Oded's work on Noise Sensitivity

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Oded Schramm Memorial conference

Sensitivity of Percolation

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This will correspond to the following phenomenon:

Property

In critical percolation, macroscopic events are of 'High Frequency'.

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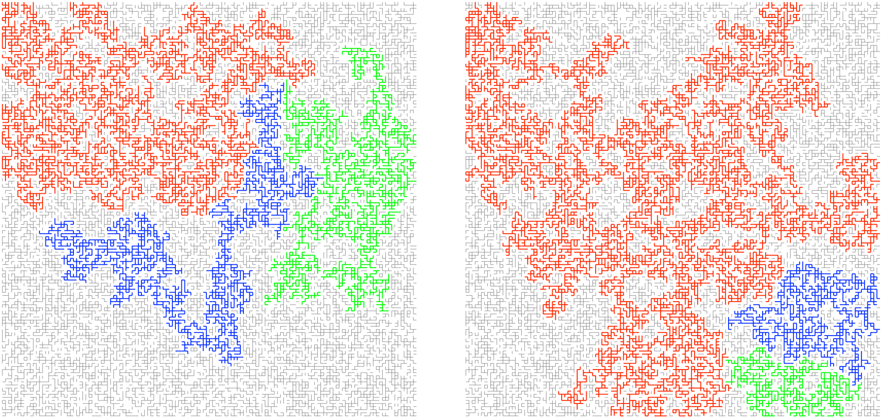

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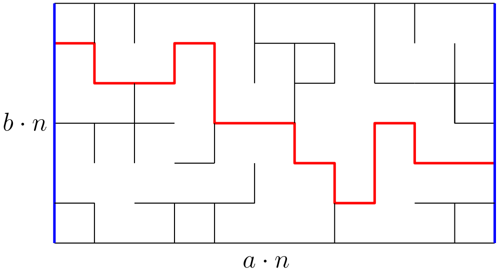

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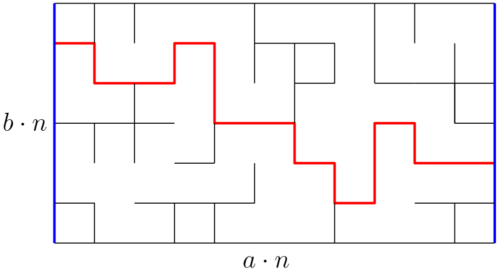

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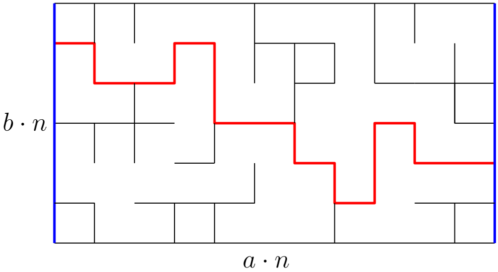

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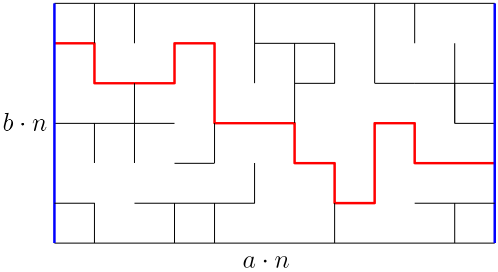

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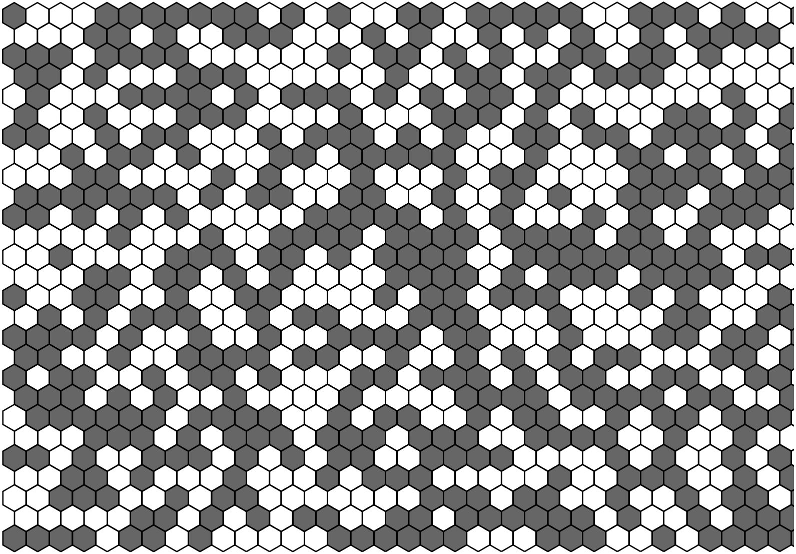
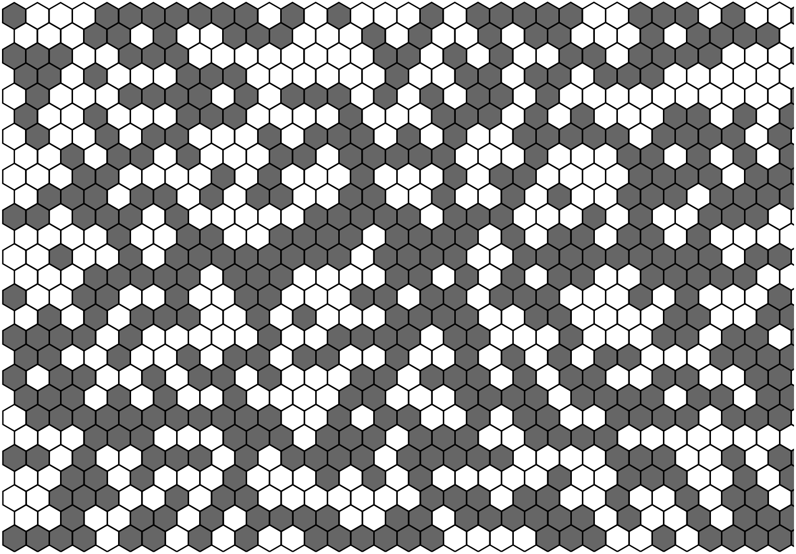


$\omega_0 \longrightarrow \omega_t$:



Noise Sensitivity

We are interested in a fast decorrelation (or fast mixing) of macroscopic properties.
This can be measured with the covariance

$$\mathrm{Cov}(f_n(\omega_0), f_n(\omega_t)) = \mathbb{E}\big[f_n(\omega_0) f_n(\omega_t)\big] - \mathbb{E}\big[f_n\big]^2 ,$$

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Defined in this way, noise sensitivity is a non-quantitative property.
We will need more detailed information on the speed at which the large scale system decorrelates.

