A Lévy-Fokker-Planck equation: entropies and convergence to equilibrium

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Outline

1 Introduction
   - Ornstein-Uhlenbeck and Fokker-Planck equations
   - Tools for the asymptotic behaviour

2 The Lévy-Fokker-Planck equation
   - The Lévy-Fokker-Planck equation

3 Results
   - The equilibrium
   - Entropies
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Ornstein-Uhlenbeck and Fokker-Planck equations

\[\begin{align*}
\left\{ \begin{array}{l}
\frac{dX_t}{dt} = \sqrt{2} dB_t - \nabla V(X_t) dt \\
X_0 = x
\end{array} \right. \\
\text{where } B_t \text{ is a standard Brownian Motion in } \mathbb{R}^n.
\]

Ito formula implies: the Semigroup \( P_t f(x) = E_x(f(X_t)) \) satisfies the PDE

\[\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} P_t f(x) = LP_t f(x) \\
P_0 f = f
\end{array} \right. \\
\text{where } L f = \Delta f - \nabla V \cdot \nabla f \text{ is the IG of } P_t. \text{ This is the Ornstein-Uhlenbeck equation.}
\]

Consider \( L^* \) or \( P_t^* \), the dual with respect to \( dx \),

\[\int L f g dx = \int f L^* g dx, \quad \text{or} \quad \int P_t f g dx = \int f P_t^* g dx, \]

then

\[ L^* g = \Delta g + \text{div}(g \cdot \nabla V). \]
The Semigroup $P_t^* f(x)$ satisfies the PDE
\[
\begin{cases}
  \frac{\partial}{\partial t} P_t^* f(x) = L^* P_t^* f(x) \\
P_0^* f = f,
\end{cases}
\]

This is the **Fokker-Planck equation**.

Let $\mu_V = e^{-V} dx$ (assume that $\mu_V$ is a probability measure), $(P_t)_{t \geq 0}$ or $L$ is self adjoint in $L^2(\mu_V)$ and the by integration by parts
\[
\int Lf \, gd\mu_V = - \int \nabla f \cdot \nabla gd\mu_V.
\]

Under smooth assumptions:
\[
\lim_{t \to \infty} P_t f(x) = \int f d\mu_V.
\]

or equivalently
\[
\lim_{t \to \infty} e^{V(x)} P_t^* g(x) = \int g dx.
\]

The good question is HOW FAST ?
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Tools for the asymptotic behaviour

- **Poincaré inequality**: a $L^2$ convergence.
  
  $$\frac{d}{dt} \var_{\mu \nu}(P_t f) = 2 \int P_t f L P_t f d\mu \nu - 0 = -2 \int |\nabla P_t f|^2 d\mu \nu,$$

  If Poincaré inequality holds
  
  $$\var_{\mu \nu}(f) \leq C \int |\nabla f|^2 d\mu \nu$$

  $$\var_{\mu \nu}(P_t f) \leq e^{-2t/C} \var_{\mu \nu}(f).$$

- **Logarithmic Sobolev inequality** a $L \log L$ convergence

  $$\frac{d}{dt} \text{Ent}_{\mu \nu}(P_t f) := \frac{d}{dt} \int P_t f \log \frac{P_t f}{\int P_t f d\mu \nu} d\mu \nu = -4 \int |\nabla \sqrt{P_t f}|^2 d\mu \nu,$$

  If Logarithmic Sobolev inequality holds

  $$\text{Ent}_{\mu \nu}(f^2) \leq C \int |\nabla f|^2 d\mu \nu$$

  $$\text{Ent}_{\mu \nu}(P_t f) \leq e^{-4t/C} \text{Ent}_{\mu \nu}(f).$$
When do we have a Poincaré or a logarithmic Sobolev inequality?

* The Gaussian measure, \( V(x) = \frac{x^2}{2} \) (Inequality proved by Gross).

* The Bakry-Emery \( \Gamma_2 \)-criterion implies that if 
  \[
  \text{Hess}(V) \geq \lambda \text{Id},
  \]
  with \( \lambda > 0 \) then logarithmic Sobolev inequality holds with 
  \( C = \frac{2}{\lambda} \) and Poincaré inequality holds with 
  \( C = \frac{1}{\lambda} \).

* There are also many technical methods to prove Poincaré or Log-Sobolev: Hardy, transportation...
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Definition of Lévy process

Lévy process \( L_t = \) process with stationary & indep increments

Fourier transform \((L_t) = e^{t \psi(\xi)}\) where \(\psi\) is characterized by the Lévy-Khinchine formula.

\[
\psi(\xi) = -\sigma \xi \cdot \xi + ib \cdot \xi + \int (e^{iz \cdot \xi} - 1 - iz \cdot \xi 1_B(\xi)) \nu(dz)
\]

where \(\nu\) is a singular measure satisfying

\[
\int_B |z|^2 \nu(dz) < +\infty \quad \int_{\mathbb{R}^d \setminus B} \nu(dz) < +\infty,
\]

\(\sigma\) is a positive definite matrix and \(b\) is a vector.
Parameters \((\sigma, b, \nu)\) characterize the Lévy process (or an infinite divisible law).

