F}$ is defined as the set of functions $f \in L^2(\mathbb{R}^2, M)$ for which the above limit exists and is finite. This expression at first sight is rather non explicit.

In [**G14**], relying mostly on techniques developed in [FOT94, Kah85], we make explicit the Liouville Dirichlet form (8.8), namely:

Theorem 8.3 (Liouville Dirichlet form). *For $\gamma \in [0, 2[$, the Liouville Dirichlet form $(\Sigma, \mathcal{F})$ takes on the following explicit form: its domain is*

$$\mathcal{F} = \Big\{ f \in L^2(D, M) \cap H^1_{loc}(D, dx); \nabla f \in L^2(D, dx) \Big\},$$

and for all functions $f, g \in \mathcal{F}$:

$$\Sigma(f,g) = \int_D \nabla f(x) \cdot \nabla g(x)\, dx.$$

Furthermore, it is strongly local and regular.

Notice that understanding rigorously this theorem is not obvious since the Liouville measure M and the Lebesgue measure dx are singular. The domain $\mathcal{F}$ is thus composed of the functions $u \in L^2(D, M)$ such that there exists a function $f \in H^1_{loc}(D, dx)$ satisfying $\nabla f \in L^2(D, dx)$ and $u(x) = f(x)$ for $M(dx)$-almost every x. It is a consequence of the general theory developped in [FOT94] (see chapter 6) and of the tools developped in [**G13**] that the definition of $(\Sigma, \mathcal{F})$ actually makes sense: in particular, a.s. in X, if f, g in $H^1_{loc}(D, dx)$ are such that $f(x) = g(x)$ for $M(dx)$-almost every x then $\nabla f(x) = \nabla g(x)$ for dx-almost every x.

Remark 8.2. This study of the Liouville Brownian motion through its Dirichlet form raises the natural question: "Is it possible to construct the Liouville Brownian motion from the only use of [FOT94]?" Using [FOT94], one may indeed define an "abstract" **Hunt process**, but this would require at least our Proposition 8.2. Moreover, this abstract construction would rigorously define a Hunt process living in the space $\mathbb{S}^2 \setminus N$ where N is some non-explicit polar set. To our knowledge, there is no general theory on Dirichlet forms which enables to get rid of this polar set. In conclusion, without using the tools developped in [**G13**], one may construct the Liouville Brownian motion in a non explicit way living in $\mathbb{S}^2 \setminus N$ and for starting points in $\mathbb{S}^2 \setminus N$ where N is a polar set (depending on the randomness of X).

The Liouville Dirichlet form obtained above is sufficiently regular to apply the **geometric theory of Dirichlet forms** developed for example in [Sto10, Stu98] whose aim is to recover the "metric" of a space X out of a stochastic process W_t leaving on X. The main motivation of their work is to obtain analogs or Riemannian structures on spaces which are too "rough" to carry such structures. We realized in [**G14**] that the sphere $\mathbb{S}^2$ equipped with the exponential of a Gaussian Free Field seems to be "too far" from a smooth structure in order to apply their theory. Indeed, it can be seen (Proposition 4.1) that their "natural metric" associated to the Liouville Dirichlet form is degenerate (identically 0).

Finally, we study in [**G14**] the **resolvent operator** associated to the Liouville Brownian motion and show that it is sufficiently regular to imply (using very nontrivial results from [FOT94]) that for any time $t > 0$, P_t^X is absolutely continuous w.r.t the Liouville measure M_γ (i.e. that there exists a **Liouville heat-kernel**).

Bibliography presented for the HDR

1. — List of publications/preprints

[G1] C. Garban and J. Trujillo-Ferreras, *The expected area of a filled planar Brownian loop is Pi/5*, Communications in Mathematical Physics, **264** (2006), no. 3, 797–810. 14 pages.

[G2] C. Garban, O. Schramm and S. Rohde, *Continuity of the SLE trace in simply connected domains*, Israel Journal of Mathematics, Volume 187, Number 1 (2012), 23-36. 14 pages.

[G3] C. Garban, G. Pete and O. Schramm, *The Fourier Spectrum of Critical Percolation*, Acta Mathematica, Volume 205, Number 1, 19–104 (2010). 85 pages.

