Magnetization field at criticality in the Ising model

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ENS Lyon and CNRS

Joint work with Federico Camia (Vrije Universiteit Amsterdam)
and Charles Newman (NYU)

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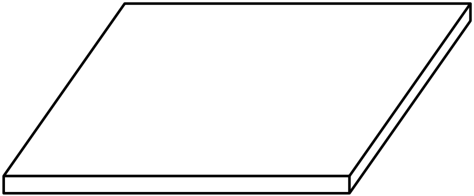
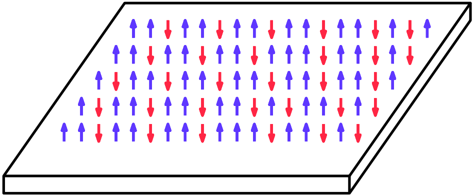
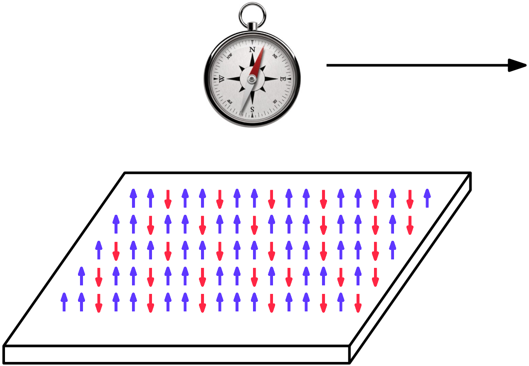
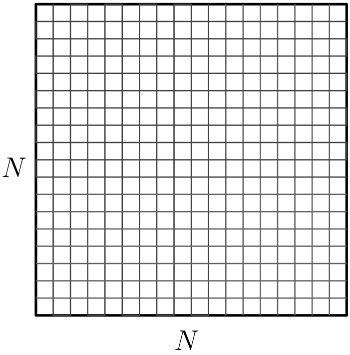






What are the fluctuations of the magnetization field ?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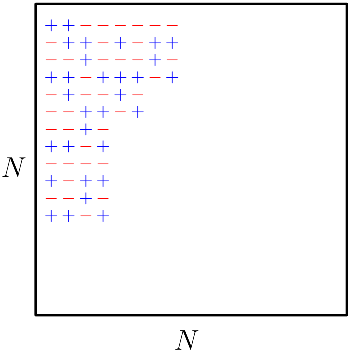
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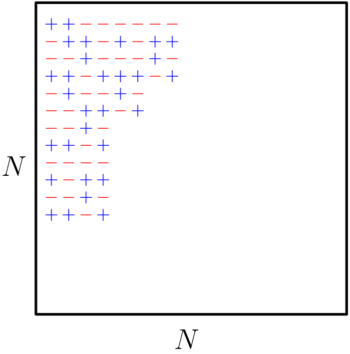
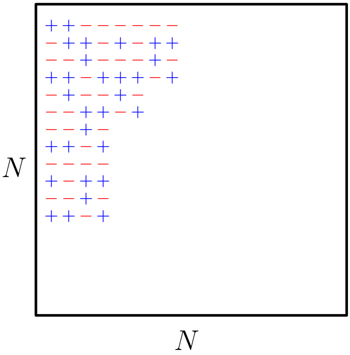


- If the spins are **i.i.d**

$$\frac{1}{N} \sum \sigma_x \to \mathcal{N}(0,1)$$



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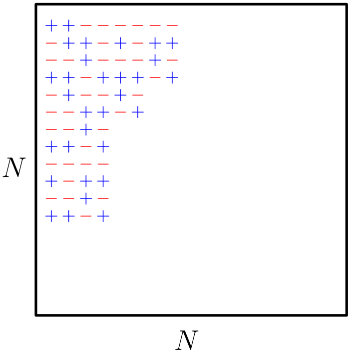
  $$\frac{1}{N}\sum \sigma_x \to \mathcal{N}(0,1)$$

  $$\frac{1}{N}\sum \sigma_x \delta_x \quad \to \quad \mathcal{W},$$ **two-dimensional white noise**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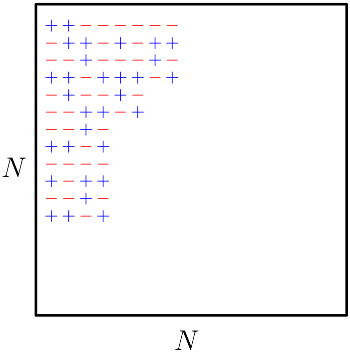
Ising model



