A Lévy-Fokker-Planck equation: entropies and convergence to equilibrium

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Outline

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Ornstein-Uhlenbeck and Fokker-Planck equations

$$\begin{cases} dX_t = 2dB_t - \nabla V(X_t)dt \\ X_0 = x \end{cases}$$

where B_t is a standard Brownian Motion in $\mathbb{R}^n$.

Ito formula implies : the Semigroup $P_t f(x) = E_x(f(X_t))$ satisfies the PDE

$$\begin{cases} \frac{\partial}{\partial t} P_t f(x) = L P_t f(x) \\ P_0 f = f, \end{cases}$$

where $Lf = \Delta f - \nabla V \cdot \nabla f$ is the IG of P_t. This is the **Ornstein-Uhlenbeck equation**.
Consider L^* or P_t^*, the dual with respect to dx,

$$\int Lfgdx = \int fL^*gdx, \quad \text{or} \quad \int \mathrm{P_t fgdx} = \int \mathrm{fP_t^* gdx},$$

then

$$L^*g = \Delta g + \mathrm{div}(g.\nabla V).$$

The Semigroup $P_t^* f(x)$ satisfies the PDE

$$\begin{cases} \frac{\partial}{\partial t} P_t^* f(x) = L^* P_t^* f(x) \\ P_0^* f = f, \end{cases}$$

This is the **Fokker-Planck equation**.
Let $\mu_V = e^{-V} dx$ (**assume that μ_V is a probability measure**), $(P_t)_{t \geq 0}$ or L is self adjoint in $L^2(\mu_V)$ and the by integration by parts

$$\int Lf \, g d\mu_V = - \int \nabla f \cdot \nabla g d\mu_V.$$

Under smooth assumptions :

$$\lim_{t \to \infty} P_t f(x) = \int f d\mu_V.$$

or equivalently

$$\lim_{t \to \infty} e^{V(x)} P_t^* g(x) = \int g dx.$$

The good question is HOW FAST ?

Outline

Tools for the asymptotic behaviour

- **Poincaré inequality** : a L^2 convergence.

$$\frac{d}{dt} var_{\mu_V}(P_t f) = 2 \int P_t f L P_t f d\mu_V - 0 = -2 \int |\nabla P_t f|^2 d\mu_V,$$

If Poincaré inequality holds

$$var_{\mu_V}(f) \leq C \int |\nabla f|^2 d\mu_V$$

$$var_{\mu_V}(P_t f) \leq e^{-2t/C} var_{\mu_V}(f).$$

- **Logarithmic Sobolev inequality** a $L \log L$ convergence

$$\frac{d}{dt} Ent_{\mu_V}(P_t f) := \frac{d}{dt} \int P_t f \log \frac{P_t f}{\int P_t f d\mu_V} d\mu_V = -4 \int |\nabla \sqrt{P_t f}|^2 d\mu_V,$$

If Logarithmic Sobolev inequality holds

$$Ent_{\mu_V}(f^2) \leq C \int |\nabla f|^2 d\mu_V$$

$$Ent_{\mu_V}(P_t f) \leq e^{-4t/C} Ent_{\mu_V}(f).$$

When do we have a Poincaré or a logarithmic Sobolev inequality ?

* The Gaussian measure, $V(x) = x^2/2$ (Inequality proved by Gross).

* The Bakry-Emery Γ_2-criterion implies that if

$$\mathrm{Hess(V)} \geq \lambda \mathrm{Id},$$

with $\lambda > 0$ then logarithmic Sobolev inequality holds with $C = 2/\lambda$ and Poincaré inequality holds with $C = 1/\lambda$.

* There are also many technical methods to prove Poincaré or Log-Sobolev : Hardy, transportation...

