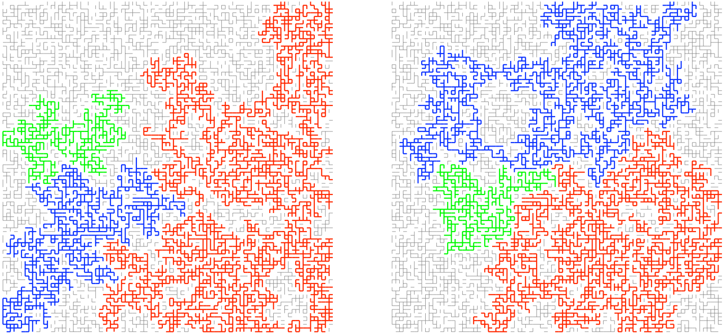
Lectures on noise sensitivity and percolation

Christophe Garban and Jeffrey E. Steif



Clay summer school, Buzios 2010

Boolean functions

Definition

A Boolean function is a function

$$f : \{-1,1\}^n \to \{0,1\} \text{ OR } \{-1,1\}$$

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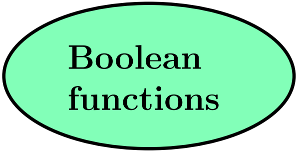
Example: **Majority**

$$f(x_1, \ldots, x_n) = \mathrm{sign}(\sum_{i=1}^{n} x_i)$$


Boolean
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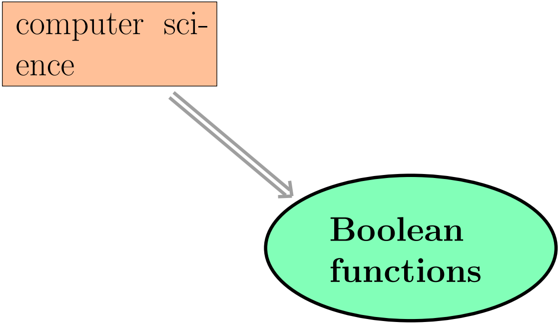
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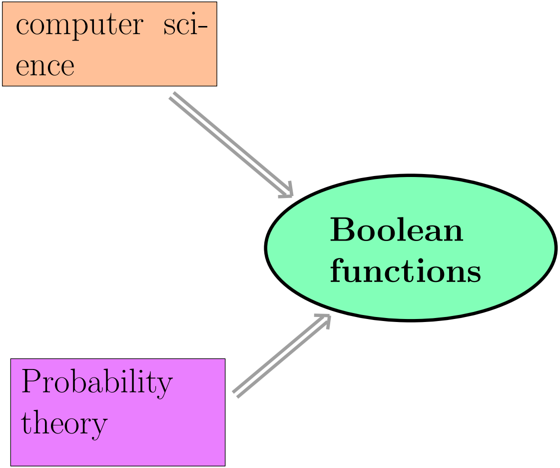
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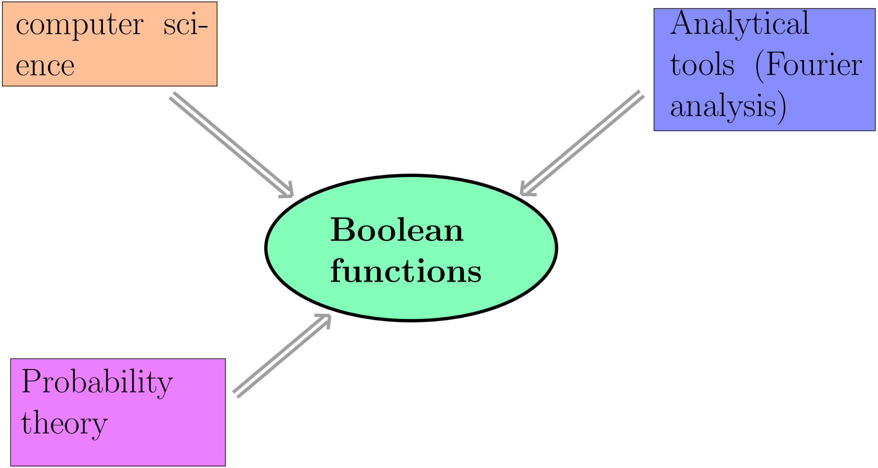
Probability theory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Analytical tools (Fourier analysis)
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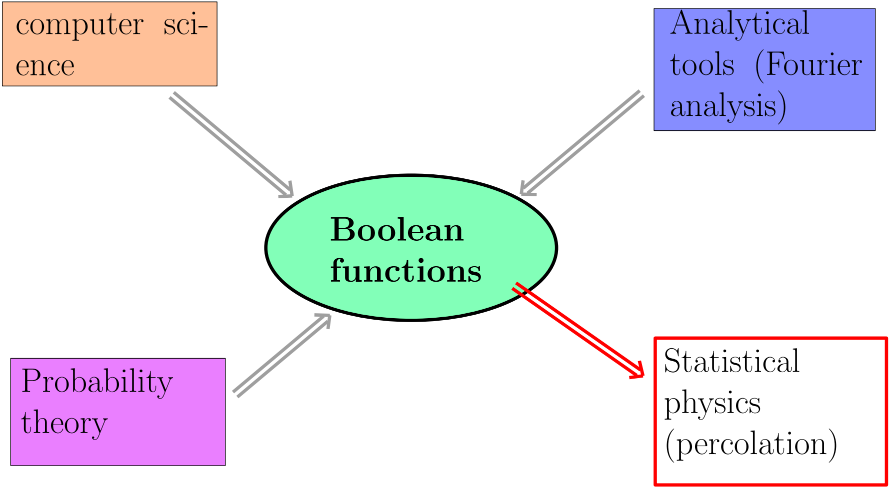
computer science
Analytical tools (Fourier analysis)
Boolean functions
Probability theory
Statistical physics (percolation)