To each configuration $\sigma \in \{-1,1\}^{N^2}$, one associates the Hamiltonian

$$H(\sigma) := -\sum_{i\sim j} \sigma_i \sigma_j$$

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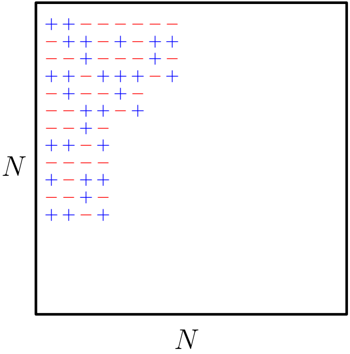

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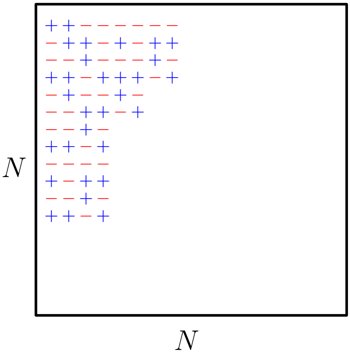
And we define:

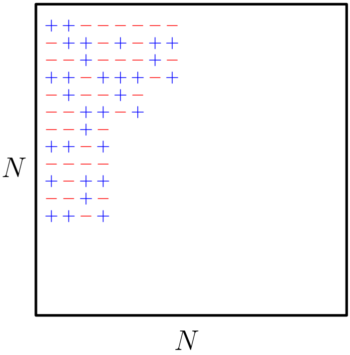
$$P_\beta(\sigma) \propto e^{-\beta H(\sigma)}$$

- If $\beta < \beta_c$

$$\frac{1}{N}\sum \sigma_x \to \mathcal{N}(0, \sigma_\beta^2)$$

$$\frac{1}{N}\sum \sigma_x \, \delta_x \to \sigma_\beta \mathcal{W}$$





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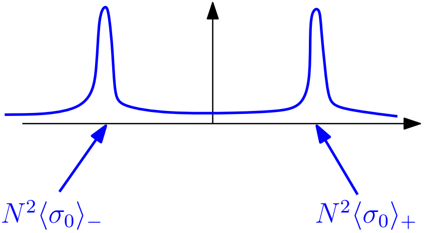

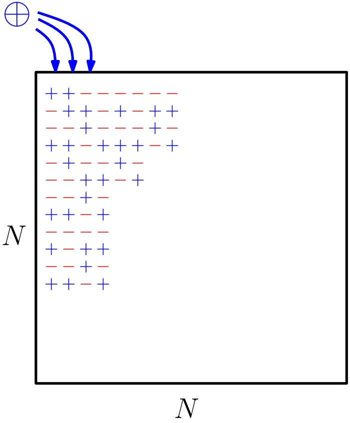

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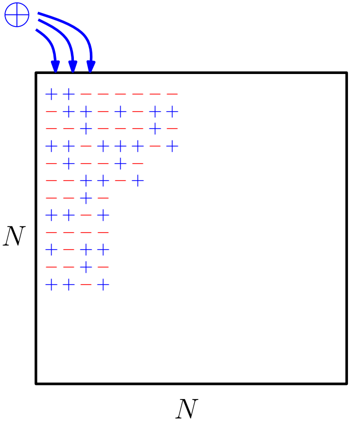
$$\frac{\sum \sigma_x - N^2 \mathbb{E}\left[\sigma_0\right]}{N} \to \mathcal{N}(0, \sigma_\beta^2)$$

- As $\beta \to \beta_c$, $\sigma_\beta \nearrow \infty$



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And now: what about $\beta = \beta_c$??

Which normalization ?

