# Near-critical Ising model

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ENS Lyon and CNRS

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# Plan

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   - Notion of **correlation length** $L(p)$

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3. Near-critical Ising model as the external magnetic field varies
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Consider your favorite statistical physics model, for example:

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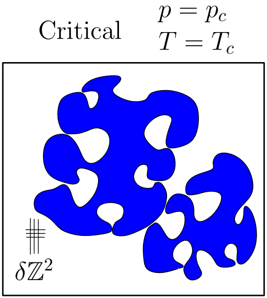

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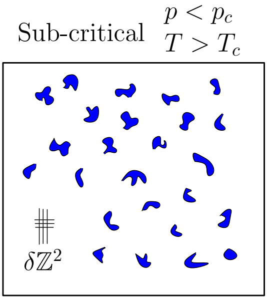


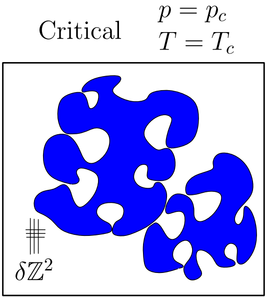

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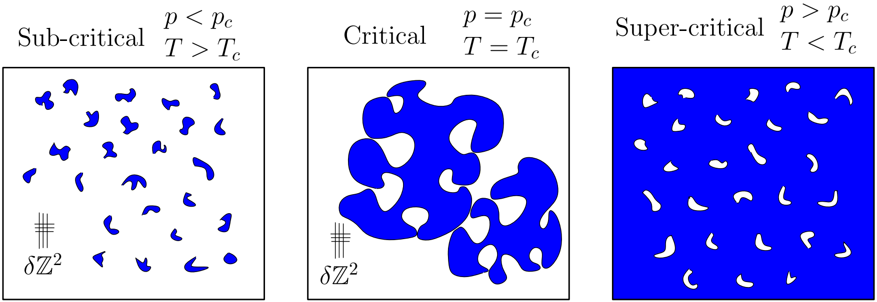

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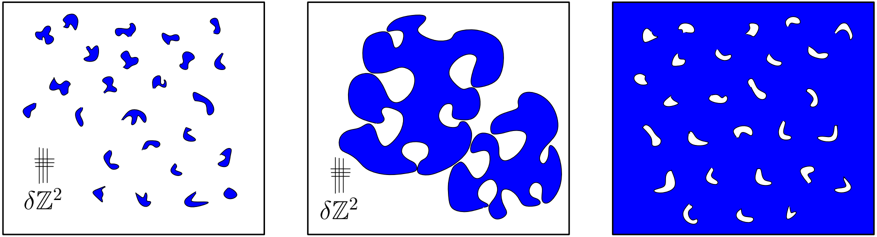

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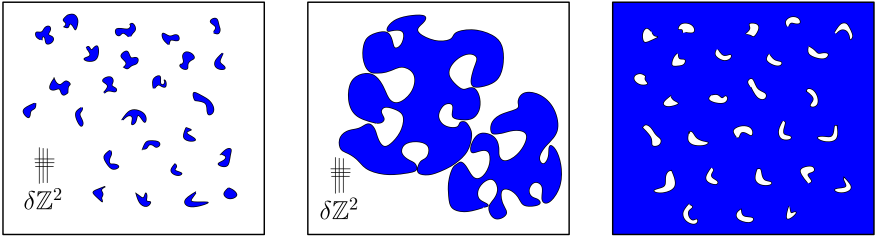
Sub-critical | $T = T_c$ and $h = 0$ | $T = T_c$ and $h > 0$



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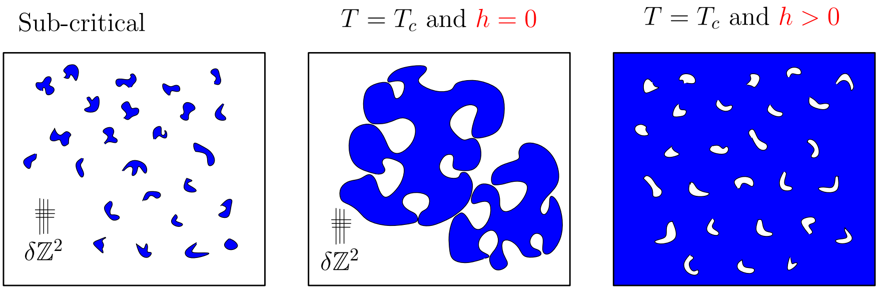
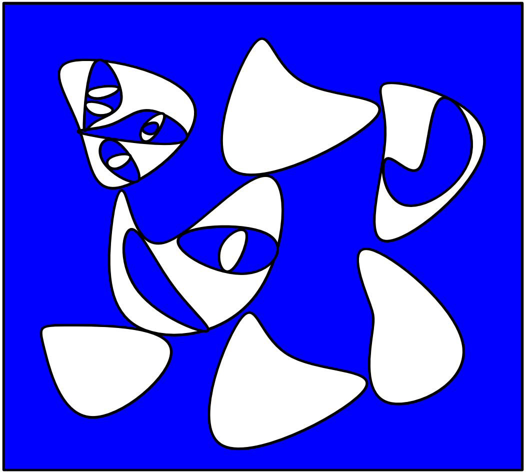



What happens if $T \approx T_c$ or $h \approx 0$ ??