[G4] ——, *Pivotal, cluster and interface measures for critical planar percolation*, J. Amer. Math. Soc. 26, no. 4, 939–1024 (2013). 85 pages.

[G5] ——, *The scaling limits of near-critical and dynamical percolation*, arXiv:1305.5526, 2013. Submitted. 86 pages

[G6] ——, *The scaling limits of the Minimal Spanning Tree and Invasion Percolation in the plane.* arXiv:1309.0269. 50 pages.

[G7] C. Garban, Appendix to the paper "On the scaling limits of planar percolation" by Oded Schramm and Stanislav Smirnov. Ann. Probab. 39 (2011), no. 5, 1768–1814. 4 pages.

[G8] E. Broman, C. Garban and J. Steif, *Exclusion Sensitivity of Boolean Functions*, Probab. Theory and Related Fields, 155, issue 3-4 , 621–663, (2013). 43 pages.

[G9] H. Duminil-Copin, C. Garban and G. Pete, *The near-critical planar FK-Ising model*, arXiv:1111.0144, to appear in Communications in Mathematical Physics, 35 pages.

[G10] F. Camia, C. Garban and C. Newman, *Planar Ising magnetization field I. Uniqueness of the critical scaling limit*, arXiv:1205.6610, to appear in Ann. Probab. 35 pages.

[G11] ——, *The Ising magnetization exponent is 1/15*, arXiv:1205.6612, to appear in Probab. Theory and Related Fields, 12 pages.

[G12] ——, *Planar Ising magnetization field II. Properties of the critical and near-critical scaling limits*, arXiv:1307.3926, 2013. 18 pages.

[G13] C. Garban, R. Rhodes and V. Vargas, *Liouville Brownian motion*, arXiv:1301.2876. Submitted, 70 pages.

[G14] ——, *On the heat kernel and the Dirichlet form of Liouville Brownian Motion. Preprint.* arXiv:1302.6050. Submitted, 31 pages

[G15] N. Berestycki, C. Garban and A. Sen, *Coalescing Brownian flows: a new approach*, arXiv:1307.4313, 2013. Submitted, 36 pages.

2. — Papers in preparations discussed in this HDR

[G16] C. Garban and G. Pete. *The scaling limit of dynamical FK-percolation.* In preparation.

[G17] N. Berestycki, C. Garban and A. Sen, *Coalescing flows on Sierpinski fractals are black noises.* In preparation.

3. — Surveys / Books / PhD thesis

[G18] C. Garban, *Oded Schramm's contributions to Noise Sensitivity*, Annals of probability, Volume 39, Number 5 (2011), 1702–1767. 66 pages.

[G19] ——, *Quantum gravity and the KPZ formula*, Bourbaki seminar (after Duplantier-Sheffield), no 1052 (mars 2012) arXiv:1206.0212. 38 pages.

[G20] C. Garban and J. Steif, *Lectures on noise sensitivity and percolation*, "Probability and Statistical Physics in Two and more Dimensions". Proceedings of the Clay Mathematics Institute Summer School and XIV Brazilian School of Probability (Buzios, Brazil), Clay Mathematics Proceedings 15 (2012), 49-154. 106 pages.

[G21] ——, *Noise sensitivity of Boolean functions and percolation*, Book in progress, to be published in Cambridge University Press. 2013

Rest of the Bibliography

[ABG12] L. Addario-Berry, N. Broutin and C. Goldschmidt. The continuum limit of critical random graphs. *Probab. Theory Related Fields* **152** (2012), no. 3-4, 367–406. arXiv:0903.4730

[ABGM13] L. Addario-Berry, N. Broutin, C. Goldschmidt and G. Miermont. The scaling limit of the minimum spanning tree of the complete graph. *Preprint*, arXiv:1301.1664.

[Ai98] M. Aizenman. Scaling limit for the incipient spanning clusters. *Mathematics of multiscale materials (Minneapolis, MN, 1995-1996)*, 1–24, IMA Vol. Math. Appl., 99, Springer, New York, 1998.

