A Lévy-Fokker-Planck equation: entropies and convergence to equilibrium

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Outline
Ornstein-Uhlenbeck and Fokker-Planck equations

\[
\begin{cases}
   dX_t = 2dB_t - \nabla V(X_t)dt \\
   X_0 = x
\end{cases}
\]

where $B_t$ is a standard Brownian Motion in $\mathbb{R}^n$.

Ito formula implies: the Semigroup $P_tf(x) = E_x(f(X_t))$ satisfies the PDE

\[
\begin{cases}
   \frac{\partial}{\partial t} P_tf(x) = LP_tf(x) \\
   P_0f = f,
\end{cases}
\]

where $Lf = \Delta f - \nabla V \cdot \nabla f$ is the IG of $P_t$. This is the Ornstein-Uhlenbeck equation.

Consider $L^*$ or $P^*_t$, the dual with respect to $dx$,

\[
\int Lfgdx = \int fL^*gdx, \quad \text{or} \quad \int P_tfgdx = \int fP^*_tgdxd,
\]

then

\[
L^*g = \Delta g + \text{div}(g.\nabla V).
\]
The Semigroup $P^*_t f(x)$ satisfies the PDE
\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} P^*_t f(x) = L^* P^*_t f(x) \\
P^*_0 f = f,
\end{array} \right.
\]
This is the Fokker-Planck equation.
Let $\mu_V = e^{-V} \, dx$ (assume that $\mu_V$ is a probability measure), $(P_t)_{t \geq 0}$ or $L$ is self adjoint in $L^2(\mu_V)$ and the by integration by parts
\[
\int Lf \, g d\mu_V = - \int \nabla f \cdot \nabla g d\mu_V.
\]
Under smooth assumptions :
\[
\lim_{t \to \infty} P_t f(x) = \int f d\mu_V.
\]
or equivalently
\[
\lim_{t \to \infty} e^{V(x)} P^*_t g(x) = \int g dx.
\]
The good question is HOW FAST?
Tools for the asymptotic behaviour

- **Poincaré inequality**: a $L^2$ convergence.

\[
\frac{d}{dt} \text{var}_{\mu_V}(P_t f) = 2 \int P_t f L P_t f d\mu_V - 0 = -2 \int |\nabla P_t f|^2 d\mu_V,
\]

If Poincaré inequality holds

\[
\text{var}_{\mu_V}(f) \leq C \int |\nabla f|^2 d\mu_V
\]

\[
\text{var}_{\mu_V}(P_t f) \leq e^{-2t/C} \text{var}_{\mu_V}(f).
\]

- **Logarithmic Sobolev inequality**: a $L \log L$ convergence

\[
\frac{d}{dt} \text{Ent}_{\mu_V}(P_t f) := \frac{d}{dt} \int P_t f \log \frac{P_t f}{\int P_t f d\mu_V} d\mu_V = -4 \int |\nabla \sqrt{P_t f}|^2 d\mu_V,
\]

If Logarithmic Sobolev inequality holds

\[
\text{Ent}_{\mu_V}(f^2) \leq C \int |\nabla f|^2 d\mu_V
\]

\[
\text{Ent}_{\mu_V}(P_t f) \leq e^{-4t/C} \text{Ent}_{\mu_V}(f).
\]
When do we have a Poincaré or a logarithmic Sobolev inequality?

* The Gaussian measure, \( V(x) = \frac{x^2}{2} \) (Inequality proved by Gross).

* The Bakry-Emery \( \Gamma_2 \)-criterion implies that if

\[
\text{Hess}(V) \geq \lambda \text{Id},
\]

with \( \lambda > 0 \) then logarithmic Sobolev inequality holds with \( C = \frac{2}{\lambda} \) and Poincaré inequality holds with \( C = \frac{1}{\lambda} \).