# Notion of correlation length (informal)



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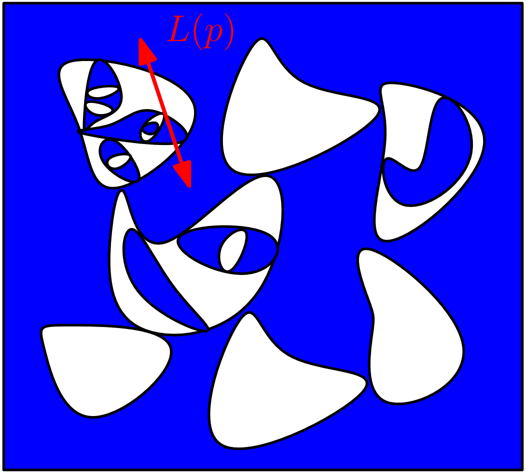


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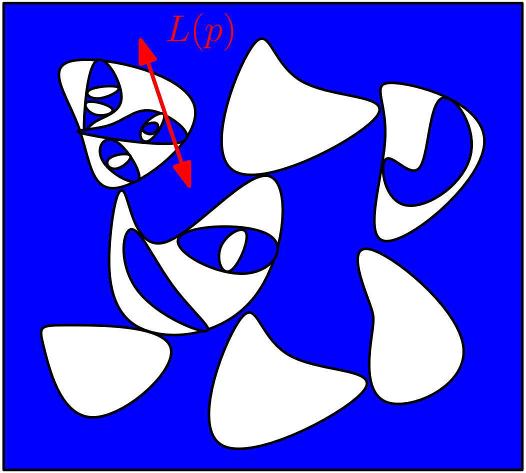


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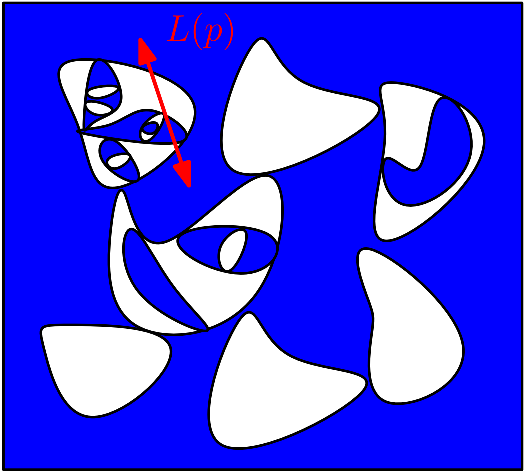


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Example (critical percolation):

**Theorem** (Smirnov-Werner 2001):

$$L(p) = \left|\frac{1}{p-p_c}\right|^{4/3+o(1)}$$

# The models we shall consider

Percolation:

$$P_p(\omega) = p^o \, (1-p)^c$$

$o = o(\omega) =$ Nb of open ed

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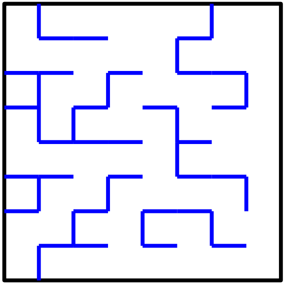

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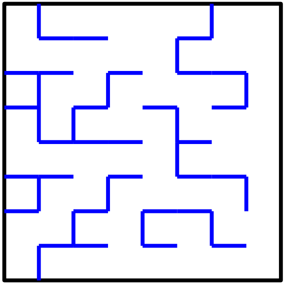

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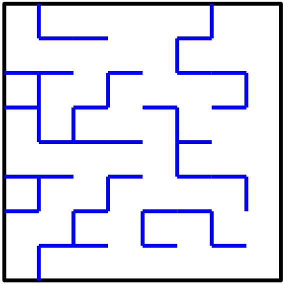

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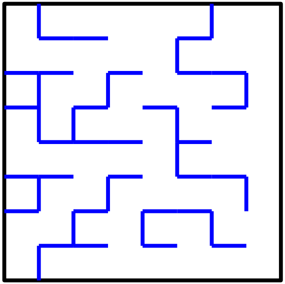

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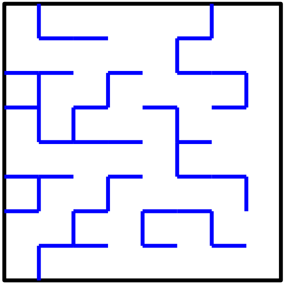
Theorem (Kesten 1980)

$$p_c(\mathbb{Z}^2) = \frac{1}{2}$$



Theorem (Beffara, Duminil-Copin 2010)

$$p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$$

# Notion of correlation length (precise definition)

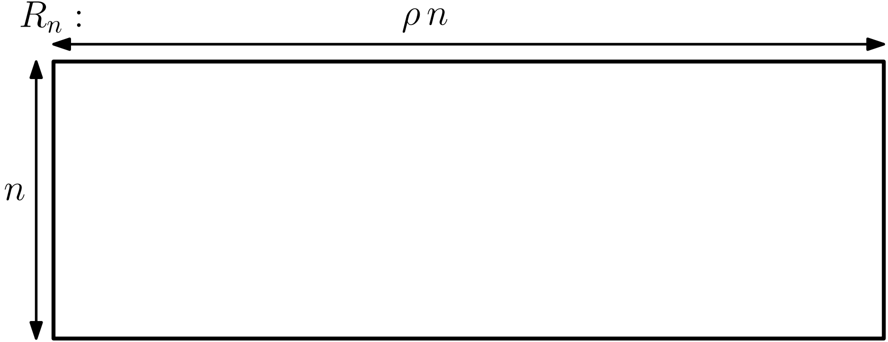
### Definition

Fix $\rho > 0$.