Harmonic Analysis of Boolean functions

We consider the larger space $L^2(\{-1,1\}^n)$ of real-valued functions from n bits into $\mathbb{R}$, endowed with the scalar product:

$$\begin{aligned}\langle f, g\rangle &= \sum_{x_1,\ldots,x_n} 2^{-n} f(x_1,\ldots,x_n) g(x_1,\ldots,x_n) \\ &= \mathbb{E}\big[fg\big]\end{aligned}$$

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One has at our disposal a natural basis for this space isomorphic to $\mathbb{R}^{2^n}$: the so-called characters of the group $\{-1,1\}^n$.
For any subset $S \subset \{1,\ldots,n\}$, consider the function χ_S defined by

$$\chi_S(x_1,\ldots,x_n) := \prod_{i\in S} x_i$$

The set of these 2^n functions forms an orthonormal basis of $L^2(\{-1,1\}^n)$.

Fourier-Walsh expansion

Thus, any Boolean function $f : \{-1,1\}^n \to \{0,1\}$ can be decomposed as

$$f = \sum_{S \subset [n]} \widehat{f}(S)\, \chi_S$$

where $\widehat{f}(S)$ are the Fourier-Walsh coefficients of f.
They satisfy

$$\widehat{f}(S) = \langle f, \chi_S \rangle = \mathbb{E}\big[f \chi_S\big]$$

Note in particular that the coefficient $\widehat{f}(\emptyset) = \mathbb{E}\big[f\big]$ corresponds to the mean $\mathbb{E}\big[f\big]$.

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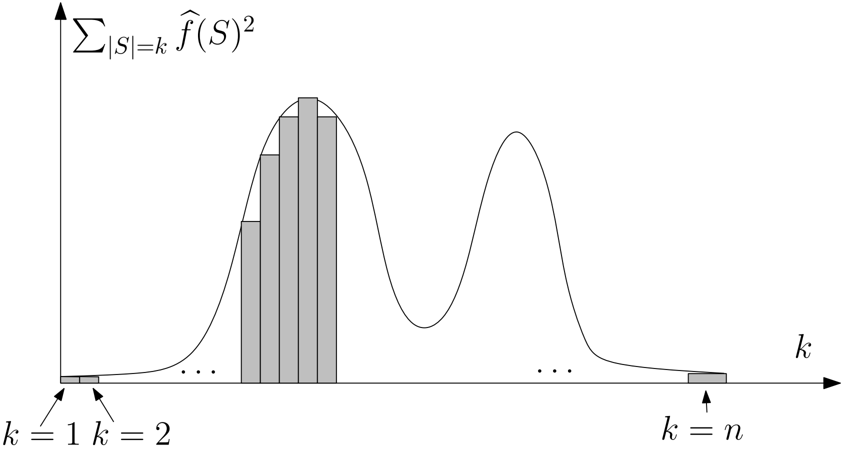
$$\begin{aligned}\mathbb{E}\big[f(\omega_0)\,f(\omega_t)\big] &= \mathbb{E}\Big[\Big(\sum_{S_1}\widehat{f}(S_1)\chi_{S_1}(\omega_0)\Big)\Big(\sum_{S_2}\widehat{f}(S_2)\chi_{S_2}(\omega_t)\Big)\Big] \\ &= \sum_{S}\widehat{f}(S)^2\,\mathbb{E}\big[\chi_S(\omega_0)\chi_S(\omega_t)\big] \\ &= \sum_{S}\widehat{f}(S)^2\,e^{-t\,|S|}\end{aligned}$$

Therefore our covariance can be written

$$\mathbb{E}\big[f(\omega_0)\,f(\omega_t)\big]-\mathbb{E}\big[f(\omega)\big]^2=\sum_{S\neq\emptyset}\widehat{f}(S)^2\,e^{-t|S|}$$

Energy spectrum of a Boolean function

If $f : \{-1,1\}^n \to \{0,1\}$ is a Boolean function, its "sensitivity" is controlled by its Energy Spectrum:



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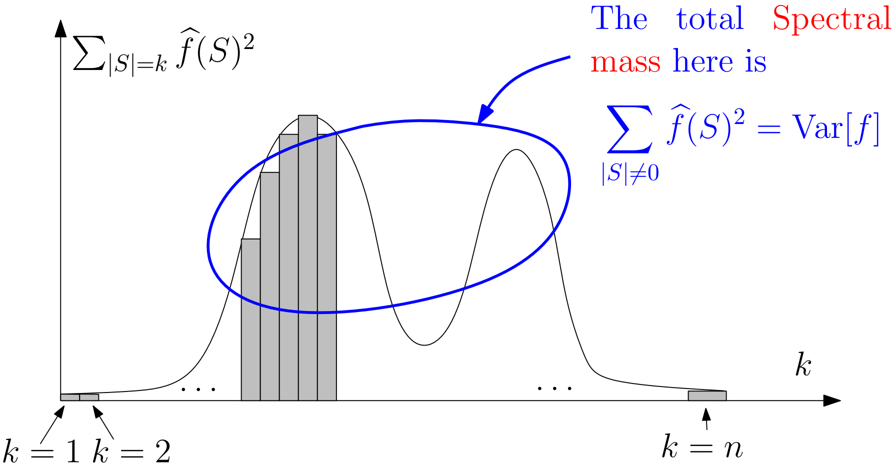