A concrete situation : VOTING SCHEMES

Imagine one has n people labelled $1, \dots, n$ which are deciding between candidates A and B according to a certain procedure or *voting scheme*. This procedure can be represented by a Boolean function

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For instance, you may think of

$$\begin{cases} A &= \text{Al Gore} \\ B &= \text{Bush} \\ n &\approx 10^8 \end{cases}$$

Noise stability

Suppose the election is "well-balanced" between A and B. One may thus consider the actual configuration of votes as a **random**

$$\omega = (x_1, \ldots, x_n) \in \{-1, 1\}^n ,$$

sampled according to the **uniform measure**. The outcome of the election should be $f(\omega)$.

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In fact due to inevitable errors in the recording of the votes, the outcome is $f(\omega^\epsilon)$ instead. Here ω^ϵ is a "slight **perturbation**" of ω.

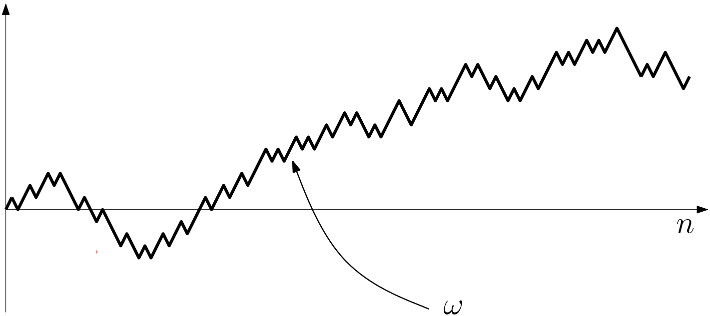
Informal definition

Noise stability corresponds to

$$\mathbb{P}\big[f(\omega) \neq f(\omega^\epsilon)\big] \quad \text{being "small"}.$$

Case of the majority function

If $f(\omega) = \mathrm{sign}(\sum x_i)$,



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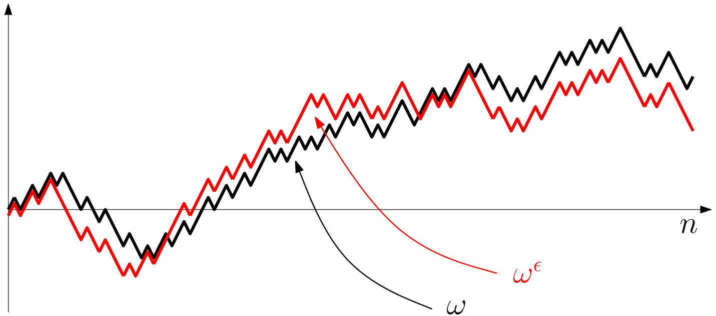
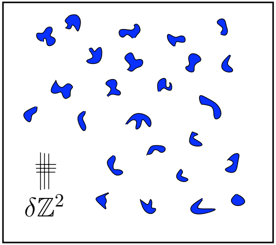