For any $n \geq 0$, let $R_n$ be the rectangle $[0, \rho n] \times [0, n]$. If $p > p_c$, then define for all $\epsilon > 0$ and all "boundary conditions" $\xi$ around $R_n$,

$$L^{\xi}_{\rho,\epsilon}(p) := \inf_{n>0} \left\{ \mathbb{P}^{\xi}_{p}\big(\text{there is a left-right crossing in } R_n\big) > 1 - \epsilon \right\}$$

# Estimating the correlation length, case of critical percolation



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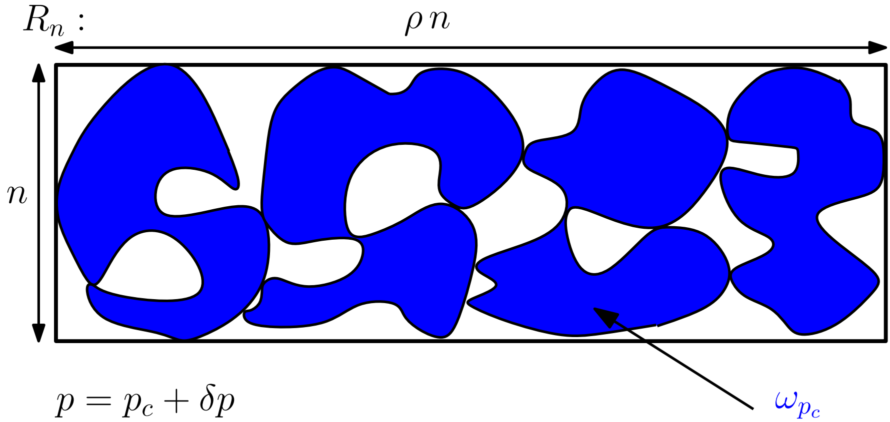


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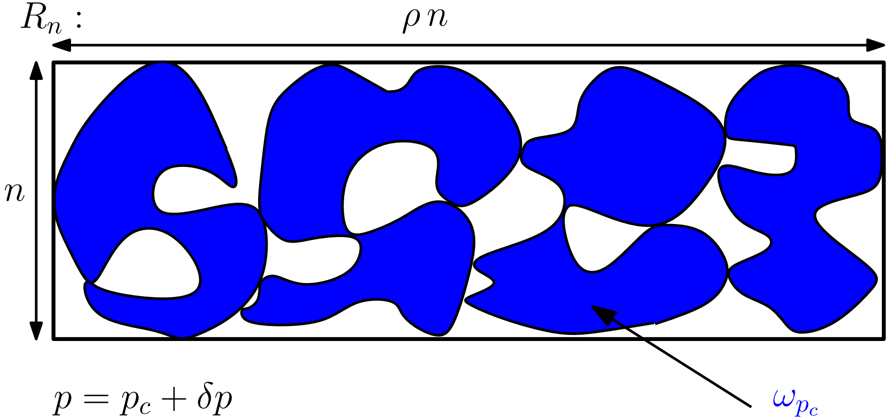


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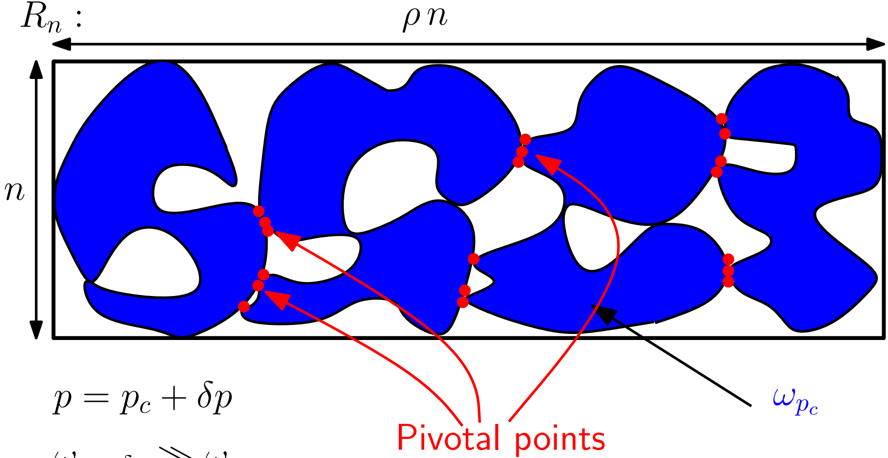


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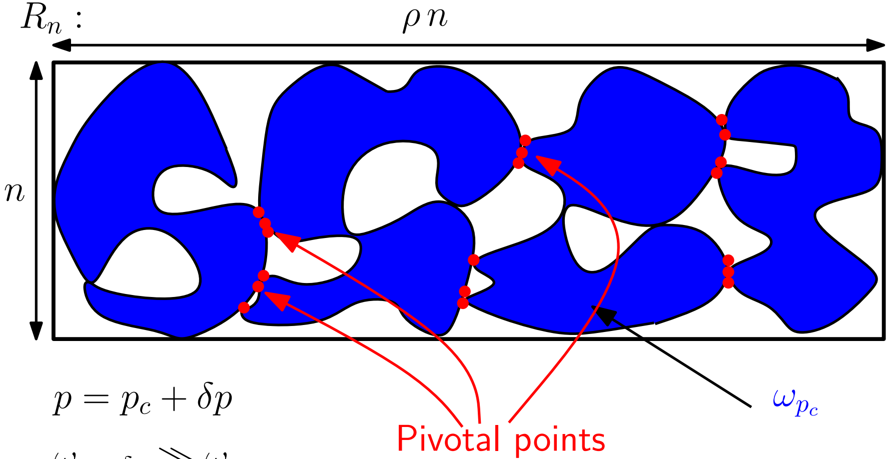