[AB99] M. Aizenman and A. Burchard. Hölder regularity and dimension bounds for random curves. *Duke Math. J.*, 99(3):419–453, 1999.

[AB+99] M. Aizenman, A. Burchard, C. M. Newman and D. B. Wilson. Scaling limits for minimal and random spanning trees in two dimensions. *Random Structures Algorithms* **15** (1999), 319–365.

[AM94] K. S. Alexander and S. Molchanov. Percolation of level sets for two-dimensional random fields with lattice symmetry. *J. Stat. Phys.* **77** (1994), 627–643.

[ARV11] R. Allez, R. Rhodes,V. Vargas: Lognormal $\star$-scale invariant random measures, to appear in *Probability Theory and Related Fields*, arXiv:1102.1895v1.

[ABL12] O. Angel, N. Berestycki, and V. Limic. Global divergence of spatial coalescents. *Probab. Theory Related Fields*, 152(3-4):625–679, 2012.

[Arr79] R. Arratia. Coalescing brownian motions on the line. Ph.D. Thesis, 1979.

[Arr81] R. Arratia. Coalescing brownian motions and the voter model on $\mathbb{Z}$. Unpublished partial manuscript. Available from rarratia@math.usc.edu., 1981.

[Bar98] M. T. Barlow. Diffusions on fractals. In *Lectures on probability theory and statistics (Saint-Flour, 1995)*, volume 1690 of *Lecture Notes in Math.*, pages 1–121. Springer, Berlin, 1998.

[BP88] M. T. Barlow and E. Perkins. Brownian motion on the Sierpi?ski gasket. *Probab. Theory Related Fields* 79 (1988), no. 4, 543–623.

[BD12a] V. Beffara and H. Duminil-Copin. Smirnov's fermionic observable away from criticality. *Annals of Probability* 40, no. 6, 2667–2689. 2012.

[BD12b] V. Beffara and H. Duminil-Copin. The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$. *Probab. Theory Related Fields*, 153, no. 3-4, 511–542, 2012.

[BM+06] S. Belhaouari, T. Mountford, Rongfeng Sun, and G. Valle. Convergence results and sharp estimates for the voter model interfaces. *Electron. J. Probab.*, 11:no. 30, 768–801, 2006.

[BKS99] I. Benjamini, G. Kalai, and O. Schramm. Noise sensitivity of Boolean functions and applications to percolation. *Inst. Hautes Études Sci. Publ. Math.*, (90):5–43 (2001), 1999.

[Ben10] I. Benjamini. Random planar metrics. Proceedings of the International Congress of Mathematicians. Volume IV, 2177Ð2187, Hindustan Book Agency, New Delhi, 2010.

[BS09] I. Benjamini and O. Schramm. KPZ in one dimensional random geometry of multiplicative cascades, Communications in Mathematical Physics, vol. 289, no 2, 653-662, 2009.

[Ber13] N. Berestycki. Diffusion in planar Liouville quantum gravity, arXiv:1301.3356.

[BC+01] C. Borgs, J. T. Chayes, H. Kesten, and J. Spencer. The birth of the infinite cluster: finite-size scaling in percolation. *Comm. Math. Phys.*, 224(1):153–204, 2001. Dedicated to Joel L. Lebowitz.

[CFN06] F. Camia, L. R. Fontes and C. M. Newman. The scaling limit geometry of near-critical 2D percolation. *J. Stat. Phys.* 125, no. 5-6, 1159–1175, 2006.

[CN06] F. Camia and C. M. Newman. Two-dimensional critical percolation: the full scaling limit. *Comm. Math. Phys.*, 268(1):1–38, 2006.

[Cer06] R. Cerf. The Wulff crystal in Ising and percolation models. Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July 6–24, 2004. With a foreword by Jean Picard. Lecture Notes in Mathematics, 1878. Springer-Verlag, Berlin, 2006.

[CM10] R. Cerf and R. J. Messikh. On the 2D Ising Wulff crystal near criticality. Ann. Probab. 38 (2010), no. 1, 102–149.

[CDH13] D. Chelkak, H. Duminil-Copin, and C. Hongler. Crossing probabilities in topological rectangles for the critical planar FK-Ising model. Preprint.

