L^q-functional inequalities and weighted porous medium equation

Ivan Gentil

Meknes, November 2006

Introduction

Consider **Ornstein-Uhlenbeck** equation with the function φ:

$$\frac{d}{dt}u(t,x) = \Delta u - \nabla\varphi(x)\cdot\nabla u := Lu, \quad \text{i.c.} \quad \mathrm{u}(0,\mathrm{x}) = \mathrm{u}_0(\mathrm{x}).$$

- $L^*(u) = \Delta u + \mathrm{div}(\mathrm{u}\nabla\varphi(\mathrm{x}))$ in $L^2(dx)$, $\int uLvdx = \int vL^*udx$ and L^* is called **Fokker Planck** equation.
- $u(t,x) \to 0$ if $\int e^{-\varphi} = \infty$, ex: $\varphi = 0$, **heat equation**.
- $u(t,x) \to \int u_0 d\mu_\varphi$ if $\int e^{-\varphi} < \infty$ where

$$d\mu_\varphi(x) = \frac{exp(-\varphi(x))}{Z_\varphi}dx, \quad Z_\varphi = \int exp(-\varphi(x))dx$$

ex: $\nabla\varphi(x) = x$, classical Ornstein-Uhlenbeck equation.

The "good" space to study Orstein-Uhlenbeck semi-group is $L^2(\mu_\varphi)$ because L is symmetric:

$$\int f\, Lg\, d\mu_\varphi = \int g\, Lf\, d\mu_\varphi = -\int \nabla f \cdot \nabla g d\mu_\varphi,$$

the total mass is conserved: $\int u(t,x) d\mu_\varphi(x) = \int u_0 d\mu_\varphi$.
<u>How converge, in the second case, the O.U. equation?</u>

Theorem

Poincaré inequality *or* **spectral gap** *inequality:*

$$\forall f, \quad \mathbf{Var}_{\mu_\varphi}(f) := \int \left(f - \int f d\mu_\varphi\right)^2 d\mu_\varphi \leqslant C \int |\nabla f|^2 d\mu_\varphi,$$

is **equivalent** *to the exponential L^2-convergence of u to $\int u_0 d\mu_\varphi$:*

$$\int \left(u(t,x) - \int u_0 d\mu_\varphi\right)^2 d\mu_\varphi \leqslant e^{-t/C} \mathbf{Var}_{\mu_\varphi}(u_0).$$

Example

- *Dimension n:* $\mathrm{Hess}(\varphi) \geqslant \lambda \mathrm{Id}$ *with* $\lambda > 0$ *then* $C > 2/\lambda$.
- *Dimension 1: if* $\varphi(x) = |x|^\alpha$ *with* $\alpha \geqslant 1$.

Then Poincaré inequality holds

$$\mathbf{Var}_{\mu_\varphi}(f) := \int \left(f - \int f d\mu_\varphi \right)^2 d\mu_\varphi \leqslant C \int |\nabla f|^2 d\mu_\varphi,$$

Remark

We get the same result for Logarithmic Sobolev inequality for the probability measure μ_φ.

As **Porous medium equation** ($u_t = \Delta u^m$) can be considered as a generalization of heat equation.
We consider here the generalization of Ornstein-Uhlenbeck equation **weighted porous medium equation (WPME)**.

Let $m \geqslant 1$:

$$\frac{d}{dt}u(t,x) = \Delta u^m - \nabla\varphi(x)\cdot\nabla u^m = L(u^m), \quad \text{i.c.} \quad \mathrm{u}(0,\mathrm{x}) = \mathrm{u}_0(\mathrm{x}) > 0.$$

Questions are:

- Existence.
- Asymptotic behaviour (L^2 convergence), as Ornstein-Uhlenbeck equation, link between the asymptotic behaviour and φ.
- The link between some functional inequalities as Poincaré...

Existence

Theorem

Let u_0 be a $\mathcal{C}^\infty$ positive initial condition on $L^{m+1}_{\mu_\psi}(\mathbb{R}^d)$ then there exists a unique classical solution of the weighed porous medium equation (WPME).

- This is not a difficult problem but we did not found any reference.
- Based on a course given by J.L. Vásquez in Montréal (1990):
  - On a bounded domain $\Omega \subset \mathbb{R}^n$ for bounded initial condition.
  - L^1-contraction principle gives uniqueness of the solution
  - Extension on $\mathbb{R}^n$ for all positive initial condition.