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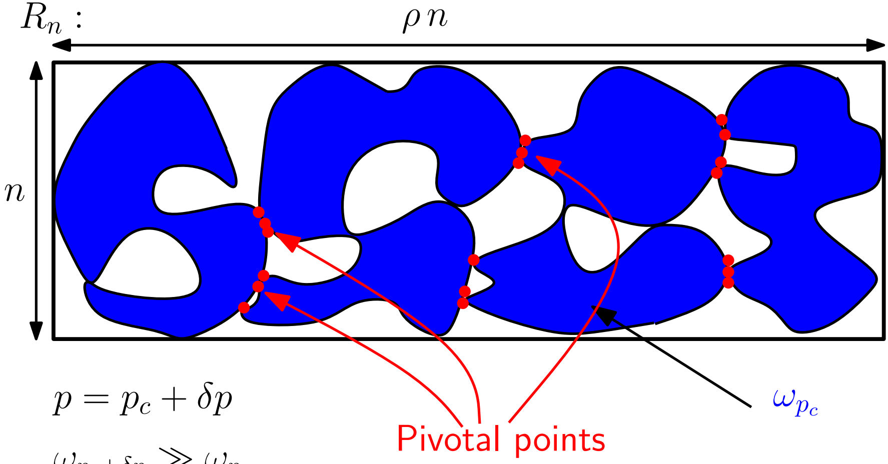


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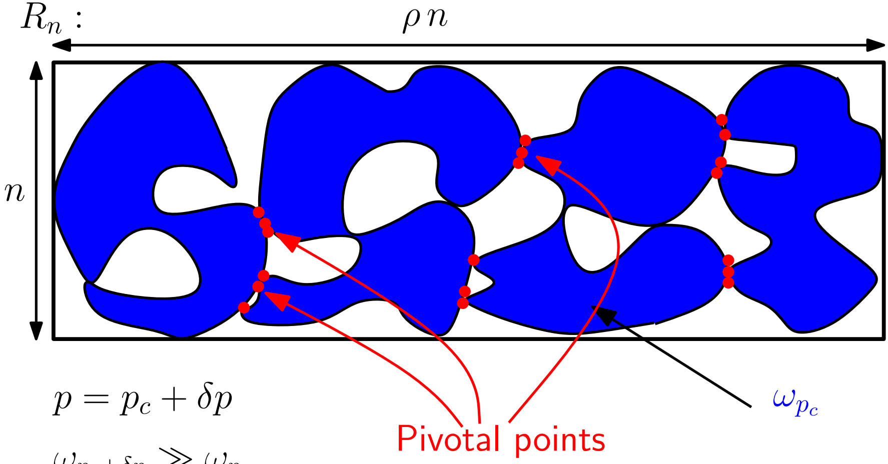


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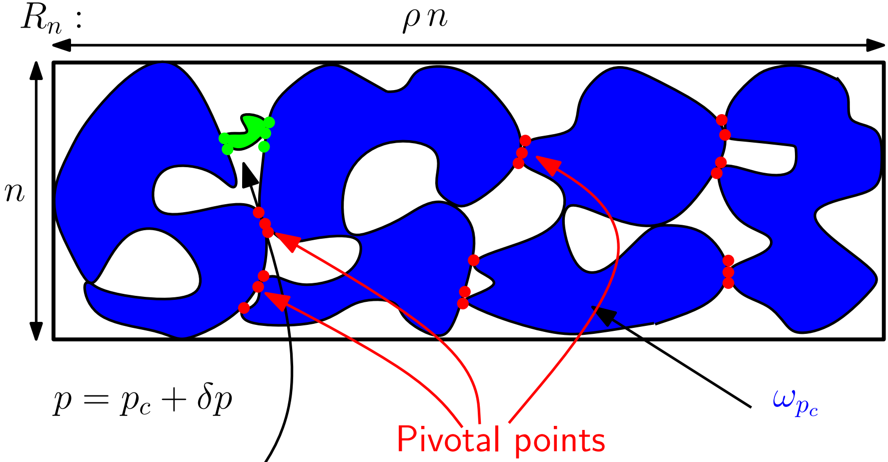


Difficulty !

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# Sharp threshold

To analyze the behavior of the correlation length, it is useful to rely on Russo's formula: if $\phi_n(p) := \mathbb{P}_p\big($ there is a left-right crossing in $R_n\big)$, then

$$\frac{d}{dp}\phi_n(p) = \mathbb{E}_p\big(\text{ Number of pivotal points in } \omega_p\big)$$
$$= \sum_{x\in R_n} \mathbb{P}_p\big(x \text{ is a pivotal point }\big)$$

This point of view also leads to the identity

$$|p - p_c|\; L(p)^2 \alpha_4(L(p)) \asymp 1$$

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But this does not match with related results known since Onsager which suggest that $L(p)$ should instead scale like $\left|\frac{1}{p-p_c}\right| \ll \left|\frac{1}{p-p_c}\right|^{24/13}$ !!