[CD+13] D. Chelkak, H. Duminil-Copin, C. Hongler, A. Kemppainen and S. Smirnov. Convergence of Ising interfaces to Schramm's SLEs. Preprint.

[CHI12] D. Chelkak, C. Hongler, and K. Izyurov. Conformal invariance of spin correlations in the planar Ising model. Preprint, *arXiv:1202.2838*, 2012.

[CS12] D. Chelkak and S. Smirnov. Universality in the 2D Ising model and conformal invariance of fermionic observables. Invent. Math. 189, no. 3, 515–580, 2012.

[DGS07] Y. Deng, T. M. Garoni, and A. D. Sokal. Critical Speeding-Up in the Local Dynamics of the Random-Cluster Model. *Phys. Rev. Lett.*, 98(23), 2007.

[Du09] J. Dubédat. SLE and the free field: partition functions and couplings. *J. Amer. Math. Soc.*, 22(4):995–1054, 2009.

[Du11] J. Dubédat. Exact bosonization of the Ising model. Preprint, *arXiv:1112.4399*, 2011.

[Dum13] H. Duminil-Copin. Limit of the Wulff Crystal when approaching criticality for site percolation on the triangular lattice. *arXiv:1305.6034*, 2013.

[DG] H. Duminil-Copin and C. Garban. Critical exponents in FK-Ising percolation. In preparation.

[DHN11] H. Duminil-Copin, C. Hongler, and P. Nolin. Connection probabilities and RSW-type bounds for the two-dimensional FK Ising model. *Comm. Pure Applied Math.*, 64:1165–1198, 2011.

[DS12] H. Duminil-Copin and S. Smirnov. Conformal invariance of lattice models. Probability and Statistical Physics in Two and More Dimensions, Editors David Ellwood, Charles Newman, Vladas Sidoravicius, Wendelin Werner, Clay Mathematics Proceedings, vol. 15, Amer. Math. Soc., Providence, RI, 2012.

[DS11] B. Duplantier and S. Sheffield. Liouville quantum gravity and KPZ. *Invent. Math.*, 185(2):333–393, 2011.

[EK86] S. N. Ethier and T. G. Kurtz. *Markov processes.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. Characterization and convergence.

[EMS09] S.N. Evans, B. Morris, and A. Sen. Coalescing systems of non-Brownian particles, 2009. To appear in Probability Theory and Related Fields, arXiv:0912.0017.

[FFS92] R. Fernández, J. Fröhlich, and A. D. Sokal. *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory.* Springer-Verlag, Berlin, 1992.

[FINR04] L. R. G. Fontes, M. Isopi, C. M. Newman, and K. Ravishankar. The Brownian web: characterization and convergence. *Ann. Probab.*, 32(4):2857–2883, 2004.

[FOT94] M. Fukushima, Y. Oshima, M. Takeda Dirichlet Forms and Symmetric Markov Processes, De Gruyter Studies in Mathematics 19, Walter de Gruyter, Berlin and Hawthorne, New York, 1994.

[Gri67] R. B. Griffiths. Correlations in Ising ferromagnets. II. External magnetic fields. *J. Math. Phys.*, 8:484–489, 1967

[GHS70] R. B. Griffiths, C. A. Hurst, and S. Sherman. Concavity of magnetization of an Ising ferromagnet in a positive external field. *J. Math. Phys.*, 11:790–795, 1970.

[Gri95] G. Grimmett. The stochastic random-cluster process and the uniqueness of random-cluster measures. *Ann. Probab.*, 23(4):1461–1510, 1995.

[Gri99] G. Grimmett. *Percolation.* Grundlehren der mathematischen Wissenschaften 321. Springer-Verlag, Berlin, second edition, 1999.

[Gri06] G. Grimmett. *The Random-Cluster Model*, volume 333 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006.

[HPS97] O. Häggström, Y. Peres, and J. E. Steif. Dynamical percolation. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(4):497–528, 1997.

[HPS12] A. Hammond, G. Pete, and O. Schramm. Local time on the exceptional set of dynamical percolation, and the Incipient Infinite Cluster. *Preprint*, arXiv:1208.3826, 2012.