* There are also many technical methods to prove Poincaré or Log-Sobolev: Hardy, transportation...
Definition of Lévy process

Lévy process $L_t = \text{process with stationary \& indep increments}$

Fourier transform $(L_t) = e^{t\psi(\xi)}$ where $\psi$ is characterized by the Lévy-Khinchine formula.

$$
\psi(\xi) = -\sigma \xi \cdot \xi + ib \cdot \xi + \int \left( e^{iz \cdot \xi} - 1 - iz \cdot \xi 1_B(\xi) \right) \nu(dz)
$$

where $\nu$ is a singular measure satisfying

$$
\int_{B} |z|^2 \nu(dz) < +\infty \quad \int_{\mathbb{R}^d \setminus B} \nu(dz) < +\infty,
$$

$\sigma$ is a positive definite matrix and $b$ is a vector.

Parameters $(\sigma, b, \nu)$ characterize the Lévy process (or a infinite divisible law).

▶ For all $t > 0$ the law of $L_t$ is an infinite divisible law:

$$
\mu = \mu_n \ast \cdots \ast \mu_n.
$$

$n$ times
**Associated infinitesimal generators** as for the Brownian Motion.

\[
I(u) = \text{div} (\sigma \nabla u) + b \cdot \nabla u + \int (u(x + z) - u(x) - \nabla u(x) \cdot z 1_B(z)) \nu(dz)
\]

These operators appear everywhere.

- Laws with **heavy tails** (decrease as power laws)
- Example: \((\sigma, b, \nu) = (0, 0, \frac{1}{|z|^\alpha} dz)\), the \(\alpha \in (0, 2)\) stable process. In that case \(\psi(\xi) = |\xi|^\alpha\).
- The case \(\alpha = 2\) is the Brownian motion, \(I = \Delta\).
The Lévy-Fokker-Planck equation

Replace $\Delta$ by $l$ a IG of a Lévy process in the Fokker-Planck equation:

\[
\begin{aligned}
\frac{\partial}{\partial t} u &= l(u) + \text{div}(ux) \\
u(0, x) &= f(x)
\end{aligned}
\]  
(LFP)

The goal is to understand the asymptotic behaviour of the semigroup.

Remark: We fix now $(\sigma, b, \nu)$ and assume that $V = x^2 / 2$.

Starting point of this work: Biler and Karch (2001)

Questions:

- Find a steady state as $e^{-V}$ as for the classical case $\Delta$.
- Find the asymptotic behaviour of the Lévy-Fokker-Planck equation (LFP).
- Find conditions to get an asymptotic behaviour using inequalities as Poincaré or logarithmic Sobolev.
An equilibrium $u_\infty \overset{\text{def}}{=} \text{a stationary solution of the LFP}$

$u_\infty$ can be seen as an \textbf{invariant measure} $\mu_V$ in the case of the Laplacian.

\begin{center}
\underline{Proposition (Existence of an equilibrium)}
\end{center}

\textbf{Assume that}

\begin{equation}
\int_{\mathbb{R}^d \setminus B} \ln |z| \nu(dz) < +\infty. \quad \text{(Con 1)}
\end{equation}

\textbf{There then exists an positive equilibrium $u_\infty$ :}

\[
I(u_\infty) + \text{div}(u_\infty x) = 0.
\]

\textbf{Moreover, $u_\infty dx$ is an \textit{infinite divisible law} whose characteristic exponent $A$ is}

\[
A(\xi) = \int_0^1 \psi(s\xi) \frac{ds}{s}.
\]
The condition
\[ \int_{\mathbb{R}^d \setminus B} \ln |z| \nu(dz) < +\infty. \] (Con1)
is satisfied for the \( \alpha \)-stable Lévy process. In that case \( u_\infty \) is the infinite divisible law of the Lévy process, \( A = \psi / \lambda \).