To avoid boundary issues, consider our system on the torus $\mathbb{Z}^2/N\mathbb{Z}^2$. We have:

$$\mathrm{Var}(\sum \sigma_x) \;\; = \;\; \sum_{x,y} \mathbb{E}\big[\sigma_x \sigma_y\big]$$

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Hence it is natural to look at the random variable $m^{(N)} := \frac{\sum \sigma_x}{N^{15/8}}$.

Question

*Does $m^{(N)}$ have a (**unique**) scaling limit ?*

"Subtle" issue of renormalization

Theorem (McCoy, Wu, 1967)

As $N \to \infty$

$$\left\langle \sigma_{0,0}\ \sigma_{N,N} \right\rangle \sim c\, N^{-1/4}$$

where $c = 2^{1/12} e^{3\zeta'(-1)}$

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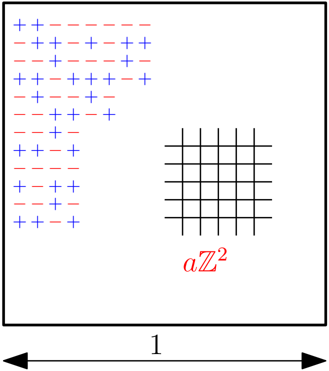
Proposition (Rotational invariance of the two-point function)

$$\frac{\langle \sigma_0\ \sigma_x \rangle}{\langle \sigma_0\ \sigma_{\|x\|_2} \rangle} \xrightarrow[\|x\|_2 \to \infty]{} 1$$

Notations / definitions

Rescaled lattice $a\,\mathbb{Z}^2$, $a \ll 1$.



Definition (Renormalized magnetization field)

$$\Phi^a := \sum_{x \in a\,\mathbb{Z}^2} \delta_x \, \sigma_x \, a^{\frac{15}{8}}$$

$$m^a := \langle \Phi^a, 1_{[0,1]^2} \rangle$$

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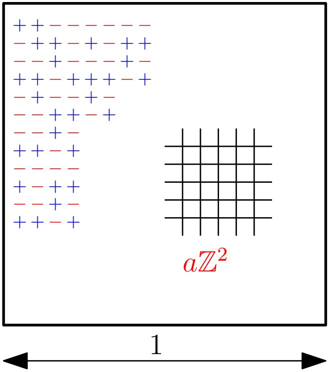

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Question

The field $\Phi^a \in \mathcal{D}'$.
Is it the case that Φ^a converges as $a \to 0$ to some random distribution Φ ?

Road map

- First part :

Theorem (Camia, G., Newman)

(i) *The magnetization field* Φ^a *on* $a\mathbb{Z}^2$ *has a unique scaling limit as the mesh* $a \longrightarrow 0$.

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- Second part: properties of the limiting objects Φ and m
  (i) Conformal covariance of Φ_Ω
  (ii) **Tail behavior** of m etc ...

- Applications (near-critical Ising model in the h-direction ...)

Behavior of the magnetization at β_c for small external field

Theorem (Camia, G., Newman)

Consider Ising model on $\mathbb{Z}^2$ *at* β_c *with a positive* **external magnetic field** $h > 0$, *then*

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This is the analog of

(i) Onsager: $\langle \sigma_0 \rangle_\beta^+ \asymp |\beta - \beta_c|^{1/8}$ as $\beta \to \beta_c$.

and

(ii) Kesten - Smirnov/Werner: $\theta(p) = |p - p_c|^{5/36+o(1)}$ for site percolation on the triangular lattice.

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Since $\overline{B_{\mathcal{H}^{-2}}(0,1)}$ is compact in $\mathcal{H}^{-3}$, one can take subsequential scaling limits $\Phi^{a_k} \to \Phi^*$ in $\mathcal{H}^{-3}$.