**So what is wrong here !?**

# Monotone couplings of FK percolation, $q = 2$

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Most remains unknown regarding the structure of these random clouds.

# What we can prove

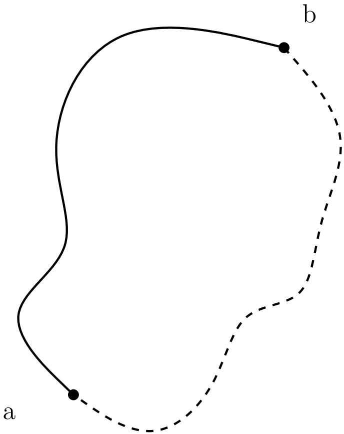
**Theorem (Duminil-Copin, G., Pete, 2011)**

*Fix $q = 2$. For every $\epsilon, \rho > 0$, there is a constant $c = c(\epsilon, \rho) > 0$ s.t.*

$$c \, \frac{1}{|p - p_c|} \leq L^{\xi}_{\rho,\epsilon}(p) \leq c^{-1} \frac{1}{|p - p_c|} \sqrt{\log \frac{1}{|p - p_c|}}$$

*for all $p \neq p_c$, whatever the choice of the boundary condition $\xi$ is.*

# Techniques behind the proof: Smirnov's observable



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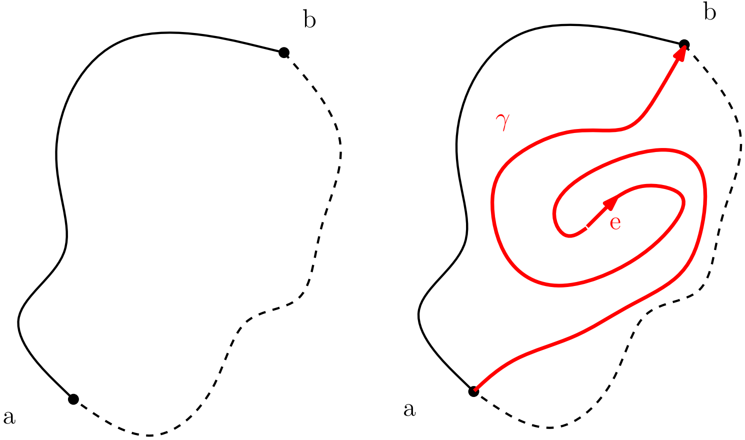

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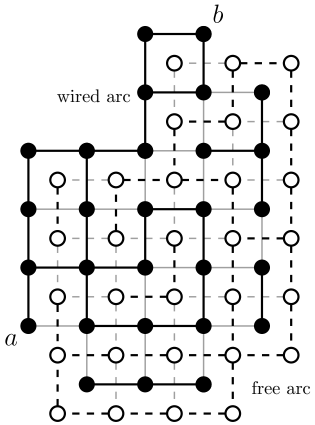


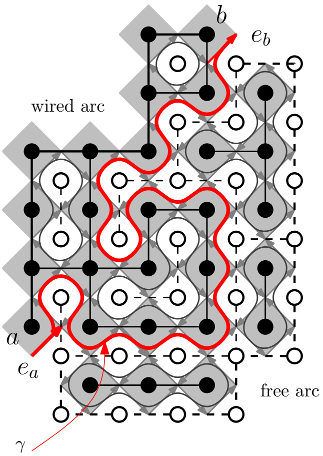

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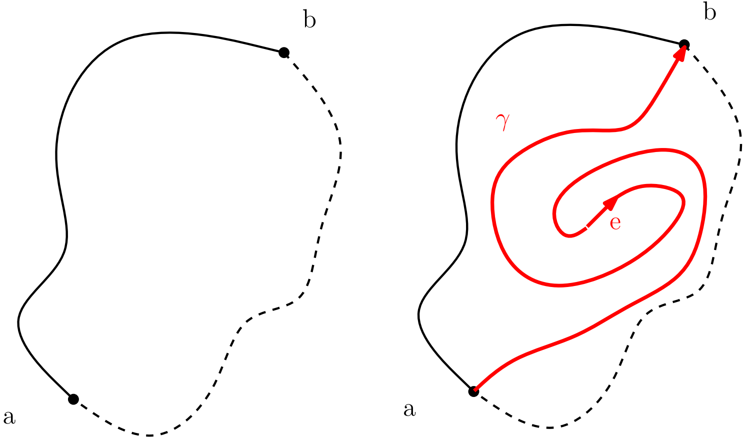


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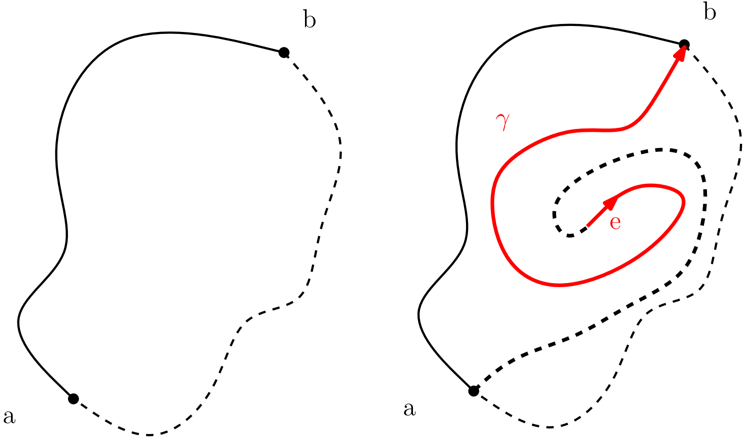