[HS94] T. Hara and G. Slade. Mean-field behaviour and the lace expansion. In *Probability and phase transition (Cambridge, 1993)*, volume 420 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 87–122. Kluwer Acad. Publ., Dordrecht, 1994.

[Jon96] O. D. Jones. Transition probabilities for the simple random walk on the Sierpiński graph. *Stochastic Process. Appl.*, 61(1):45–69, 1996.

[Kah85] J-P. Kahane. Sur le chaos multiplicatif. *Ann. Sci. Math. Québec*, 9(2):105–150, 1985.

[Kas78] Y. Kasahara. Tauberian theorems of exponential type. *J. Math. Kyoto Univ.*, 18(2): 209–219, 1978.

[KKL88] J. Kahn, G. Kalai, and N. Linial. The influence of variables on boolean functions. *29th Annual Symposium on Foundations of Computer Science*, (68–80), 1988.

[Ke80] H. Kesten. The critical probability of bond percolation on the square lattice equals 1/2. Comm. Math. Phys. 74, no. 1, 41–59, 1980.

[Ke87] H. Kesten. Scaling relations for 2D-percolation. Comm. Math. Phys. 109, no. 1, 109–156, 1987.

[KZ87] H. Kesten and Y. Zhang. Strict inequalities for some critical exponents in two-dimensional percolation. *J. Statist. Phys.* 46, no. 5-6, 1031–1055, 1987.

[KPZ88] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov. Fractal structure of 2D-quantum gravity. *Modern Phys. Lett. A*, 3(8):819–826, 1988.

[LSW02] G. F. Lawler, O. Schramm and W. Werner. One-arm exponent for critical 2D percolation. *Electron. J. Probab.* 7, no. 2, 2002.

[LS11] G.F. Lawler and S. Sheffield. A natural parametrization for the Schramm-Loewner evolution. Ann. Probab. 39 (2011), no. 5, 1896–1937.

[LT11] M. Ledoux and M. Talagrand. *Probability in Banach spaces.* Classics in Mathematics. Springer-Verlag, Berlin, 2011. Isoperimetry and processes, reprint of the 1991 edition.

[LG07] J-F. Le Gall. The topological structure of scaling limits of large planar maps. *Invent. Math.*, 169(3):621–670, 2007.

[LG13] J-F. Le Gall. Uniqueness and universality of the Brownian map. Ann. Probab. **41**, 2880–2960, 2013.

[LGM11] J-F. Le Gall and G. Miermont. Scaling limits of random trees and planar maps. arXiv:1101.4856, 2011.

[LJ06] Y. Le Jan. New developments in stochastic dynamics. In *International Congress of Mathematicians. Vol. III*, pages 649–667. Eur. Math. Soc., Zürich, 2006.

[LJL04] Y. Le Jan and S. Lemaire. Products of Beta matrices and sticky flows. *Probab. Theory Related Fields*, 130(1):109–134, 2004.

[LJR04a] Y. Le Jan and O. Raimond. Flows, coalescence and noise. *Ann. Probab.*, 32(2):1247–1315, 2004.

[LJR04b] Y. Le Jan and O. Raimond. Sticky flows on the circle and their noises. *Probab. Theory Related Fields*, 129(1):63–82, 2004.

[LyP13] R. Lyons, with Y. Peres. *Probability on trees and networks.* Book in preparation, present version is at http://mypage.iu.edu/~rdlyons, 2013.

[MS10] N. Makarov and S. Smirnov. Off-critical lattice models and massive SLEs. XVIth International Congress on Mathematical Physics, 362–371, World Sci. Publ., Hackensack, NJ, 2010.

[MW73] B. M. McCoy and T. T. Wu. *The two-dimensional Ising model.* University Press, Cambridge, Massachusetts, 1973.

[Mie13] G. Miermont. The Brownian map is the scaling limit of uniform random plane quadrangulations. Acta Math. 210, 319–401 (2013)

[NRS05] C. M. Newman, K. Ravishankar, and Rongfeng Sun. Convergence of coalescing nonsimple random walks to the Brownian web. *Electron. J. Probab.*, 10:no. 2, 21–60, 2005.