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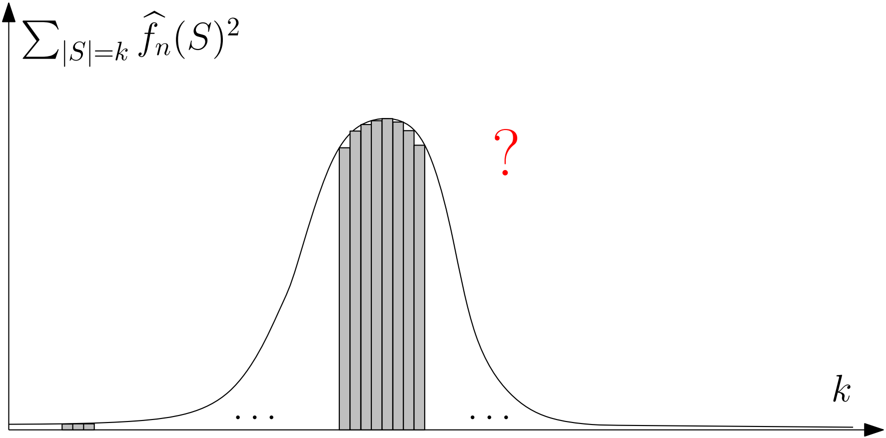

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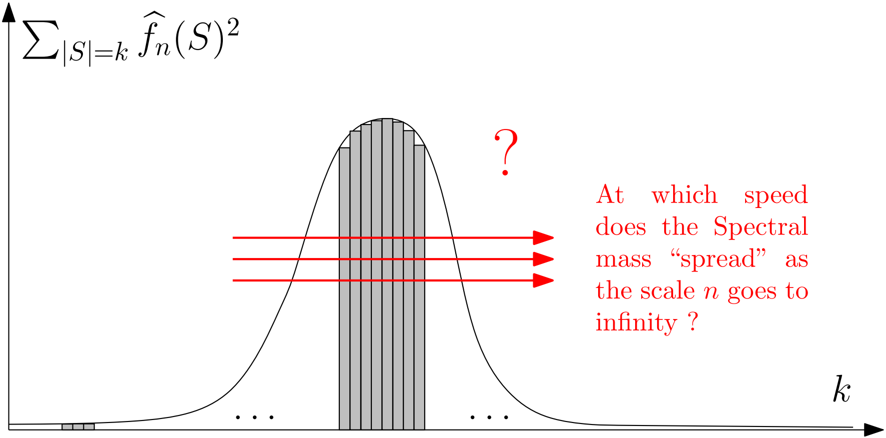

Energy Spectrum of Majority

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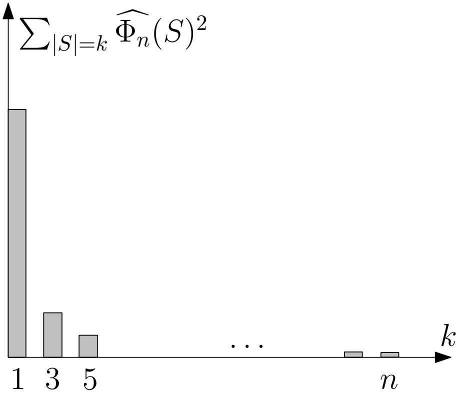
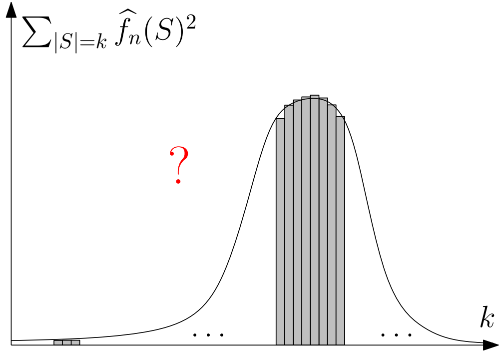


Three (very different !) approaches to Localize the Spectrum



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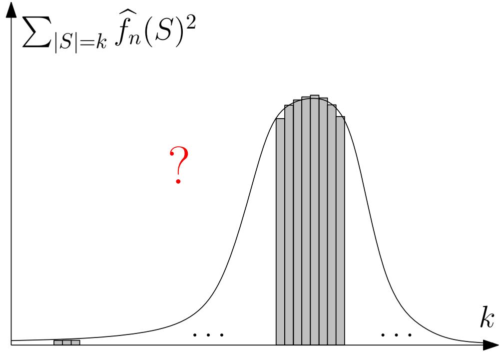

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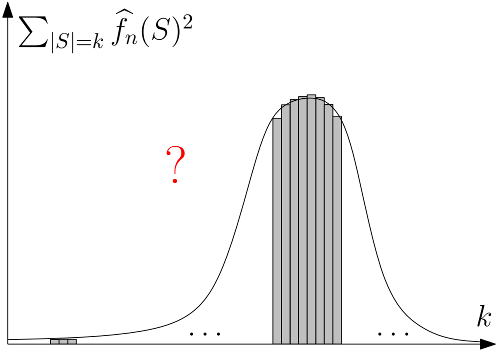

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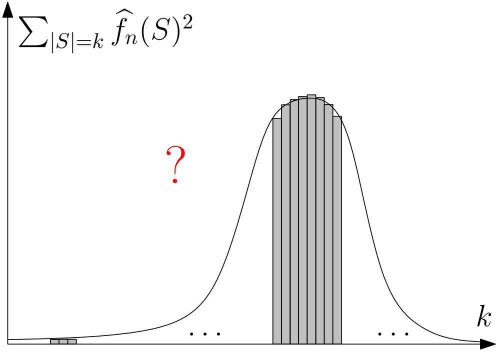

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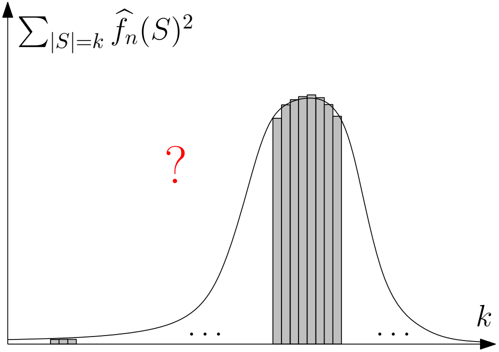