Proof: The Fourier transform \( \hat{u}_\infty \) satisfies
\[ \psi(\xi) \hat{u}_\infty + \xi \cdot \nabla \hat{u}_\infty = 0 \]
so that \( \hat{u}_\infty = \exp(-A) \) with \( A \) such that:
\[ \nabla A(\xi) \cdot \xi = \psi(\xi), \]
then
\[ A(\xi) = \int_0^1 \psi(s\xi) \frac{ds}{s}. \]
Con1 prove that \( A \) is well defined and is the characteristic exponent of a Lévy process.
For $\phi : \mathbb{R}^+ \to \mathbb{R}$ convex and smooth and $\mu$ a probability measure, consider the $\phi$-entropy

$$E_\mu^\phi(f) = \int \phi(f) d\mu - \phi \left( \int f d\mu \right)$$

**Examples**

For $\phi(x) = \frac{1}{2} x^2$ ($E_\mu^\phi$=the variance), $D_\phi(a, b) = \frac{1}{2} (a - b)^2$

For $\phi(x) = x \ln x - x - 1$ ($E_\mu^\phi$=entropy), $D_\phi(a, b) = a \ln \frac{a}{b} + b - a$

This is a natural interpolation between the variance and the Entropy.
Define also a Bregman distance

\[ D_\phi(a, b) = \phi(a) - \phi(b) - \phi'(b)(a - b) \geq 0 \]

Theorem

Let \( \mu(dx) = u_\infty(x)dx \), \( \nu \) the Lévy measure associated to \( I \) and consider \( v(t, x) = \frac{u(t,x)}{u_\infty(x)} \), then

\[
\frac{d}{dt} E_\mu^\phi(v(t, \cdot)) = - \int \int D_\phi\left(v(x + z), v(x)\right) \nu(dz) \mu(dx).
\]

Fisher information

\[
F_\mu^\phi(v) = \int \int D_\phi\left(v(x + z), v(x)\right) \nu(dz) \mu(dx).
\]

Can be seen as a Dirichlet form with respect to the measure \( u_\infty(x)dx \)
The proof is related to:

► A related equation: the Lévy-Ornstein-Ulenbeck equation (LOU)

The function \( v = u/u_\infty \) satisfies

\[
\partial_t v = \frac{1}{u_\infty} \left( I(u_\infty v) - I(u_\infty) v \right) + x \cdot \nabla v \overset{\text{def}}{=} L v.
\]

Dual operator of \( L \) wrt \( \mu = u_\infty(x) dx \)

\[
\int w_1 \left( L w_2 \right) d\mu = \int \left( \tilde{I}(w_1) - x \cdot \nabla w_1 \right) w_2 d\mu,
\]

where \( \tilde{I} \) is \( I \) with \( \tilde{\nu}(dx) = \nu(-dx) \).

Recall that in the classical case \( L \) is a self-adjoint operator with respect to \( \mu \).
Convergence towards the equilibrium

Theorem

We assume that $\nu_1$ has a density $N$ with respect to $dx$ and satisfies

$$\int_{\mathbb{R}^d \setminus B} \ln |z| \ N(z) \ dz < +\infty.$$  \hspace{1cm} \text{(Con 1)}

If $N$ is even and satisfies,

$$\forall z, \quad \int_1^{+\infty} N(sz)s^{d-1} \ ds \leq CN(z)$$

then for any smooth convex function $\Phi$ one gets :

$$\forall t \geq 0, \quad \operatorname{Ent}^\Phi_{u_\infty} \left( \frac{u(t)}{u_\infty} \right) \leq e^{-\frac{t}{c}} \operatorname{Ent}^\Phi_{u_\infty} \left( \frac{u_0}{u_\infty} \right).$$
Proof

\[
\frac{d}{dt} E_\mu^\phi(v(t)) = -F_\mu^\phi(t)
\]

it is enough to compare \( F_\mu^\phi \) with \( E_\mu^\phi \).

► A functional inequality \[\text{[Ane-Ledoux'00Wu'00,Chafaï'04]}\]

If \( \mu \) is an infinite divisible law (without gaussian part) \( \phi \) satisfies \( \phi'' > 0 \) and ...

