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ANALYSIS AND GEOMETRY OF MARKOV DIFFUSION OPERATORS

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How far can you go with the Cauchy-Schwarz inequality and integration by parts?

To Leonard Gross

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INTRODUCTION

This book is devoted to the analysis of semigroups of Markov operators in their interplay between analysis, probability theory and geometry.

Semigroups of operators on a Banach space are very general models and tools in the analysis of time evolution phenomena and dynamical systems. They have a long history in mathematics and have been studied in a number of settings, from functional analysis and mathematical physics to probability theory, Riemannian geometry, Lie groups, analysis of algorithms etc.

The part of semigroup theory investigated in this book deals with Markov diffusion semigroups and their infinitesimal generators which naturally arise as solutions of stochastic differential equations and partial differential equations. As such, the topic covers a large body of mathematics ranging from probability theory and partial differential equations to functional analysis and differential geometry for operators or processes on manifolds. Within these frameworks, research and interest have developed over the years with a wide variety of questions such as regularity and smoothing properties of differential operators, Sobolev-type estimates, heat kernel bounds, non-explosion properties, convergence to equilibrium, existence and regularity of solutions of stochastic differential equations, martingale problems, stochastic calculus of variations and so on.

This book is more precisely focused on the concrete interplay between the analytic, probabilistic and geometric aspects of Markov diffusion semigroups and generators involved in convergence to equilibrium, spectral bounds, functional inequalities and various bounds on solutions of evolution equations linked to geometric properties of the underlying structure.

One prototype example at this interface is simply the standard heat semigroup $(P_t)_{t\geq 0}$ on the Euclidean space $\mathbb{R}^n$ whose Gaussian kernel

$$u = u(t,x) = p_t(x) = \frac{1}{(4\pi t)^{n/2}}\, e^{-|x|^2/4t}, \quad t > 0,\ x \in \mathbb{R}^n,$$

is a fundamental solution of the heat equation

$$\partial_t u = \Delta u, \quad u(0,x) = \delta_0,$$

for the standard Laplace operator Δ, thus characterized as the infinitesimal generator of the semigroup $(P_t)_{t\geq 0}$.

From the probabilistic viewpoint, the family of kernels $p_t(x)$, $t > 0$, $x \in \mathbb{R}^n$, represents the transition probabilities of a standard Brownian motion $(B_t)_{t\geq 0}$ as

$$\mathbb{E}\big(f(x + B_{2t})\big) = \int_{\mathbb{R}^n} f(y)\, p_t(x-y)dy = P_t f(x), \quad t > 0,\ x \in \mathbb{R}^n,$$

for all bounded measurable functions $f : \mathbb{R}^n \to \mathbb{R}$.

The third aspect investigated in this work is geometric, and perhaps less immediately apparent than the analytic and probabilistic aspects. It aims to interpret, in some sense, the commutation of derivation and action of the semigroup as a curvature condition. For the standard Euclidean semigroup example above, the commutation $\nabla P_t f = P_t(\nabla f)$ will express a zero curvature, although this is not only the curvature of Euclidean space as a Riemannian manifold but rather as the Euclidean space equipped with the Lebesgue measure, invariant under the heat flow $(P_t)_{t\geq 0}$, and the bilinear operator $\Gamma(f,g) = \nabla f \cdot \nabla g$.

In order to develop the investigation along these lines, the exposition emphasizes the basic structure of Markov Triple [(1)] (E, μ, Γ) consisting of a (measurable) state space E, a carré du champ operator Γ and a measure μ invariant under the dynamics induced by Γ. The notion of carré du champ operator Γ associated to a Markov semigroup $(P_t)_{t\geq 0}$ with infinitesimal generator L given (on a suitable algebra $\mathcal{A}$ of functions on E) by

$$\Gamma(f,g) = \frac{1}{2}\big[\mathrm{L}(fg) - f\,\mathrm{L}g - g\,\mathrm{L}f\big],$$

will namely be a central tool of investigation, the associated Γ-calculus providing, at least at a formal level, a kind of algebraic framework to encircle the relevant properties and results.