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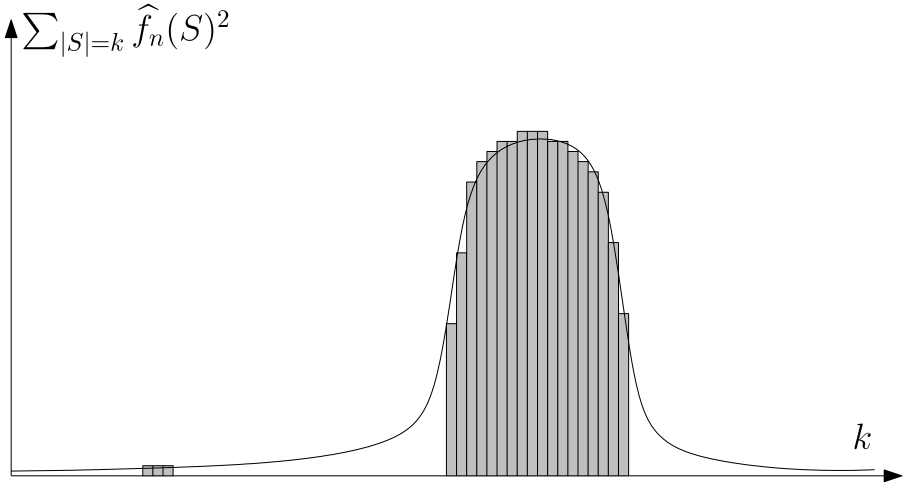

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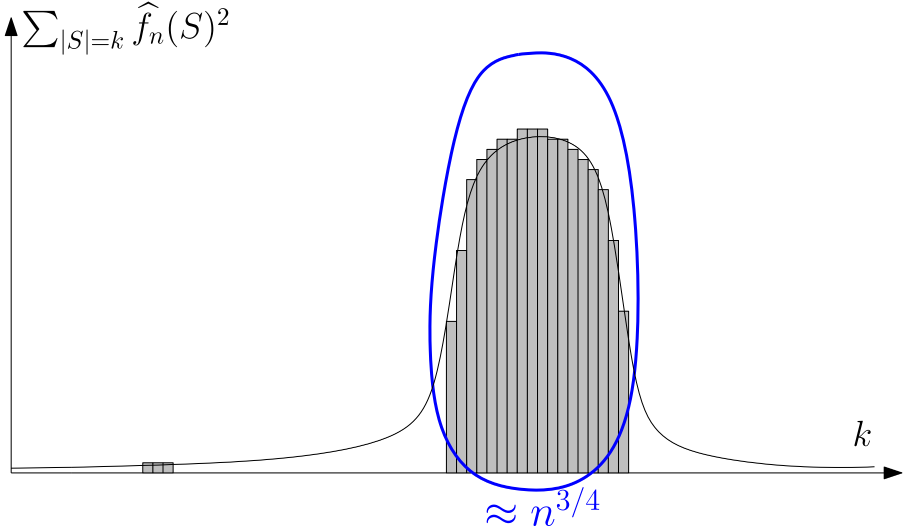

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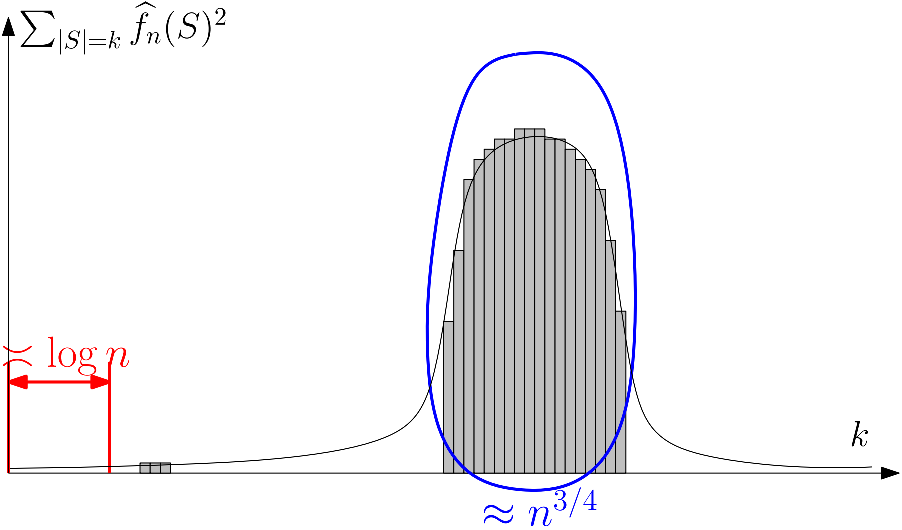

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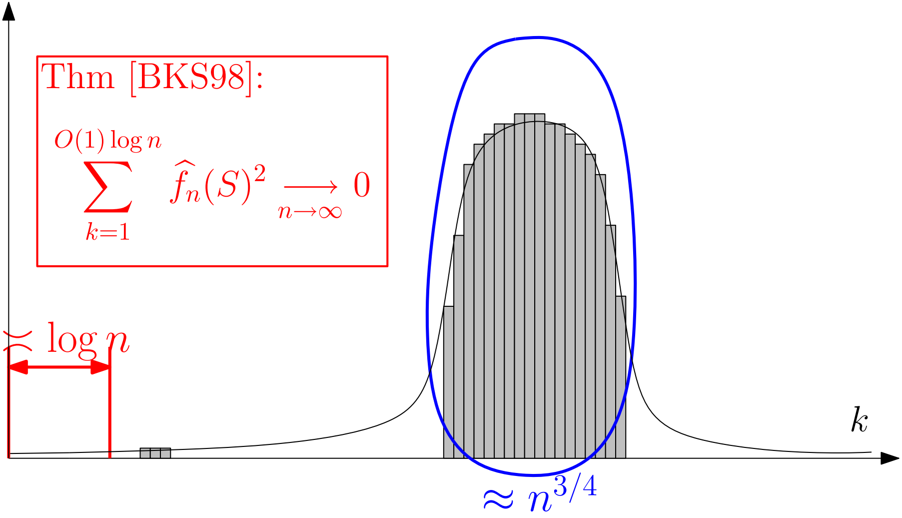