Outline

Definition of Lévy process

Lévy process L_t = process with stationary & indep increments

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Fourier transform $(L_t) = e^{t\psi(\xi)}$ where ψ is characterized by the Lévy-Khinchine formula.

$$\psi(\xi) = -\sigma\xi\cdot\xi + ib\cdot\xi + \int (e^{iz\cdot\xi} - 1 - iz\cdot\xi \mathbf{1}_B(\xi))\nu(dz)$$

where ν is a singular measure satisfying

$$\int_B |z|^2 \nu(dz) < +\infty \qquad \int_{\mathbb{R}^d\setminus B} \nu(dz) < +\infty,$$

σ is a positive definite matrix and b is a vector.
Parameters (σ, b, ν) characterize the Lévy process (or a inifinite divisible law).

- For all $t > 0$ the law of L_t is an infinite divisible law : $\mu = \underbrace{\mu_n \star \cdots \star \mu_n}_{n \text{ times}}$.

▶ **Associated infinitesimal generators** as for the Brownian Motion.

$$\begin{aligned} I(u) &= \operatorname{div}(\sigma \nabla u) + b \cdot \nabla u \\ &\quad + \int (u(x+z) - u(x) - \nabla u(x) \cdot z \mathbf{1}_B(z)) \nu(dz) \end{aligned}$$

These operators appear everywhere (mathematical finance, mechanics, fluids *etc.*)

- Laws with heavy tails (decrease as power laws)
- Example : $(\sigma, b, \nu) = (0, 0, \frac{1}{|z|^{\alpha+d}} dz)$, the $\alpha \in (0, 2)$ stable process. In that case $\psi(\xi) = |\xi|^\alpha$.
- The case $\alpha = 2$ is the Brownian motion, $I = \Delta$.

The Lévy-Fokker-Planck equation

Replace Δ by I a IG of a Lévy process in the Fokker-Planck equation :

$$\begin{cases} \frac{\partial}{\partial t} u = I(u) + \mathrm{div}(\mathrm{ux}) \\ u(0,x) = f(x) \end{cases} \tag{LFP}$$

The goal is to understand the asymptotic behaviour of the semigroup.

Remark : We assume that $V = x^2/2$.

Questions :

- Find a steady state as e^{-V} as for the classical case Δ.
- Find the asymptotic behaviour of the Lévy-Fokker-Planck equation (LFP).
- Find conditions to get an asymptotic behaviour using inequalities as Poincaré or logarithmic Sobolev.

Outline

An equilibrium $u_\infty \stackrel{def}{=}$ a stationary solution of the LFP
u_∞ can be seen as an invariant measure μ_V in the case of the Laplacian.

Proposition (Existence of an equilibrium)

Assume that

$$\int_{\mathbb{R}^d \setminus B} \ln |z| \nu(dz) < +\infty.$$

There then exists an positive equilibrium u_∞ :

$$I(u_\infty) + div(u_\infty x) = 0.$$

Moreover, $u_\infty dx$ is an infinite divisible law whose characteristic exponent A is

$$A(\xi) = \int_0^1 \psi(s\xi) \frac{ds}{s}.$$

The condition

$$\int_{\mathbb{R}^d \setminus B} \ln |z| \nu(dz) < +\infty. \qquad \text{(Con1)}$$

is satisfied for the α-stable Lévy process. In that case u_∞ is the infinite divisible law of the Lévy process, $A = \psi/\lambda$.

Proof : The Fourier transform $\hat{u}_\infty$ satisfies

$$\psi(\xi)\hat{u}_\infty + \xi \cdot \nabla \hat{u}_\infty = 0$$

so that $\hat{u}_\infty = \exp(-A)$ with A such that :

$$\nabla A(\xi) \cdot \xi = \psi(\xi),$$

then

$$A(\xi) = \int_0^1 \psi(s\xi) \frac{ds}{s}.$$

Con1 prove that A is well defined and is the characteristic exponent of a Lévy process.