These analytic, stochastic and geometric features form the basis for the investigation undertaken in this book, describing Markov semigroups through their infinitesimal generators as solutions of second order differential operators and their probabilistic representations as Markov processes, and analyzing them with respect to curvature properties. The investigation is limited to symmetric (reversible in the Markovian terminology) semigroups, although various ideas and techniques go beyond this framework. We also restrict our attention to the diffusion setting, that is when the carré du champ operator is a derivation operator in its two arguments, even when many results could

1. The terminology "Markov triple" has of course not to be confused with solutions of the Markov Diophantine equation $x^2 + y^2 + z^2 = 3xyz$!

be developed in a more general setting. These restrictions rule out many interesting fields of applications (discrete Markov chains, statistical mechanics models, most of the analysis of algorithms of interest in optimization theory or approximations of partial differential equations e.g.), but allow us to concentrate on central features in the analysis of semigroups, in the same way that ordinary differential equations are in general easier to handle than discrete sequences. Even within the field of symmetric diffusion semigroups, we do not try to cover all the possible interesting cases and have to omit (in order to keep this monograph within a reasonable size) the specific analysis related to hypoelliptic diffusions, as well as the special features of diffusions on Lie groups, together with many interesting developments arising from infinite interacting particle systems among others.

Besides, although partly, and even mainly, motivated by the analysis of the behavior of diffusion processes (that is solutions of time homogeneous stochastic differential equations), we do not either concentrate on the probabilistic aspects of the subject, such as almost sure convergence of functionals of the trajectories of the underlying Markov processes, recurrence or transience, and chose to translate most of the features of interest into functional analytic properties of the Markov structure (E, μ, Γ) under investigation.

Heat kernel bounds, functional inequalities and their applications towards convergence to equilibrium and geometric features of Markov operators are parts of the main topics of interest developed in this monograph. A particular emphasis is put on family of inequalities relating, on a Markov Triple (E, μ, Γ), functionals of functions $f : E \to \mathbb{R}$ to the energy induced by the invariant measure μ and the carré du champ operator Γ,

$$\mathcal{E}(f, f) = \int_E \Gamma(f, f) d\mu.$$

Typical functionals are the variance, entropy or $\mathbb{L}^p$-norms leading to the main functional inequalities of interest, the Poincaré or spectral gap inequality, the logarithmic Sobolev inequality and the Sobolev inequality. The goal will be in particular to establish such families of inequalities under suitable curvature conditions which may be described by the carré du champ operator Γ and its iterated Γ_2 operator.

Similar inequalities are investigated at the level of the underlying semigroup $(P_t)_{t \geq 0}$ for the heat kernel measures, comparing $P_t(\varphi(f))$ (for some $\varphi : \mathbb{R} \to \mathbb{R}$) to $P_t(\Gamma(f, f))$ or $\Gamma(P_t f, P_t f)$, and giving rise to heat kernel bounds. To this task, proofs will develop the main powerful tool of heat flow monotonicity, or semigroup interpolation, with numerous illustrative applications and strong intuitive content. To illustrate, as a wink, the principle, let us briefly present here a heat flow proof of the classical Hölder inequality much in the spirit of this book. In particular, the reduction to a quadratic bound is typical of the arguments developed in this work. Let f, g be suitable (strictly) positive functions on $\mathbb{R}^n$ and $\theta \in (0, 1)$. For fixed $t > 0$, consider, at any point (omitted), the interpolation

$$\Lambda(s) = P_s\big(e^{\theta \log P_{t-s}f + (1-\theta)\log P_{t-s}g}\big), \quad s \in [0,t],$$

where $(P_t)_{t\geq 0}$ is the standard heat semigroup on $\mathbb{R}^n$ as recalled above. Together with the heat equation $\partial_s P_s = \Delta P_s = P_s\,\Delta$, the derivative in s of Λ is given by

$$\Lambda'(s) = P_s\Big(\Delta(e^H) - e^H\big[\theta\, e^{-F}\Delta(e^F) + (1-\theta)e^{-G}\Delta(e^G)\big]\Big)$$

where $F = \log P_{t-s}f$, $G = \log P_{t-s}g$ and $H = \theta F + (1-\theta)G$. Now by standard calculus,

$$\begin{aligned} e^{-H}\Delta(e^H) - \big[\theta\, e^{-F}\Delta(e^F) + (1-\theta)e^{-G}\Delta(e^G)\big] \\ = |\nabla H|^2 - \theta|\nabla F|^2 - (1-\theta)|\nabla G|^2 \end{aligned}$$

which is negative by convexity of the square function. Hence $\Lambda(s)$, $s \in [0,t]$, is decreasing, and thus

$$\Lambda(t) = P_t\big(f^\theta g^{1-\theta}\big) \leq (P_t f)^\theta (P_t g)^{1-\theta} = \Lambda(0).$$

Normalizing by $t^{n/2}$ and letting t tend to infinity yields Hölder's inequality for the Lebesgue measure. Actually, the same argument may be performed at the level of a Markov semigroup with invariant finite discrete measure, yielding thus Hölder's inequality for arbitrary measures.