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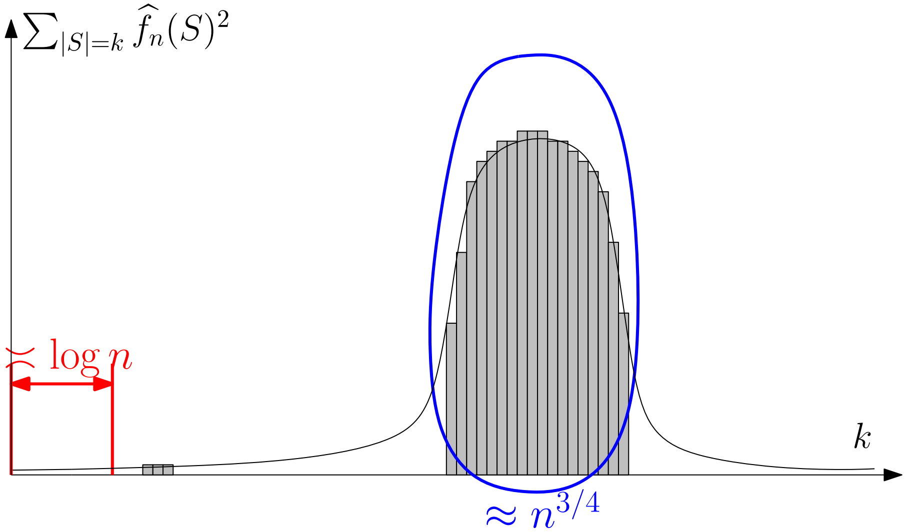

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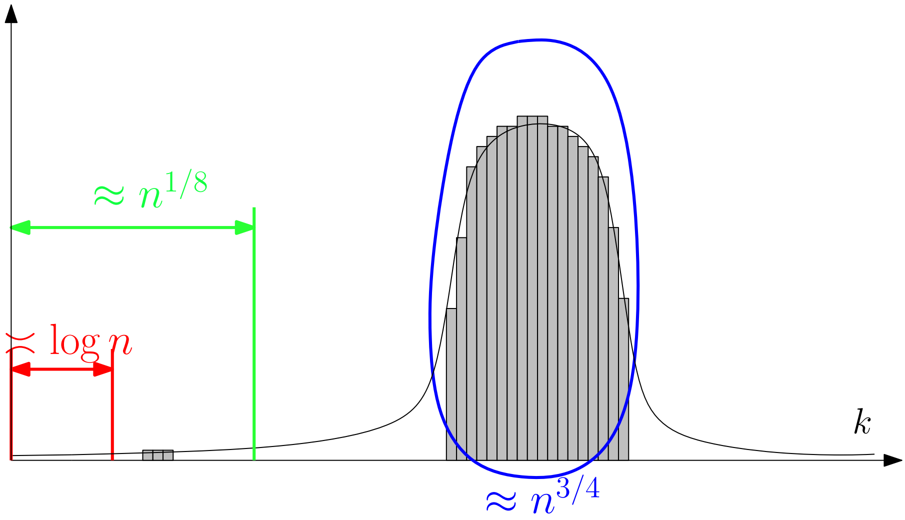

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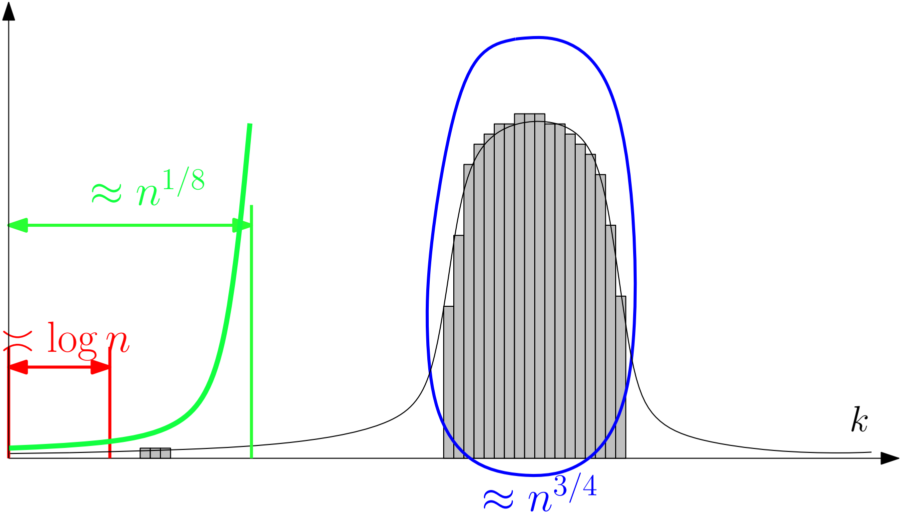

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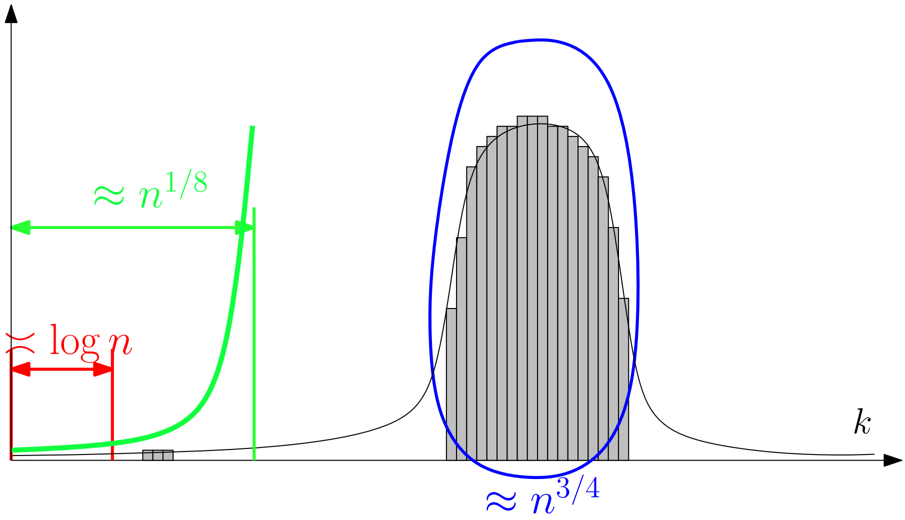