Outline

For $\phi : \mathbb{R}^+ \to \mathbb{R}$ convex and smooth and μ a probability measure, consider the ϕ-entropy

$$E_\mu^\phi(f) = \int \phi(f)d\mu - \phi\left(\int fd\mu\right)$$

- **Examples**

For $\phi(x) = \frac{1}{2}x^2$ (E_μ^ϕ=the variance), $D_\phi(a,b) = \frac{1}{2}(a-b)^2$

$$F_\mu^\phi(v) = \frac{1}{2}\iint (v(x+z) - v(x))^2 \nu(dz)\mu(dx)$$

For $\phi(x) = x\ln x - x - 1$ (E_μ^ϕ=entropy), $D_\phi(a,b) = a\ln\frac{a}{b} + b - a$

This is a natural interpolation between the variance and the Entropy.

Define also a Bregman distance

$$D_\phi(a,b) = \phi(a) - \phi(b) - \phi'(b)(a-b) \geq 0$$

Theorem

Let $\mu(dx) = u_\infty(x)dx$, ν the Lévy measure associated to I and consider $v(t,x) = \frac{u(t,x)}{u_\infty(x)}$, then

$$\frac{d}{dt}\, E_\mu^\phi(v(t,\cdot)) = -\iint D_\phi\Big(v(x+z), v(x)\Big)\nu(dz)\mu(dx).$$

- **Fisher information**

$$F_\mu^\phi(v) = \iint D_\phi\Big(v(x+z), v(x)\Big)\nu(dz)\mu(dx).$$

Can be seen as a Dirichlet form with respect to the measure $u_\infty(x)dx$

The proof is related to :

▶ A related equation : the Lévy-Ornstein-Ulenbeck equation (LOU)

The function $v = u/u_\infty$ satisfies

$$\partial_t v = \frac{1}{u_\infty}\Big(I(u_\infty v) - I(u_\infty)v\Big) + x\cdot\nabla v \stackrel{def}{=} Lv.$$

Dual operator of L wrt $\mu = u_\infty(x)dx$

$$\int w_1\Big(Lw_2\Big)d\mu = \int\Big(\check{I}(w_1) - x\cdot\nabla w_1\Big)w_2 d\mu,$$

where $\check{I}$ is I with $\check{\nu}(dx) = \nu(-dx)$.

Recall that in the classical case L is a self-adjoint operator with respect to μ.

Outline

Convergence towards the equilibrium

Theorem

We assume that ν_l *has a density* N *with respect to* dx *and satisfies*

$$\int_{\mathbb{R}^d \setminus B} \ln |z|\, N(z)\, dz < +\infty.$$

If N *is even and satisfies,*

$$\forall z, \quad \int_1^{+\infty} N(sz) s^{d-1} ds \leq C N(z)$$

then for any smooth convex function Φ *one gets :*

$$\forall t \geq 0, \quad \mathrm{Ent}^{\Phi}_{u_\infty}\left(\frac{u(t)}{u_\infty}\right) \leq e^{-\frac{t}{C}} \mathrm{Ent}^{\Phi}_{u_\infty}\left(\frac{u_0}{u_\infty}\right).$$

Proof

$$\frac{d}{dt} E_\mu^\phi(v(t)) = -F_\mu^\phi(t)$$

it is enough to compare F_μ^ϕ with E_μ^ϕ.

▶ **A functional inequality** [Wu'00,Chafaï'04]

If μ is an infinite divisible law (without gaussian part)
ϕ satisfies $\phi'' > 0$ and ...

Then $$E_\mu^\phi(f) \leq 1 \int D_\phi(v(x+z), v(x))\, \nu_\mu(dz)\mu(dx)$$

ν_μ is the derivation associated to the probability measure μ.

Example : If $\phi(x) = \ln x - x - 1$ and for the Gaussian measure this is exactly the Log-Sobolev inequality

$\rightarrow$ Generalization of Log-Sobolev inequality to the infinite divisible law.

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If

$$\nu_\mu \leq C\nu_l,$$

This is true for fractional Laplacians and for Lévy process near α-stable process.

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then

$$E_\mu^\phi(u/u_\infty) \leq E_\mu^\phi\left(\frac{u_0}{u_\infty}\right) e^{-\frac{t}{C}}$$

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