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It remains to **characterize** the subsequential scaling limits Φ^*

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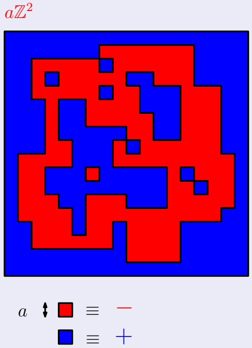
Theorem (Smirnov, Chelkak-Smirnov)



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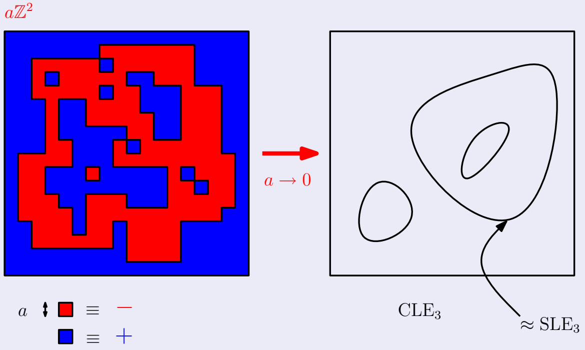

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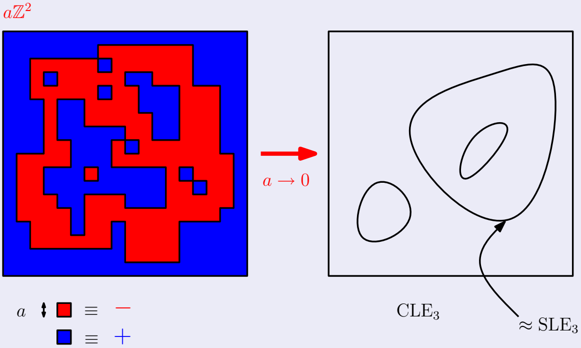

Question

Can one recover the magnetization $m = \lim_{a\to 0} m^a$ as a functional of the CLE_3 *?*

Use instead the scaling limit of FK percolation ($q = 2$)

Definition

$$\mathbb{P}_{p,q}(\omega) := \frac{1}{Z_{p,q}} p^{o(\omega)} (1-p)^{c(\omega)} q^{\#\text{clusters}}$$

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$$\begin{aligned} m^a = \sum_{x \in a\mathbb{Z}^2 \cap [0,1]^2} a^{15/8}\sigma_x \quad &= \quad \sum_{x \in \mathcal{C}_1} a^{15/8}\sigma_x + \sum_{x \in \mathcal{C}_2} a^{15/8}\sigma_x + \ldots \\ &= \quad \xi_1 \,\mathrm{Area}_a(\mathcal{C}_1) + \xi_2 \,\mathrm{Area}_a(\mathcal{C}_2) + \ldots \end{aligned}$$

where $(\xi_i)_{i \geq 1}$ are i.i.d coin flips.

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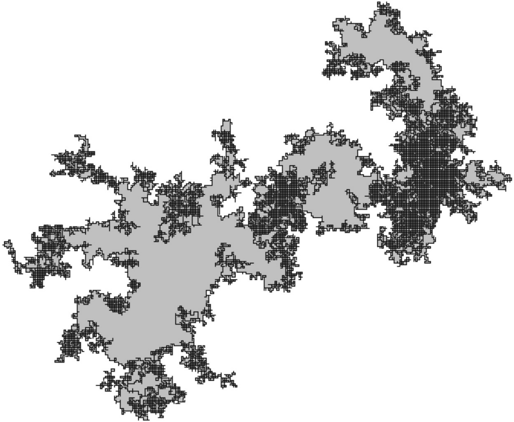
Claim

Uniformly in $a \to 0$,

$$\mathbb{E}\Big[\Big(\sum_{\mathrm{diam}(\mathcal{C}_i) \leq \epsilon} \xi_i \, \mathrm{Area}_a(\mathcal{C}_i) \Big)^2\Big] \leq C \, \epsilon^{7/4}$$