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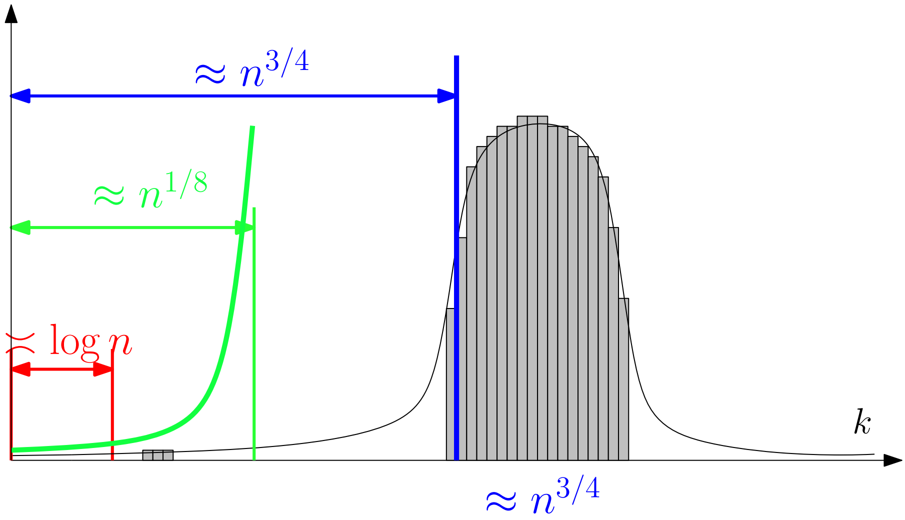

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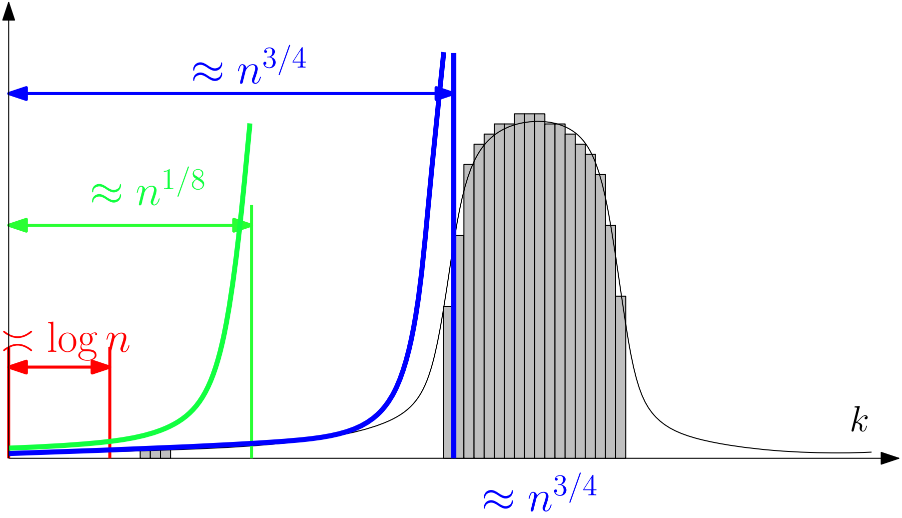

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$$\delta = \delta_{\mathcal{A}} := \sup_{i \in [n]} \mathbb{P}\big[i \in J \big] \,.$$

Examples

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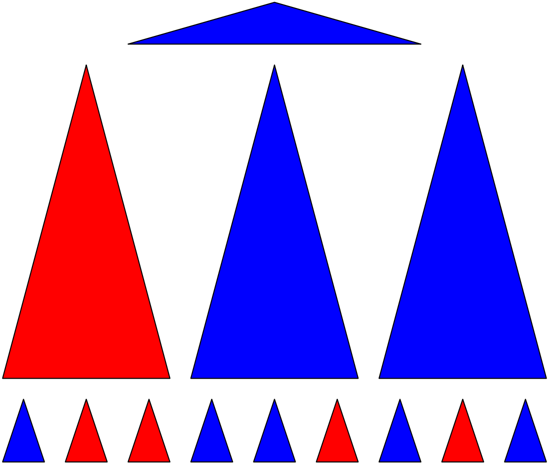
- Recursive Majority:



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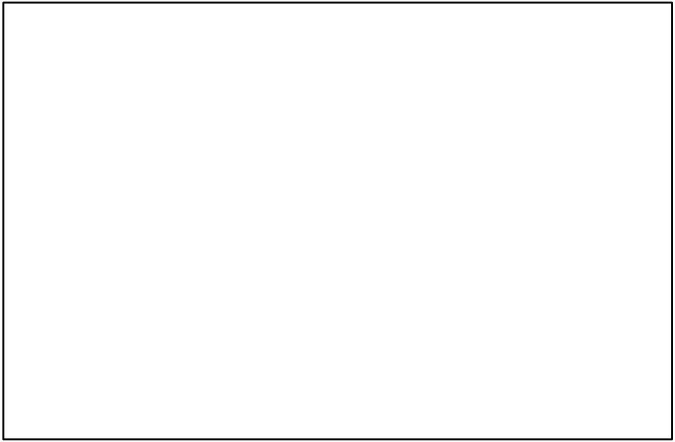

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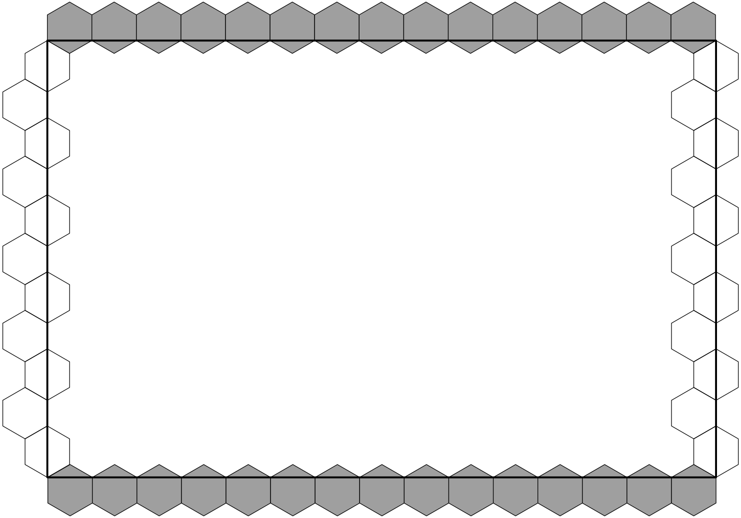