- For all \(t > 0\) the law of \(L_t\) is an infinite divisible law:

\[
\mu = \mu_n \ast \cdots \ast \mu_n.
\]

\(n\) times
Associated infinitesimal generators as for the Brownian Motion.

\[ l(u) = \text{div}(\sigma \nabla u) + b \cdot \nabla u + \int (u(x + z) - u(x) - \nabla u(x) \cdot z 1_B(z)) \nu(dz) \]

These operators appear everywhere.

- Laws with heavy tails (decrease as power laws)
- Example: \((\sigma, b, \nu) = (0, 0, \frac{1}{|z|^{\alpha+d}} dz)\), the \(\alpha \in (0, 2)\) stable process. In that case \(\psi(\xi) = |\xi|^\alpha\).
- The case \(\alpha = 2\) is the Brownian motion, \(l = \Delta\).
The Lévy-Fokker-Planck equation

Replace $\Delta$ by $I$ a IG of a Lévy process in the Fokker-Planck equation:

$$\begin{cases} \frac{\partial}{\partial t} u = I(u) + \text{div}(ux) \\ u(0, x) = f(x) \end{cases}$$

(LFP)

The goal is to understand the asymptotic behaviour of the semigroup.

Remark: We fix now $(\sigma, b, \nu)$ and assume that $V = x^2/2$.

Starting point of this work: Biler and Karch (2001)

Questions:
- Find a steady state as $e^{-V}$ as for the classical case $\Delta$.
- Find the asymptotic behaviour of the Lévy-Fokker-Planck equation (LFP).
- Find conditions to get an asymptotic behaviour using inequalities as Poincaré or logarithmic Sobolev.
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An equilibrium \( u_\infty \overset{\text{def}}{=} \) a stationary solution of the LFP
\( u_\infty \) can be seen as an \textit{invariant measure} \( \mu_V \) in the case of the
Laplacian.

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**Proposition (Existence of an equilibrium)**

Assume that

\[
\int_{\mathbb{R}^d \setminus B} \ln |z| \nu(dz) < +\infty. \quad (\text{Con 1})
\]

There then exists an \textit{positive} equilibrium \( u_\infty \):

\[
I(u_\infty) + \text{div}(u_\infty x) = 0.
\]

Moreover, \( u_\infty dx \) is an \textit{infinite divisible law} whose characteristic
exponent \( A \) is

\[
A(\xi) = \int_0^1 \psi(s\xi) \frac{ds}{s}.
\]
The condition
\[
\int_{\mathbb{R}^d \setminus B} \ln |z| \nu(dz) < +\infty. \tag{Con1}
\]
is satisfied for the $\alpha$-stable Lévy process. In that case $u_\infty$ is the infinite divisible law of the Lévy process, $A = \psi/\lambda$.

Proof: The Fourier transform $\hat{u}_\infty$ satisfies
\[
\psi(\xi) \hat{u}_\infty + \xi \cdot \nabla \hat{u}_\infty = 0
\]
so that $\hat{u}_\infty = \exp(-A)$ with $A$ such that:
\[
\nabla A(\xi) \cdot \xi = \psi(\xi),
\]
then
\[
A(\xi) = \int_0^1 \psi(s\xi) \frac{ds}{s}.
\]
Con1 proves that $A$ is well defined and is the characteristic exponent of a Lévy process.
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For $\phi : \mathbb{R}^+ \to \mathbb{R}$ convex and smooth and $\mu$ a probability measure, consider the $\phi$-entropy

$$E^\phi_\mu(f) = \int \phi(f)d\mu - \phi\left(\int fd\mu\right)$$

**Examples**

For $\phi(x) = \frac{1}{2}x^2$ ($E^\phi_\mu$=the variance), $D_\phi(a, b) = \frac{1}{2}(a - b)^2$

$$F^\phi_\mu(v) = \frac{1}{2} \iint (v(x + z) - v(x))^2 \nu(dz)\mu(dx)$$

For $\phi(x) = x \ln x - x - 1$ ($E^\phi_\mu$=entropy), $D_\phi(a, b) = a \ln \frac{a}{b} + b - a$

This is a natural interpolation between the variance and the Entropy.
Define also a Bregman distance

\[ D_\phi(a, b) = \phi(a) − \phi(b) − \phi'(b)(a − b) \geq 0 \]

---

**Theorem**

Let \( \mu(dx) = u_\infty(x)dx \), \( \nu \) the Lévy measure associated to \( I \) and consider \( v(t, x) = \frac{u(t, x)}{u_\infty(x)} \), then

\[
\frac{d}{dt} E^\phi_\mu(v(t, \cdot)) = - \int\int D_\phi\left( v(x + z), v(x) \right) \nu(dz) \mu(dx).
\]

---

**Fisher information**

\[
F^\phi_\mu(v) = \int\int D_\phi\left( v(x + z), v(x) \right) \nu(dz) \mu(dx).
\]