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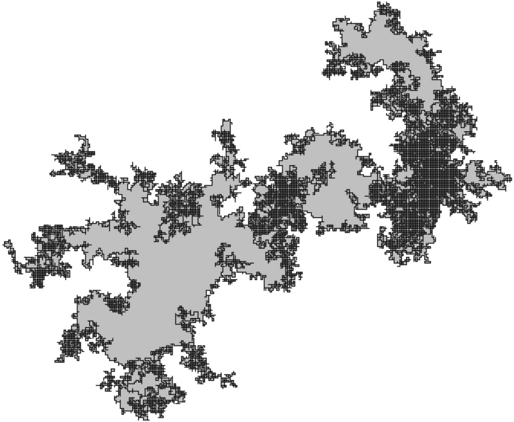
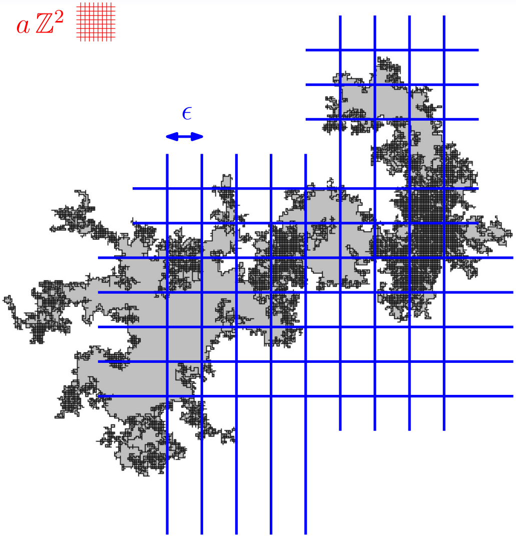
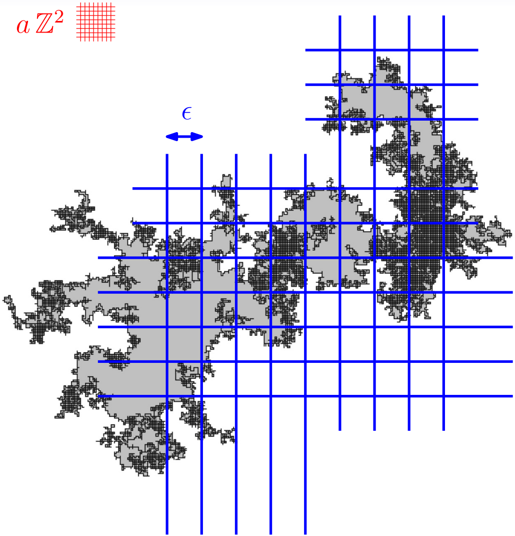




$a \ll \epsilon \ll 1$



$$a \ll \epsilon \ll 1$$

$$\#\text{points} \;=\; \sum_i X_i$$

$$\sim\; \sum_i Y_i\, K\left(\frac{\epsilon}{a}\right)^{15/}$$

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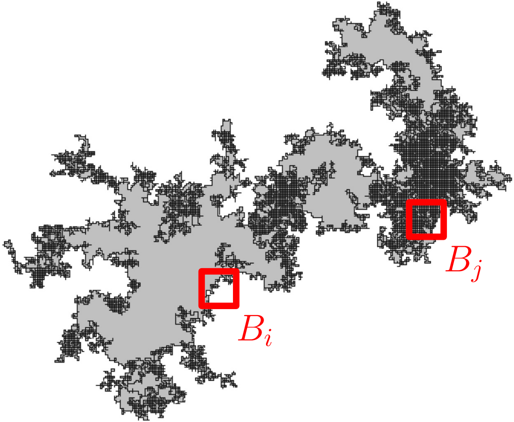

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$$\mathbb{E}\Big[\big(X - K(\frac{\epsilon}{a})^{15/8} Y\big)^2\Big] = \sum_{i,j} \mathbb{E}\Big[\big(X_i - K(\frac{\epsilon}{a})^{15/8} Y_i\big)\,\big(X_j - K(\frac{\epsilon}{a})^{15/8} Y_j\big)\Big]$$

$$= o\big(\mathbb{E}\big[X^2\big]\big)$$

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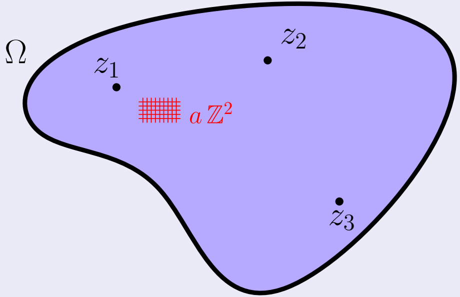