While functional inequalities and their related applications are an important focal point, they are actually also the opportunity to discuss a number of issues related to examples and properties of Markov semigroups and operators. One objective of this work is thus also to present basic tools and ideas revolving around Markov semigroups and to illustrate their usefulness in different contexts.

The monograph is structured into four main parts.

The first part, covering Chapters 1 to 3, presents some of the main features, properties and examples of Markov diffusion semigroups and operators as considered in this work. In a somewhat informal but intuitive way, Chapter 1 introduces Markov semigroups, their infinitesimal generators and associated Markov processes, stochastic differential equations and diffusion semigroups. It also describes a few standard operations and techniques while working with semigroups. Chapter 2 develops in detail the main geometric model examples which will serve as references for many developments, namely the heat semigroups and Laplacians on the flat Euclidean space, the sphere and the hyperbolic space. Sturm-Liouville operators on the line, and some of the most relevant examples (Ornstein-Uhlenbeck, Laguerre and Jacobi), are also presented therein. On the basis of these preliminary observations and examples, Chapter 3 then tries to describe a general framework of investigation. While it would not be appropriate to try to cover in a unique formal mould all the cases of interest, it is nevertheless useful to emphasize basic properties and tools in order to easily and suitably develop the Γ-calculus. In particular, it is necessary to describe with some care the various classes and algebras of functions to deal with and to show their relevance in the classical smooth settings. Note that while infinite-dimensional models would require further care in this abstract formalism, the methods and principles emphasized throughout this work are similarly relevant for them. Taking the more classical picture as granted, Chapter 3 may be skipped at first reading (or limited to the summary Section 3.4).

Part II, forming the core of the text, includes Chapters 4 to 6 and covers the three main functional inequalities of interest, Poincaré or spectral gap inequalities, logarithmic Sobolev inequalities and Sobolev inequalities. For each family, basic properties and tools are detailed, in tight connection with the reference examples of Chapter 2 and their geometric properties. Stability, perturbation and comparison properties, characterization in dimension one, concentration bounds and convergence to equilibrium are thus addressed for each family. The discussion then distinguishes between inequalities for the heat kernel measures (local) and for the invariant measure (global) which are analyzed and established under curvature hypotheses. Chapter 4 is thus devoted to Poincaré or spectral gap inequalities, closely related to spectral decompositions. Chapter 5 deals with logarithmic Sobolev inequalities, emphasized as the natural substitute for classical Sobolev-type inequalities in infinite dimension, and their equivalent hypercontractive smoothing properties. Sobolev inequalities form a main family of interest for which Chapter 6 provides a number of equivalent descriptions (entropy-energy, Nash or Gagliardo-Nirenberg inequalities) and associated heat kernel bounds. A significant account of this chapter is devoted to the rich geometric content of Sobolev inequalities, their conformal invariance, and the curvature-dimension conditions.

On the basis of the main functional inequalities of Part II, Part III consisting of Chapters 7 to 9 address several variations, extensions and related topics of interest. Chapter 7 deals with general families of functional inequalities, each of them having their own interest and usefulness. The exposition mainly emphasizes entropy-energy (on the model of logarithmic Sobolev inequalities) and Nash-type inequalities. Besides, tightness of functional inequalities is studied with the tool of weak Poincaré inequalities. Chapter 8 is an equivalent description of the various families of inequalities for functions presented so far in terms of sets and capacities for which co-area formulas provide the suitable link. The second part of this chapter is concerned with isoperimetric-type inequalities for which semigroup tools again prove most useful. The last Chapter 9 briefly presents some of the recent important developments in optimal transportation in connection with the semigroup and Γ-calculus, including in particular a discussion on the relationships between functional and transportation cost inequalities (in a smooth Riemannian setting).