Percolation

Sub-critical ($p < p_c$)



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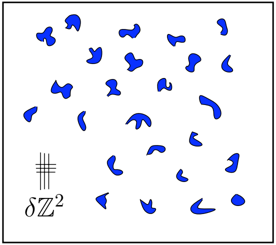

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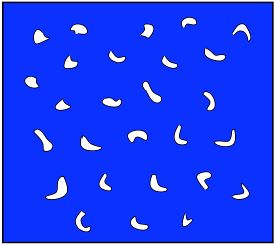
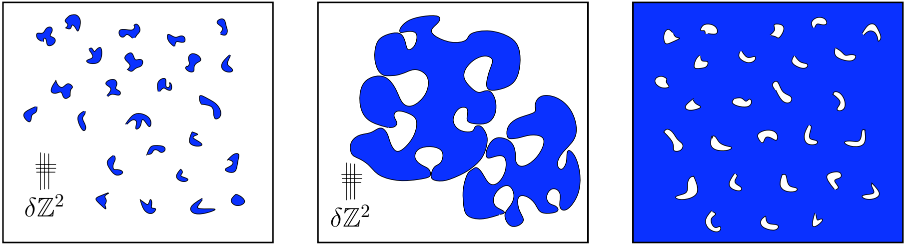


Percolation

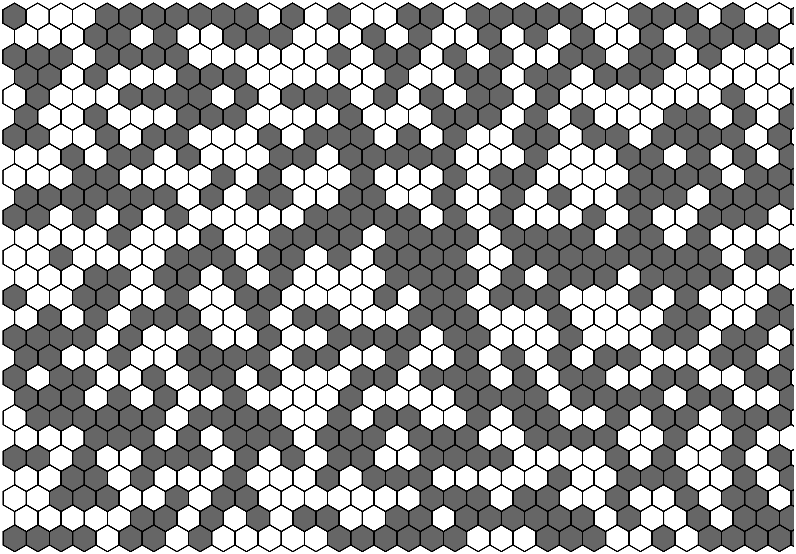
Sub-critical $(p < p_c)$ Critical (p_c) Super-critical $(p > p_c)$



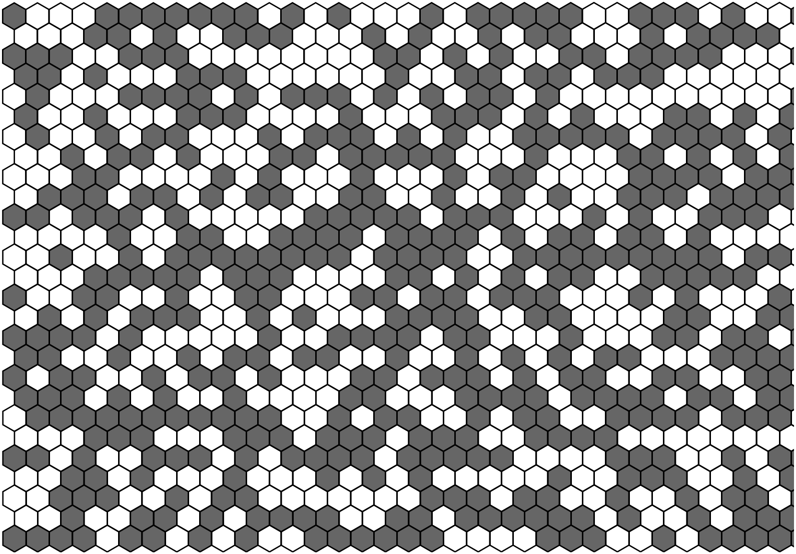
Question

How does critical percolation "react" to perturbations ?