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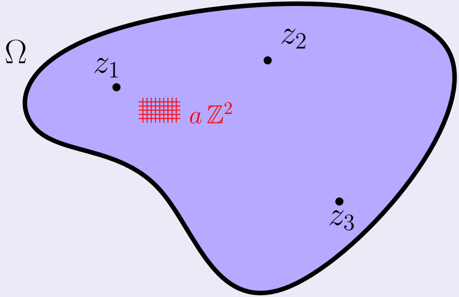

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$$z_1, \ldots, z_n \mapsto \langle z_1, \ldots, z_n \rangle_\Omega^+ ,$$

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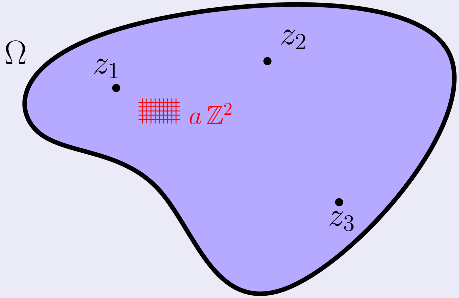

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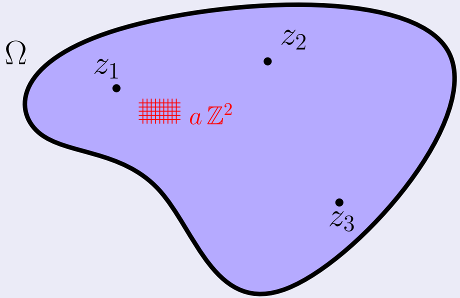
$$z_1, \ldots, z_n \mapsto \langle z_1, \ldots, z_n \rangle_\Omega^+ \,,$$

so that as the mesh $a \to 0$*, and if the points* $z_1, \ldots, z_n$ *remain "macroscopically far apart",*

$$a^{-\frac{n}{8}} \, \mathbb{E}_\Omega^+\left(\sigma_{z_1^a} \ldots \sigma_{z_n^a}\right) \sim \langle z_1, \ldots, z_n \rangle_\Omega^+$$

Proof 2: use the recent results by Dubédat or Chelkak, Hongler, Izyurov

Theorem (Dubédat or Chelkak, Hongler, Izyurov)



There exist n-point **correlation functions**

$$z_1, \ldots, z_n \mapsto \langle z_1, \ldots, z_n \rangle_\Omega^+ ,$$

so that as the mesh $a \to 0$, *and if the points* $z_1, \ldots, z_n$ *remain "macroscopically far apart",*

$$a^{-\frac{n}{8}} \, \mathbb{E}_\Omega^+ \left(\sigma_{z_1^a} \ldots \sigma_{z_n^a} \right) \sim \langle z_1, \ldots, z_n \rangle_\Omega^+$$

This suggests that one should have for each $n \geq 1$:

$$\mathbb{E}_\Omega^+ \left(m_a^n \right) \underset{a \to 0}{\longrightarrow} \iint \langle z_1, \ldots, z_n \rangle_\Omega^+ \, dz_1 \ldots dz_n$$

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The second property will follow from the following proposition:

Proposition (Camia, G., Newman)

For all fixed $t > 0$, one has

$$\sup_{a>0} \mathbb{E}_\Omega^+ \left(e^{t\, m^a} \right) < \infty$$

Second part: properties of Φ and m

- Conformal covariance properties
- Tail estimates, i.e. $\mathbb{P}\big[m > x\big]$ for $x \gg 1$.
- Analyticity of the *probability density function* of m

Conformal covariance

Theorem

(i) $m_{[0,\lambda]^2} \overset{(d)}{=} \lambda^{15/8}\, m_{[0,1]^2}$

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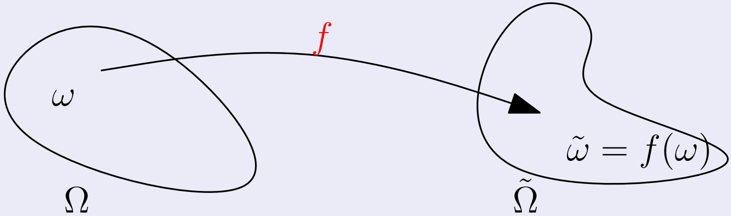