The last part of the monograph consists of three appendices, on semigroups of operators on a Banach space, elements of stochastic calculus and basics of differential and Riemannian geometry. At the interface between analysis, probability and geometry, these appendices aim to possibly supplement the reader's knowledge depending on his own background. They are not strictly necessary to the comprehension of the core of the text, but may serve as a support for the more specialized parts. It should be mentioned however that the last two sections of the third appendix on basics of Riemannian geometry actually contain material on the Γ-calculus (in a Riemannian context) which will be used in a critical way in some parts of the book.

This book is designed as to be at the same time an introduction to the subject intended to be accessible to non-specialists as well as an exposition of both basic and more advanced results of the theory of Markov diffusion semigroups and operators. We indeed chose to concentrate on those points where we feel that the techniques and ideas are central and may be used in a wider context, even though we never try to reach the widest generality. Every chapter starts at an elementary level for the notions developed in it, but may evolve to more specialized topics which in general may be skipped at first reading. It should actually be underlined that the level of exposition throughout the book is pretty much non-uniform, putting sometimes emphasis on facts or results which may appear as obvious or classical for some of the readers while developing at the same time more sophisticated issues. This choice is motivated by the wish to be accessible to readers with different backgrounds, and also by the aim of providing tools and methods to access more difficult parts of the theory or to be applied in different contexts. This delicate balance is not always reached but we hope that nevertheless the chosen style of exposition could be helpful.

The monograph is intended to students and researchers interested in the modern aspects of Markov diffusion semigroups and operators and their connections with analytic functional inequalities, probabilistic convergence to equilibrium and geometric curvature. Selected chapters may be used for advanced courses on the topic. Part I (at the exception of Chapter 3) and Part II are the main parts of the book on which readers who aim to get a flavor of Markov semigroups and their applications may concentrate. Part III tries to synthetize developments of the last decade, with selected choices in the topics put forward. The book requires from the reader only a reasonable knowledge of basic functional analysis, measure theory and probability theory. It is also expected that it may be read in a non-linear way, although the various chapters are not completely independent. For the reader not familiar with the main themes (analysis, probability and geometry), the appendices collect some basic material.

Each Chapter is divided into Sections, often themselves divided in Sub-Sections. Section 1.8 is the eighth section in Chapter 1. Theorem 4.6.2 indicates a theorem in Chapter 4, Section 6, that is in Section 4.6, and (3.2.2) is a formula in Section 3.2. An item of a given chapter is also referred to in the other chapters by the page at which it appears. There are no references to articles or books within the exposition of a given chapter. The Sections

"Notes and References" at the end of each chapter shortly describe some historical developments with pointers to the literature. References are far from exhaustive and rather limited actually. There is no claim for completeness and we apologize for omissions and errors. For books and monographs, we tried to indicate the references in historical order with respect to original editions (although links point towards the latest editions).

This book started at the occasion of lectures by the first author at Saint-Louis du Sénégal in April 2009. He thanks the organizers of this school for the opportunity to give this course and the participants for their interest. This work presents results and developments which have emerged during the last three decades. Over the years, we benefited from the vision, expertise and help of a number of friends and colleagues, among them M. Arnaudon, F. Barthe, W. Beckner, S. Bobkov, F. Bolley, C. Borell, E. Carlen, G. Carron, P. Cattiaux, D. Chafaï, D. Cordero-Erausquin, T. Coulhon, J. Demange, J. Dolbeault, K. D. Elworthy, M. Émery, A. Farina, P. Fougères, N. Gozlan, L. Gross, A. Guillin, E. Hebey, B. Helffer, A. Joulin, C. Léonard, X. D. Li, P. Maheux, F. Malrieu, L. Miclo, E. Milman, B. Nazaret, V. H. Nguyen, Z.-M. Qian, M.-K. von Renesse, C. Roberto, M. de la Salle, L. Saloff-Coste, K.-T. Sturm, C. Villani, F.-Y. Wang, L. Wu, B. Zegarlinski. We wish to thank them for their helpful remarks and constant support. F. Bolley, S. Campese and C. Léonard went through parts of the manuscript at several stages of the preparation, and we warmly thank them for all their corrections and comments that help in improving the exposition.

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We apologize for all the errors, and invite the readers to report remarks, mistakes and misprints. A list of errata and comments will be maintained online.

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