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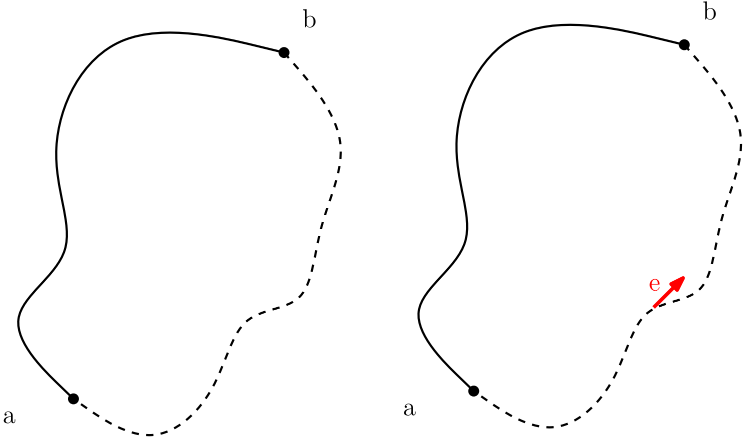


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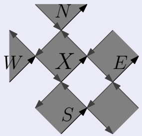
$$|F_p(e)| = \mathbb{P}_{p,2}(e \in \gamma)$$

## “Near-harmonicity” of Smirnov’s observable

### Theorem (Smirnov, exact harmonicity at criticality)

*For* $q = 2$ *and* $p = p_c(2) = \sqrt{2}/(1+\sqrt{2})$*, once restricted to a proper sub-lattice (NE pointing edges), the observable* $F_{p_c}$ *is* **harmonic***:*

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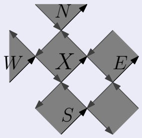



## Theorem (Beffara, Duminil-Copin)

*When $p < p_c$, the observable $F_p$ is now* **massive harmonic***: namely*

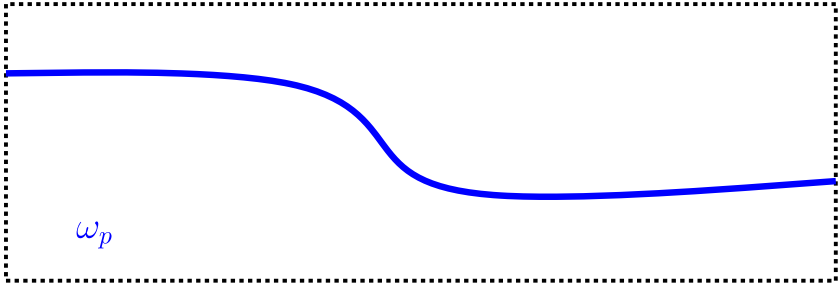
$$\Delta F_p(e_X) = m(p)\, F_p(e_X)\,,$$

*where the* **mass** $m(p) \asymp |p - p_c|^2$.

# Upper-bound on the correlation length

Fix $\rho, \epsilon > 0$. For any $p > p_c$, we want to find a scale $n$ so that the rectangle $R_n$ is crossed horizontally with high probability ($> 1 - \epsilon$).

$R_n$ :



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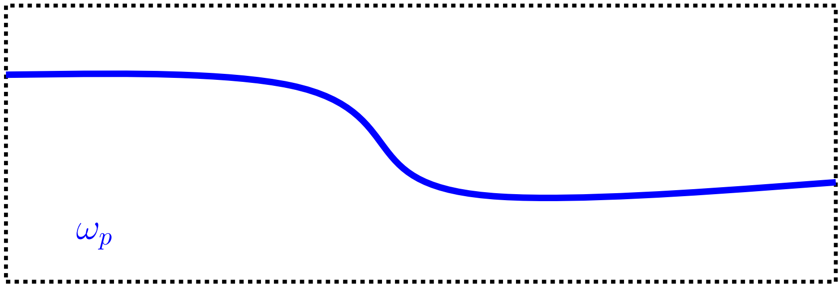


We want this probability under $P_p$ to be $> 1 - \epsilon$

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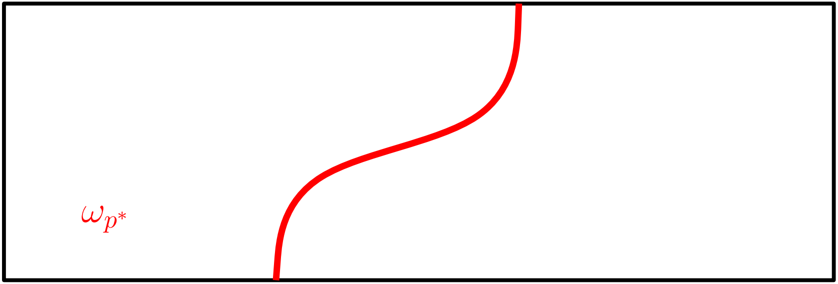


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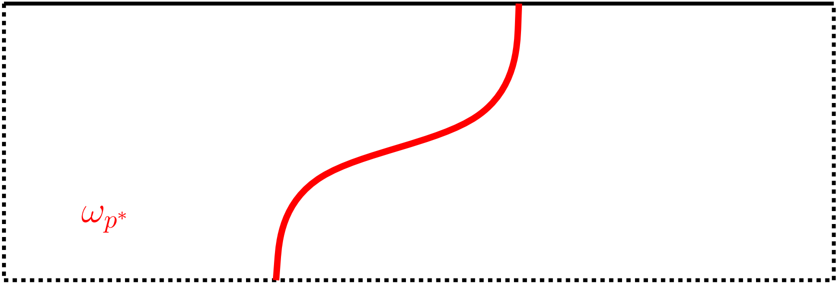