Then \( E_\mu^\phi(f) \leq 1 \int D_\phi(v(x + z), v(x)) \nu_\mu(dz) \mu(dx) \)

\( \nu_\mu \) is the derivation associated to the probability measure \( \mu \).

Example : If \( \phi(x) = \ln x - x - 1 \) and for the Gaussian measure this is exactly the Log-Sobolev inequality

→ Generalization of Log-Sobolev inequality to the infinite divisible law.
Proof

\[
\frac{d}{dt} E_\mu(\nu(t)) = -F_\mu(t)
\]

it is enough to compare \( F_\mu \) with \( E_\mu \).

► A functional inequality [Ane-Ledoux’00Wu’00,Chafaï’04]

\[\text{If } \mu \text{ is an infinite divisible law (without gaussian part)}\]

\[\phi \text{ satisfies } \phi'' > 0 \text{ and ...}\]

\[\text{Then } E_\mu(f) \leq \int D_\phi(\nu(x + z), \nu(x)) \nu_\mu(dz) \mu(dx)\]

\[\text{If } \nu_\mu \leq C \nu_1,\]

This is true for fractional Laplacians and for Lévy process near \( \alpha \)-stable process.
Proof

\[ \frac{d}{dt} E_\mu^\phi(v(t)) = -F_\mu^\phi(t) \]

it is enough to compare \( F_\mu^\phi \) with \( E_\mu^\phi \).

► A functional inequality [Ane-Ledoux’00Wu’00,Chafaï’04]

If \( \mu \) is an infinite divisible law (without gaussian part) \( \phi \) satisfies \( \phi'' > 0 \) and ... Then

\[ E_\mu^\phi(f) \leq C \int D_\phi(v(x + z), v(x)) \nu(dz)\mu(dx) \]

If

\[ \nu_\mu \leq C \nu_\nu, \]

This is true for fractional Laplacians and for Lévy process near \( \alpha \)-stable process.
Proof

\[ \frac{d}{dt} E_\mu^\phi (\nu(t)) = -F_\mu^\phi (t) \]

it is enough to compare \( F_\mu^\phi \) with \( E_\mu^\phi \).

▶ **A functional inequality** [Ane-Ledoux’00Wu’00,Chafaï’04]

If \( \mu \) is an infinite divisible law (without gaussian part)
\( \phi \) satisfies \( \phi'' > 0 \) and ...

Then \( E_\mu^\phi (f) \leq C F_\mu^\phi (f) \)

If \( \nu_\mu \leq C \nu_1 \),

This is true for fractional Laplacians and for Lévy process near \( \alpha \)-stable process.
Proof

\[ \frac{d}{dt} E_{\mu}^{\phi}(v(t)) = -F_{\mu}^{\phi}(t) \]

it is enough to compare \( F_{\mu}^{\phi} \) with \( E_{\mu}^{\phi} \).

A functional inequality [Ane-Ledoux’00, Wu’00, Chafaï’04]

If \( \mu \) is an infinite divisible law (without gaussian part)
\( \phi \) satisfies \( \phi'' > 0 \) and ...

Then \( E_{\mu}^{\phi}(f) \leq CF_{\mu}^{\phi}(f) \)

If
\[ \nu_{\mu} \leq C \nu_{I}, \]
then
\[ E_{\mu}^{\phi}(u/u_{\infty}) \leq E_{\mu}^{\phi} \left( \frac{u_0}{u_{\infty}} \right) e^{-\frac{t}{c}} \]

This is true for fractional Laplacians and for Lévy process near \( \alpha \)-stable process.
• Conclusion
  - Family of Entropies $\rightarrow$ Associated Fisher information
  - Sufficient condition for exponential decay to the equilibrium

• Perspectives
  - $V(x) = x^2/2 \rightarrow$ General potential ?
  - What happens if the Lévy measure $\nu$ has atoms ?