[N08a] P. Nolin. Near-critical percolation in two dimensions. *Electron. J. Probab.* 13, no. 55, 1562–1623, 2008.

[N08b] P. Nolin. Critical exponents of planar gradient percolation. *Ann. Probab.* 36 no. 5, 1748–1776, 2008.

[NW09] P. Nolin and W. Werner. Asymmetry of near-critical percolation interfaces. *J. Amer. Math. Soc.*, 22(3):797–819, 2009.

[On44] L. Onsager. Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev. (2)* 65, 117–149, 1944.

[RSV] R. Rhodes,J. Sohier,V. Vargas. $\star$-scale invariant random measures, to appear in Annals of Probability, available on arxiv.

[RV10] R. Robert and V. Vargas. Gaussian multiplicative chaos revisited, *Annals of Probability*, **38** 2 (2010) 605-631.

[RV11] R. Rhodes and V. Vargas. KPZ formula for log-infinitely divisible multifractal random measures, *ESAIM Probability and Statistics*, **15** (2011) 358.

[RV13] R. Rhodes and V. Vargas. Gaussian multiplicative chaos and applications: a review. arXiv:1305.6221, 2013.

[Sa97] C. Sabot. Existence and uniqueness of diffusions on finitely ramified self-similar fractals. (English, French summary) *Ann. Sci. École Norm. Sup.* (4) 30 (1997), no. 5, 605–673.

[Sch00] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* 118, 221–288, 2000.

[Sch07] O. Schramm. Conformally invariant scaling limits: an overview and a collection of problems. *International Congress of Mathematicians. Vol. I*, 513–543, Eur. Math. Soc., Zürich, 2007.

[SchSm11] O. Schramm and S. Smirnov. On the scaling limits of planar percolation. With an appendix by Christophe Garban. *Ann. Probab.* 39, no. 5, 1768–1814, 2011.

[SchSt10] O. Schramm and J. E. Steif. Quantitative noise sensitivity and exceptional times for percolation. *Ann. of Math. (2)* 171, no. 2, 619–672, 2010.

[Sh07] S. Sheffield. Gaussian free fields for mathematicians, *Probab. Th. Rel. Fields*, **139**, 2007. 521–541.

[Sh09] S. Sheffield. Exploration trees and conformal loop ensembles. (English summary) *Duke Math. J.* 147 (2009), no. 1, 79–129.

[Sh10] S. Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. ArXiv:1012.4797, 2010.

[Sm01] S. Smirnov. Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits. *C. R. Acad. Sci. Paris Sér. I Math.* 333, no. 3, 239–244, 2001.

[Sm10] S. Smirnov. Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model. Ann. of Math. (2) 172, no. 2, 1435–1467, 2010.

[SW01] S. Smirnov and W. Werner. Critical exponents for two-dimensional percolation. *Math. Res. Lett.*, 8(5-6):729–744, 2001.

[Sto10] P. Stollmann. A dual characterization of length spaces with application to Dirichlet metric spaces, Studia Mathematica. 2010.

[Stu98] K. T. Sturm. The geometric aspect of Dirichlet forms, New directions in Dirichlet forms, 233-277, AMS/IP Stud. Adv. Math., 8, Amer. Math. Soc., Providence, RI, 1998.

[TW98] B. Tóth and W. Werner. The true self-repelling motion. *Probab. Theory Related Fields*, 111(3):375–452, 1998.

[Tu40] J. W. Tukey. *Convergence and uniformity in topology.* Annals of Mathematics Studies, no. 2. Princeton University Press, Princeton, N. J., 1940.

[Tsi04] B. Tsirelson. Scaling limit, noise, stability. In *Lectures on probability theory and statistics*, volume 1840 of *Lecture Notes in Math.*, pages 1–106. Springer, Berlin, 2004.

[We07] W. Werner. *Lectures on two-dimensional critical percolation.* IAS Park City Graduate Summer School, 2007.

[Wu66] T. T. Wu. Theory of Toeplitz determinants and the spin correlations of the two-dimensional Ising model. I. *Phys. Rev.*, 149:380–401, 1966.