Convergence to the equilibrium

Theorem (As Poincaré inequality)

L^q-Poincaré inequality with $q = 2/(m+1)$ $u_0 \in L^2(\mu_\varphi)$,

$$\forall f \geqslant 0,\ \mathbf{Var}_{\mu_\varphi}(f^q)^{1/q} := \left(\int f^{2q} d\mu - \left(\int f^q d\mu_\varphi \right)^2 \right)^{1/q} \leqslant C_P \int |\nabla f|^2 d\mu_\varphi,$$

is **equivalent** *to the polynomial L^2-convergence,*

$$\int \left(u - \int u_0 d\mu_\varphi \right)^2 d\mu_\varphi \leqslant \frac{1}{\left(\mathbf{Var}_{\mu_\varphi}(f)^{-(m-1)/2} + \frac{4mC_P(m-1)}{(m+1)^2} t \right)^{2/(m-1)}}.$$

Proof: Take $F(t) = \mathbf{Var}_{\mu_\varphi}(u)$, we get

$$F'(t) = -\frac{8m}{(m+1)^2}\int |\nabla u^{\frac{m+1}{2}}|^2 d\mu_\varphi$$

L^q-Poincaré inequality implies

$$\frac{\partial}{\partial t}\mathbf{Var}_{\mu_\psi}(u) \leq -\frac{8C_P m}{(m+1)^2}\left(\mathbf{Var}_{\mu_\psi}(u)\right)^{\frac{m+1}{2}}.$$

On the other side, the L^2-convergence implies that

$$F'(0) \leqslant -\frac{8mC_P}{(m+1)^2}F(0)^{1+2/(m-1)},$$

implies L^q-Poincaré.

Condition and example

The main difficulties is to prove such inequalities and Tools are **Capacity-measure inequalities**.

Let μ a probability measure and ν a positive measure on M. If $A \subset \Omega \subset M$,

$$Cap_\nu(A,\Omega) := \inf \left\{ \int |\nabla f|^2 d\nu;\ f \in \mathcal{C}^1(M),\ \mathbf{1}_A \leqslant f \leqslant \mathbf{1}_\Omega \right\}.$$

Let $q \in (0,1)$ and defined

$$\beta_P = \sup \left\{ \sum_{k\in\mathbb{Z}} \frac{\mu(\Omega_k)^{1/(1-q)}}{Cap_\nu(\Omega_k,\Omega_{k+1})^{q/(1-q)}} \right\}^{(1-q)/q} \in [0,+\infty],$$

where the supremum is taken over all $\Omega \subset M$ with $\mu(\Omega) = 1/2$ and all sequences $(\Omega_k)_{k\in\mathbb{Z}}$ such that for all $k \in \mathbb{Z}$, $\Omega_k \subsetneq \Omega_{k+1} \subset \Omega$.

Theorem

Let μ a probability measure and ν a positive measure on M.

- *Let $q \in [1/2, 1)$ and C_P the best constant of*

$$\left(\int f^{2q} d\mu - \left(\int f^q d\mu \right)^2 \right)^{1/q} \leqslant C_P \int |\nabla f|^2 d\nu,$$

  implies that $\beta_P \leqslant C_P$.
- *Let $q \in (0, 1)$ and assume that $\beta_P < +\infty$. Then (μ, ν) satisfies a L^q-Poincaré inequality with constant C which satisfies $C \leqslant C_2 \beta_P$ where C_2 is a constant which depend on q.*

Conclusion:

$$\beta_P \sim C_P,$$

The goal now is to compute β_P!

Theorem (Maz'ja)

Let $q \in [1/2, 1)$. Then for all $\Omega \in M$ and $(\Omega_k)_{k\in\mathbb{Z}}$ such that $\Omega_k \subset \Omega_{k+1} \subset \Omega$ one get

$$\sum_{k\in\mathbb{Z}} \frac{\mu(\Omega_k)^{1/(1-q)}}{Cap_\nu(\Omega_k, \Omega_{k+1})^{q/(1-q)}} \leqslant \frac{1}{1-q} \int_0^{\mu(\Omega)} \left(\frac{t}{\Phi(t)}\right)^{q/(1-q)} dt,$$

where

$$\Phi(t) = \inf\left\{Cap_\nu(A, \Omega);\ A \subset \Omega,\ \mu(A) \geqslant t\right\} \quad \text{i.e.} \quad \Phi(\mu(A)) \leqslant \mathrm{Cap}_\nu(A, \Omega).$$

Then

- Tools as Hardy inequalities or weak Poincaré inequality give the result in dimension 1.
- Tensorization property gives the result in dimension n.
- Perturbation property extend the result.

Example

Let $\varphi(x) = \sum_{i=1}^{n} \log(1 + |x_i|^{1+\alpha}) + W(x_1, \cdots, x_n)$ *and with* W *bounded*

$$d\mu_\varphi(x_1, \cdots, x_n) = \left(\prod_{i=1}^{n} \frac{1}{1 + |x_i|^{1+\alpha}} \right) e^{W(x_1, \cdots, x_n)} dx_1 \cdots dx_n,$$

The measure μ_φ *satisfies a* L^q*-Poincaré inequality with* $q \in [1/2, 1)$ *if* $\alpha > 2q/(1-q)$*.*
Then the **weighted porous medium**

$$\frac{d}{dt} u(t,x) = \Delta u^m - \nabla \varphi(x) \cdot \nabla u^m = L(u^m), \quad \text{i.c.} \quad u(0,x) = u_0(x) > 0.$$

associated converge in L^2 *if* $m > (\alpha + 4)/\alpha$*.*