ω:



$\omega \longrightarrow \omega^{\epsilon}$:



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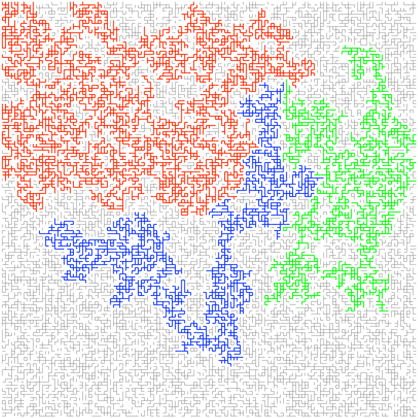

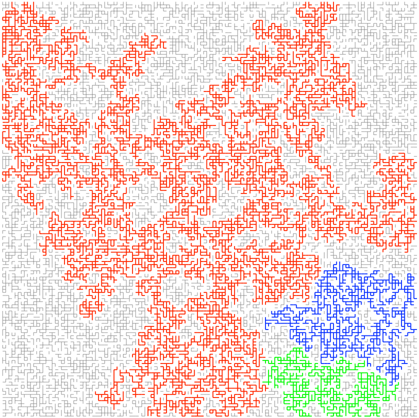

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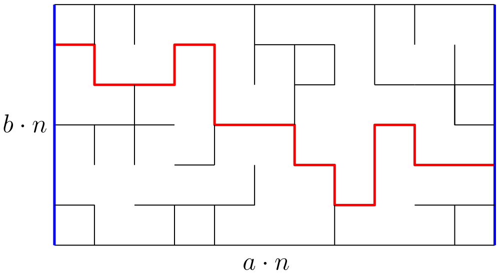

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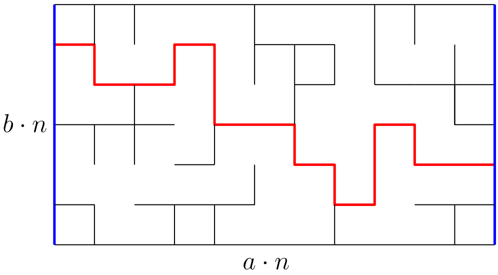

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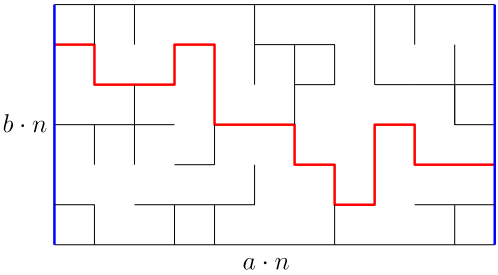

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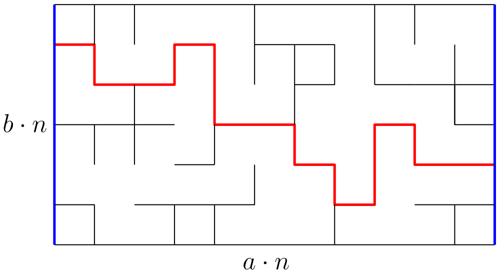

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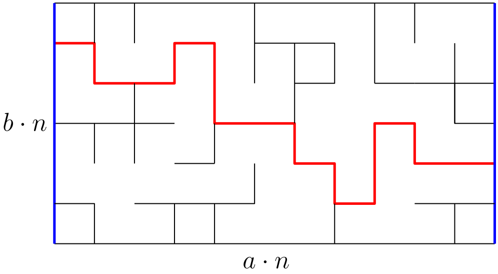

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Informal definition

Noise sensitivity corresponds to $f_n(\omega)$ and $f_n(\omega^\epsilon)$ being very little correlated (i.e. $\mathrm{Cov}(f_n(\omega), f_n(\omega^\epsilon))$ being very small).