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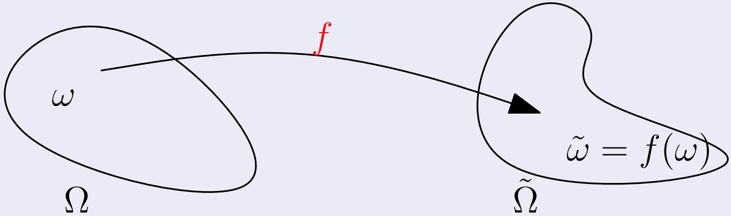

(ii) $f_* \Phi(\omega)$ *and* $\Phi(\tilde\omega)$ *are a.c.*

Furthermore, $\forall \begin{cases} \phi \in C_c^\infty(\Omega) \\ \tilde\phi = \phi \circ f^{-1} \in C_c^\infty(\tilde\Omega) \end{cases}$ *, one has*

$$\left\langle \Phi(\tilde\omega),\, \tilde\phi \right\rangle = \left\langle \Phi(\omega),\, \phi(z)\,|f'(z)|^{15/8} \right\rangle$$

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This suggests

$$\mathbb{P}\Big[m^\xi > x\Big] \approx e^{-O(1) x^{16}}$$

Tail behavior: main theorem

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There exists a universal constant $c > 0$ such that for **any** *boundary conditions ξ around $[0,1]^2$, as $x \to \infty$:*

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$\Uparrow$ (by a Tauberian Theorem)

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COROLLARY: m^{ξ} is indeed not Gaussian.

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Theorem (GHS inequality, Griffiths, Hurst, Sherman, 1970)

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With $h := t a^{15/8}$, this gives

$$\sup_{a>0} \left\{ \log \mathbb{E}\left[e^{t m^a}\right] \right\} \leq \sup_{a>0} \left(t\mathbb{E}\left[m^a\right] + \frac{t^2}{2}\mathrm{Var}[m^a] \right) < \infty$$

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With $h := ta^{15/8}$, this gives

$$\sup_{a>0}\left\{\log \mathbb{E}\left[e^{tm^a}\right]\right\} \leq \sup_{a>0}\left(t\mathbb{E}\left[m^a\right] + \frac{t^2}{2}\mathrm{Var}[m^a]\right) < \infty$$

Using the estimate with $\tilde{t} > t > 0$, we get

Fact

$$\mathbb{E}\left[e^{tm}\right] = \lim_{a\to 0} \mathbb{E}\left[e^{tm^a}\right] < \infty$$

Get a hand on the exponential moments

Main tool: **use scaling !**

$$\log \mathbb{E}\left[e^{tm}\right] \quad = \quad \log \mathbb{E}\left[e^{t/\lambda^{15/8} m_\lambda}\right]$$

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&\sim \lambda_t^2 \lim_{L\to\infty} \frac{1}{L^2} \log \mathbb{E}\left[e^{m_L}\right] \\
&\sim t^{16/15} C
\end{aligned}$$

Analyticity of the *density function* of m

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Theorem

For all $t \in \mathbb{R}$ and any boundary condition ξ, one has

$$|\mathbb{E}^{\xi}(e^{i\,t\,m})| \leq e^{-c\,|t|^{\frac{16}{15}}} .$$

In particular, the density function $f = f_{\Omega}$ is an **entire** *function on the whole plane $\mathbb{C}$!*

Further applications

A first one:

Theorem

Consider Ising model on $\mathbb{Z}^2$ at β_c with a positive external magnetic field $h > 0$, then

$$\langle \sigma_0 \rangle \asymp h^{\frac{1}{15}}$$

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A second one:

Theorem

Consider Ising model on $a\mathbb{Z}^2$ with external magnetic field $h\, a^{15/8}$, then there is a **scaling limit** *as $a \to 0$ towards a* **massive** *near-critical model.*