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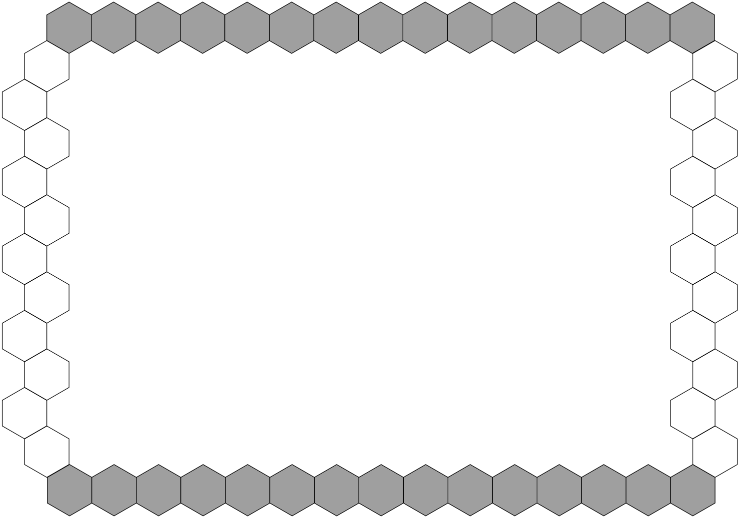

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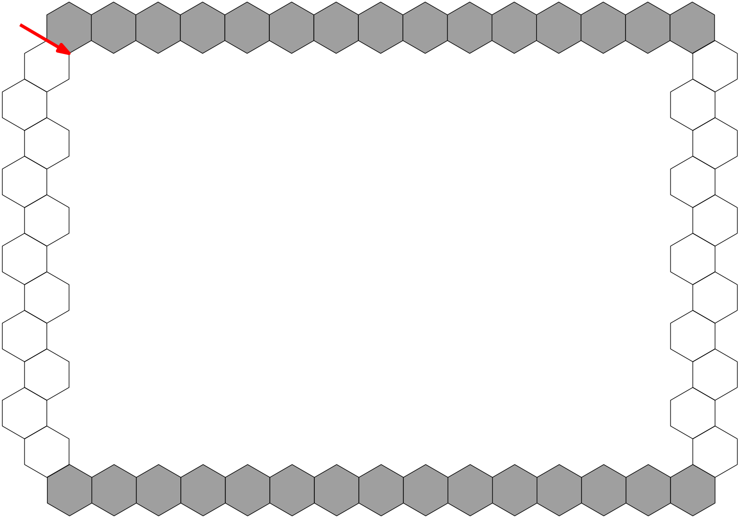

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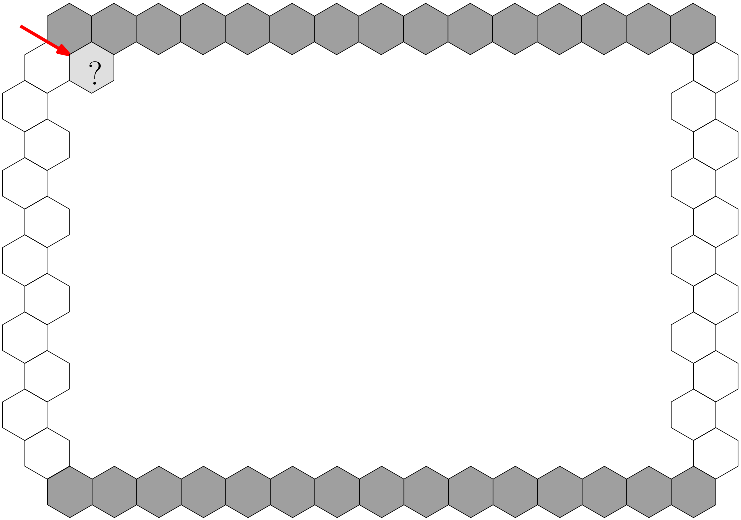

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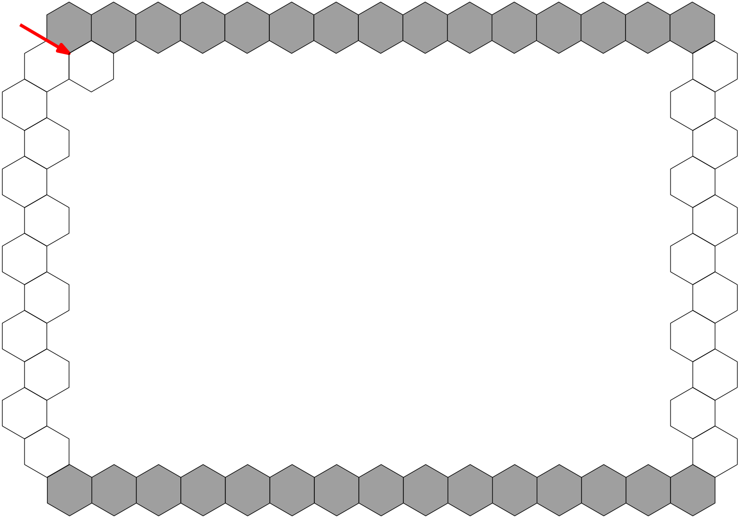

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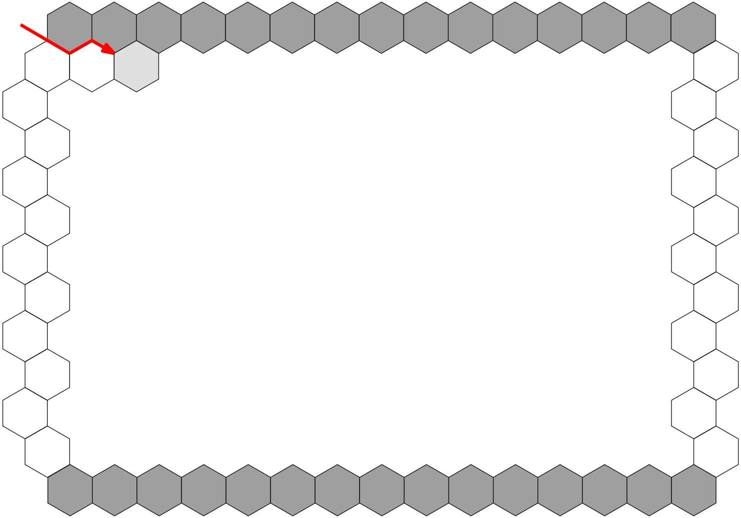