Can be seen as a Dirichlet form with respect to the measure \( u_\infty(x)dx \).
The proof is related to:

- A related equation: the Lévy-Ornstein-Ulenbeck equation (LOU)

The function \( v = u/u_\infty \) satisfies

\[
\partial_t v = \frac{1}{u_\infty} \left( I(u_\infty v) - I(u_\infty) v \right) + x \cdot \nabla v \overset{\text{def}}{=} L v.
\]

Dual operator of \( L \) wrt \( \mu = u_\infty(x) dx \)

\[
\int w_1 \left( L w_2 \right) d\mu = \int \left( \tilde{I}(w_1) - x \cdot \nabla w_1 \right) w_2 d\mu,
\]

where \( \tilde{I} \) is \( I \) with \( \tilde{\nu}(dx) = \nu(-dx) \).

Recall that in the classical case \( L \) is a self-adjoint operator with respect to \( \mu \).
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Convergence towards the equilibrium

**Theorem**

*We assume that $\nu_I$ has a density $N$ with respect to $dx$ and satisfies*

$$
\int_{\mathbb{R}^d \setminus B} \ln |z| \ N(z) \ dz < +\infty. \quad (\text{Con 1})
$$

*If $N$ is even and satisfies,*

$$
\forall z, \quad \int_1^{+\infty} N(sz)s^{d-1} \ ds \leq CN(z) \quad (\text{Con 2})
$$

*then for any smooth convex function $\Phi$ one gets :*

$$
\forall t \geq 0, \quad \text{Ent}_{u_\infty}^\Phi \left( \frac{u(t)}{u_\infty} \right) \leq e^{-\frac{t}{c}} \text{Ent}_{u_\infty}^\Phi \left( \frac{u_0}{u_\infty} \right).
$$
Proof

\[ \frac{d}{dt} E_\mu^\phi(v(t)) = -F_\mu^\phi(t) \]

it is enough to compare \( F_\mu^\phi \) with \( E_\mu^\phi \).

\[ \text{A functional inequality} \ [\text{Ane-Ledoux'00, Wu'00, Chafaï'04}] \]

If \( \mu \) is an infinite divisible law (without gaussian part) \( \phi \) satisfies \( \phi'' > 0 \) and ...

Then \( E_\mu^\phi(f) \leq 1 \int D_\phi(v(x + z), v(x)) \nu_\mu(dz) \mu(dx) \)

\( \nu_\mu \) is the derivation associated to the probability measure \( \mu \).

Example: If \( \phi(x) = \ln x - x - 1 \) and for the Gaussian measure this is exactly the Log-Sobolev inequality

→ Generalization of Log-Sobolev inequality to the infinite divisible law.
Proof

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**A functional inequality** [Ane-Ledoux’00, Wu’00, Chafaï’04]

If \( \mu \) is an infinite divisible law (without gaussian part) \( \phi \) satisfies \( \phi'' > 0 \) and ...

Then \( E_\mu^\phi(f) \leq \int D_\phi(v(x + z), v(x)) \nu_\mu(dz) \mu(dx) \)

If \( \nu_\mu \leq C\nu_1 \),

This is true for fractional Laplacians and for Lévy process near \( \alpha \)-stable process.
Proof

\[ \frac{d}{dt} E_\mu^\phi(v(t)) = -F_\mu^\phi(t) \]

it is enough to compare \( F_\mu^\phi \) with \( E_\mu^\phi \).

▸ A functional inequality [Ane-Ledoux’00, Wu’00, Chafaï’04]

If \( \mu \) is an infinite divisible law (without gaussian part) \( \phi \) satisfies \( \phi'' > 0 \) and ...

Then

\[ E_\mu^\phi(f) \leq C \int D_\phi(v(x + z), v(x)) \nu(dz) \mu(dx) \]

If

\[ \nu_\mu \leq C \nu_l, \]

This is true for fractional Laplacians and for Lévy process near \( \alpha \)-stable process.
Proof

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► A functional inequality [Ane-Ledoux’00, Wu’00, Chafaï’04]

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▶ A functional inequality [Ane-Ledoux’00, Wu’00, Chafaï’04]

If \( \mu \) is an infinite divisible law (without gaussian part)
\( \phi \) satisfies \( \phi'' > 0 \) and ...

Then \( E_\mu^\phi(f) \leq CF_\mu^\phi(f) \)

If
\[ \nu_\mu \leq C\nu_1, \]

then
\[ E_\mu^\phi(u/u_\infty) \leq E_\mu^\phi \left( \frac{u_0}{u_\infty} \right) e^{-\frac{t}{\alpha}} \]

This is true for fractional Laplacians and for Lévy process near \( \alpha \)-stable process.
• **Conclusion**
  - Family of Entropies $\rightarrow$ Associated Fisher information
  - Sufficient condition for exponential decay to the equilibrium

• **Perspectives**
  - $V(x) = x^2/2 \rightarrow$ General potential ?
  - What happens if the Lévy measure $\nu$ has atoms ?