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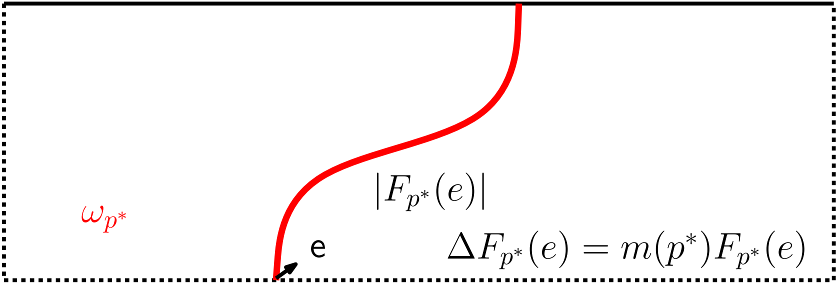


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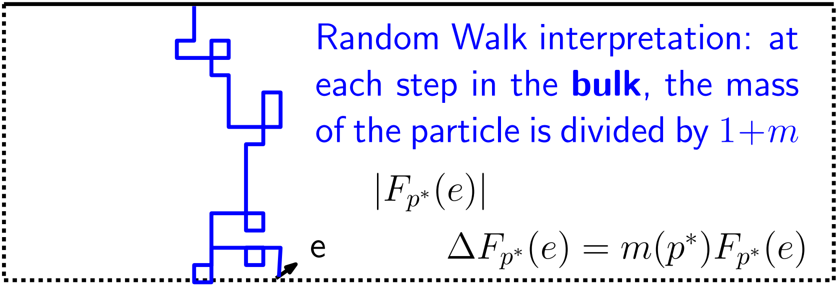

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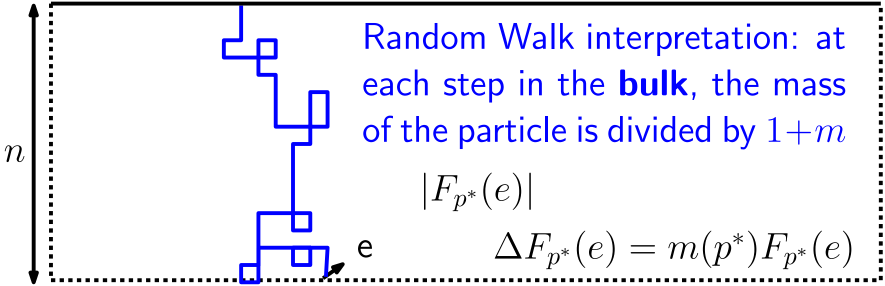


Recall $m(p^*) \asymp |p^* - p_c|^2 \asymp |p - p_c|^2$

If scale $n \gg |p - p_c|^{-1}$, then the RW does more than $|p - p_c|^{-2}$ steps and thus, in average its mass goes to zero.

## Lower-bound on the correlation length

Fix $\rho, \epsilon > 0$. For any $p > p_c$, we want to find scales $n$ so that the rectangle $R_n$ is NOT crossed horizontally with high probability (i.e. with prob $< 1 - \epsilon$).

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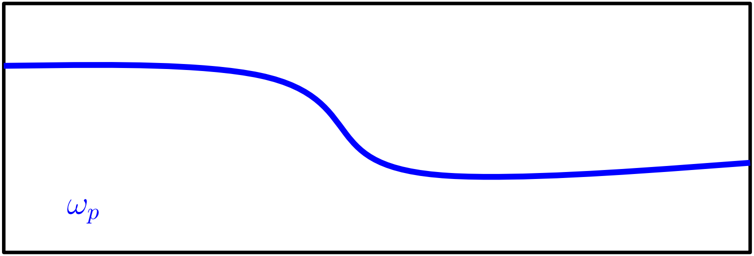

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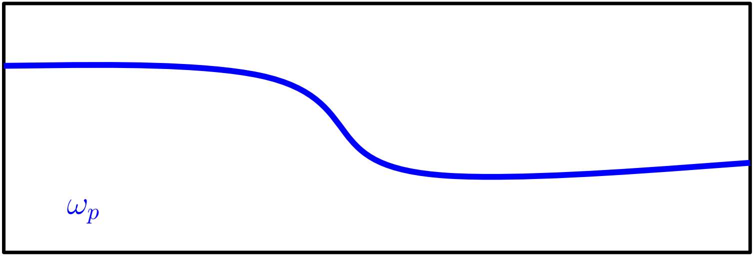


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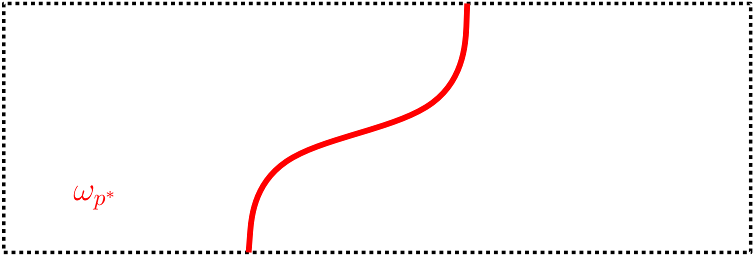