Applications to dynamical percolation

Informal definition

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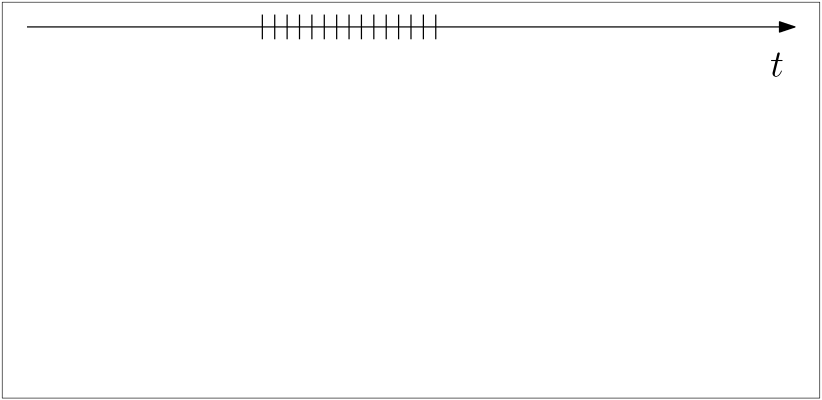

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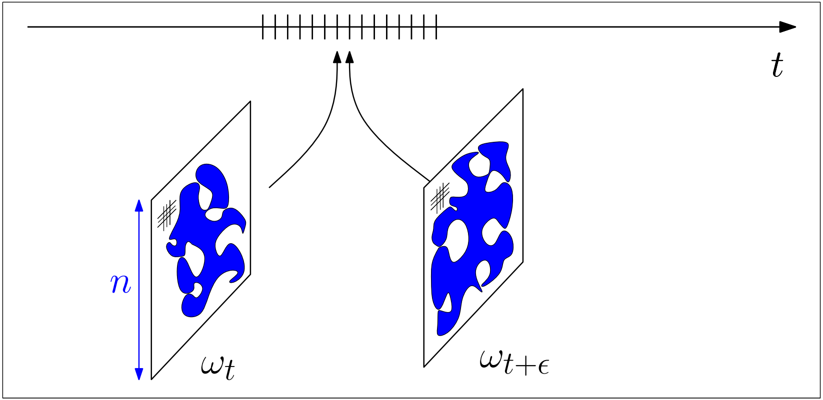

Applications to Sub-Gaussian fluctuations

Informal definition (First Passage Percolation)

Let $0 < a < b$. Define the **random metric** *on the graph $\mathbb{Z}^d$ as follows: for each edge $e \in \mathbb{E}^d$, fix its length τ_e to be a with probability 1/2 and b with probability $1/2$.*

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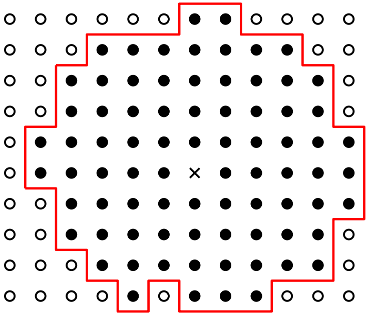

It is well-known that the random ball

$$B_\omega(R) := \{x \in \mathbb{Z}^d, \underset{\omega}{\mathrm{dist}}(0, x) \leq R\}$$

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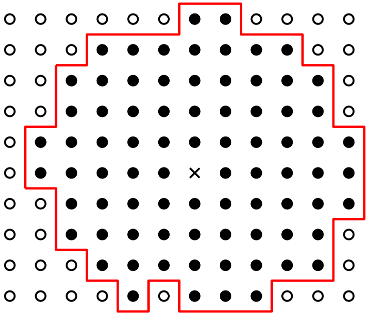

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Question

What are the fluctuations around this asymptotic shape ?

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Fact

f being noise sensitive will correspond to f being of "High frequency".

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- **Discrete Fourier analysis**
- **Hypercontractivity**
- **Randomized algorithms**
- Viewing the "frequencies of percolation" as **random fractals** of the plane.