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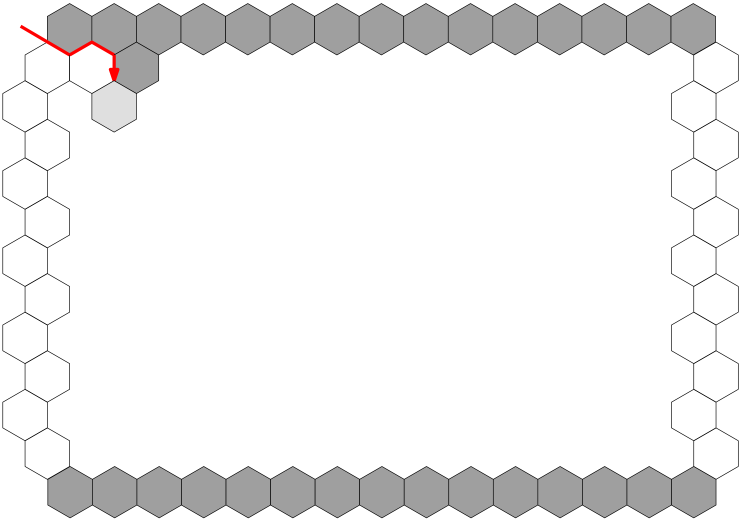

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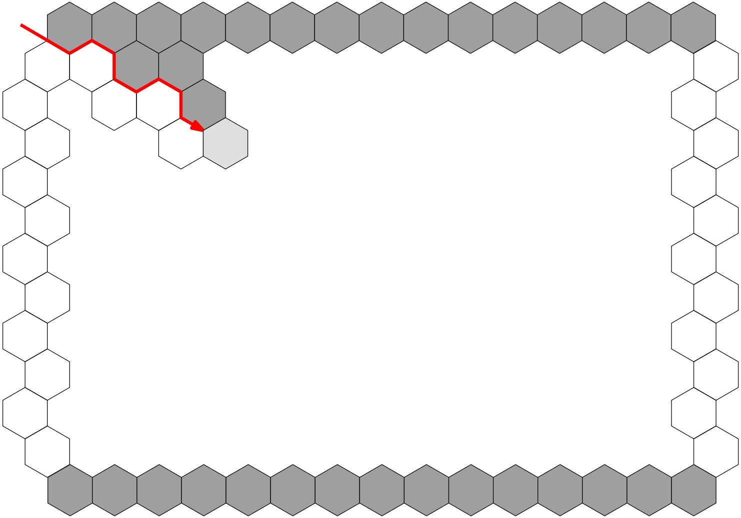

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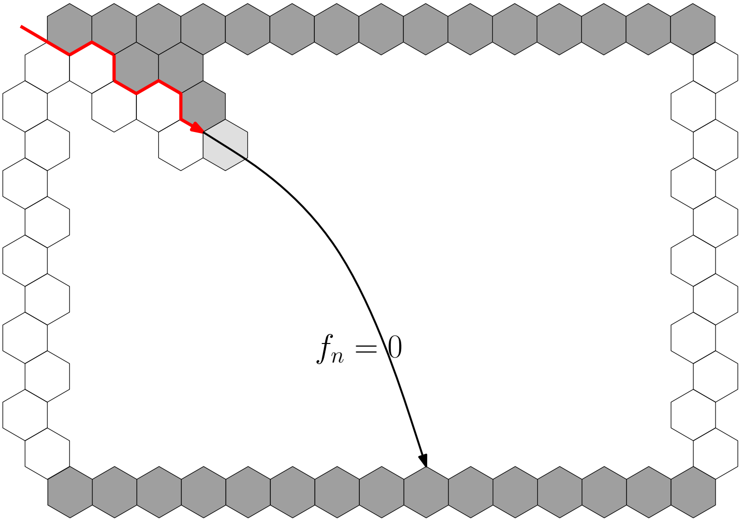

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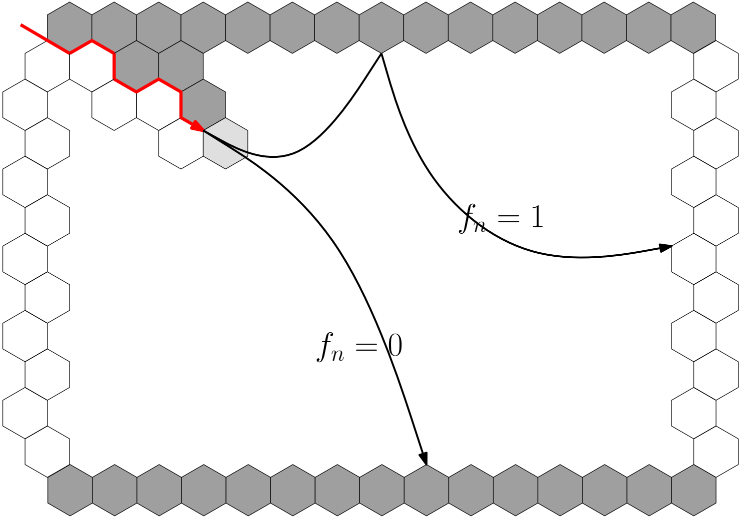

Revealment for percolation

Proposition (Schramm, Steif, 2005)

On the triangular lattice, a slight modification of the above randomized algorithm gives a small revealment for the left-right Boolean functions f_n of order

$$\delta_n \approx n^{-1/4}$$

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Let $f : \{-1, 1\}^n \to \mathbb{R}$ be a real-valued function.

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Let $f : \{-1, 1\}^n \to \mathbb{R}$ be a real-valued function. Let $\mathcal{A}$ be a randomized algorithm computing f whose revealment is $\delta = \delta_{\mathcal{A}}$.

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Then, for any $k = 1, 2, \ldots$ the Fourier coefficients of f satisfy

$$\sum_{|S|=k} \widehat{f}(S)^2 \leq k\,\delta\,\|f\|^2$$