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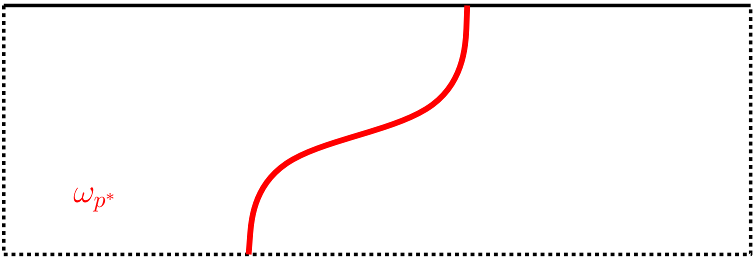


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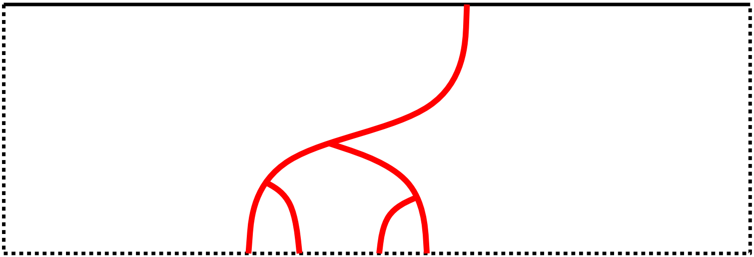

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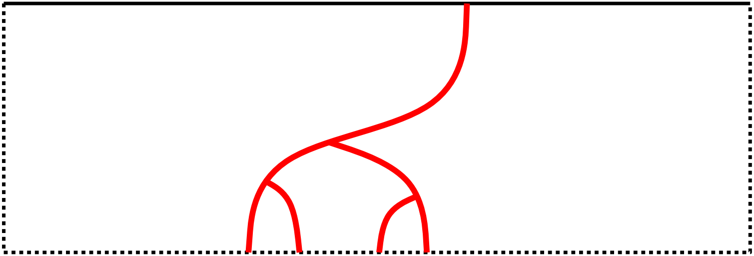

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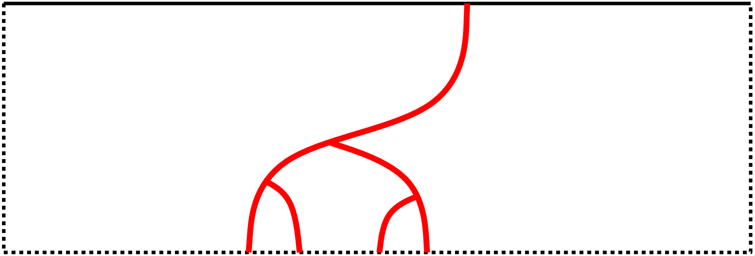

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Using the RSW proof at $p_c$ from Duminil-Copin, Hongler, Nolin, $E_{p^*}(N^2) \leq E_{p_c}(N^2) < c^{-1}n$

# Ising model

To each configuration $\sigma \in \{-1, 1\}^{N^2}$, one associates the Hamiltonian

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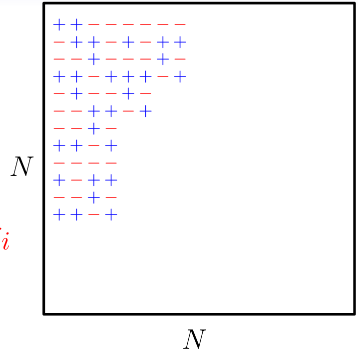

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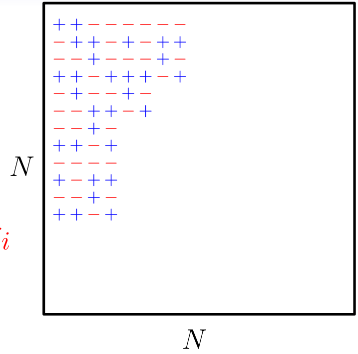


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The Ising model is intimately related with FK percolation ($q = 2$) via the following identity: If $h = 0$,

$$E_\beta(\sigma_x \sigma_y) = P_{p,q=2}[x \leftrightarrow y] \text{ with } 1 - p = e^{-2\beta}$$

# Classical near-critical results

**Theorem (Kesten - Smirnov/Werner)**

*For site percolation on the triangular lattice,*

$$\theta(p) := \mathbb{P}\big[0 \leftrightarrow \infty\big] = |p - p_c|^{5/36+o(1)} \quad \text{as } p \searrow p_c$$

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**Theorem (Onsager, 1944)**

*For Ising model on $\mathbb{Z}^2$:*

$$\langle \sigma_0 \rangle_\beta^+ \asymp |\beta - \beta_c|^{1/8} \quad \text{as } \beta \searrow \beta_c$$

# Average magnetization under small external field

## Theorem (Camia, G., Newman)

*Consider Ising model on* $\mathbb{Z}^2$ *at* $\beta_c$ *with a positive* **external magnetic field** $h > 0$*, then*

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- Lower bound: prove that the correlation length $L(h) \asymp h^{-8/15}$ and conclude using

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- Upper bound: rely on a kind of strong "convexity" property satisfied by the Ising model, namely the **GHS inequality**.

**Theorem (GHS inequality, Griffiths, Hurst, Sherman, 1970)**

$Z_{\beta,h} := \sum_\sigma e^{-\beta H(\sigma) + h \sum \sigma_x}$ *is such that*

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