Errata of the book
Analysis and Geometry of Markov Diffusion Operators
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- Page 30, line 8: instead of $\int (LP_t)^2 d\mu$ read $\int (LP_t f)^2 d\mu$, thanks to Kevin Tanguy.
- Page 35, line 10: instead of
$$\sum_{y \in E} \mu(x) \mathrm{L}(x, y) = 0$$
read
$$\sum_{x \in E} \mu(x) \mathrm{L}(x, y) = 0,$$
thanks to Michał Strzelecki.
- Page 36, line 4: instead of $\hat{L}$ read K, thanks again to Michał Strzelecki.
- Page 39, line -1: instead of $F(x, T)$ read $F(T, x)$, thanks to Michał Strzelecki.
- Page 42, line -4: the matrix $\mathfrak{g}$ is supposed to be also definite-positive, thanks to Michał Strzelecki.
- Page 49, line -13: instead of $g = \psi'(f)$ read $g = \psi(f)$, thanks to Michał Strzelecki.
- Page 79, line -5: instead of
$$\sum_{i,j=1}^{n} (\partial_{ij} f)^2 \geqslant \frac{1}{n} (\sum_{i=1}^{n} \partial_i^2 f)^2$$
read
$$\sum_{i,j=1}^{n} (\partial_{ij}^2 f)^2 \geqslant \frac{1}{n} (\sum_{i=1}^{n} \partial_{ii}^2 f)^2,$$
thanks to Michał Strzelecki.

- Page 90, line 11: instead of

$$k_3(t,d) = \frac{1}{4\pi t)^{3/2}} \frac{d}{\sinh(d)} \exp\Big(-t - \frac{d^2}{4t}\Big)$$

read (add a ()

$$k_3(t,d) = \frac{1}{(4\pi t)^{3/2}} \frac{d}{\sinh(d)} \exp\Big(-t - \frac{d^2}{4t}\Big)$$

- Page 95, Proposition 2.4.1. The proposition schould be replace by the following with its proof.

Proposition 2.4.1 Let $\mathbf{L}f = f'' + a(x)f'$ be defined on $\mathcal{C}_c^\infty(0,\infty)$, where a is a smooth function on $(0,\infty)$. Then the operator $\mathbf{L}$ is symmetric with respect to the measure $d\mu = e^A dx$ where $A' = a$. Moreover, as soon as there exist two constants $C > 0$ and $c \geqslant 3/4$ such that

$$a'(x) + \frac{a^2(x)}{2} \geqslant \frac{c}{x^2} - C,\ x > 0,$$

then $\mathbf{L}$ is essentially self-adjoint.

Proof

$\lhd$ We briefly outline the arguments. The fact that $\mathbf{L}$ is symmetric with respect to $d\mu = e^A dx$ is immediate (see Section 2.6). Remove then the gradient in $\mathbf{L}$ according to the technique described in Sect. 1.15.7, p. 65. The problem is reduced to proving that the operator $\mathbf{L}_1 f = f'' - Kf$ where $K = \frac{a'}{2} + \frac{a^2}{4}$ is essentially self-adjoint on $(0,\infty)$ with respect to the Lebesgue measure. To this task, according to Proposition A.5.3, p. 482, it is enough to show that for some $\lambda \in \mathbf{R}$, the equation $f'' = (\lambda + K)f$ (understood in the distributional sense) has no solution in $\mathcal{L}^2(dx) = \mathcal{L}^2((0,\infty), dx)$ except 0. By the hypothesis, λ may be chosen so that $\lambda + K \geqslant K_0(x)$ where $K_0(x) = \frac{c}{x^2}$. Any solution f on $(0,\infty)$ of $f'' = (K+\lambda)f$ is as smooth as K. Assuming that f is not identically 0, up to a sign change, let $f(x_0) > 0$ for some $x_0 > 0$. Now, if $f'(x_0) > 0$, it is easy to see from $f'' \geqslant K_0(x)f$ that f increases on (x_0,∞), and is therefore convex on this interval. Being convex it grows at least linearly at infinity and therefore is not in $\mathcal{L}^2(dx)$.

On the other hand, if $f'(x_0) < 0$, from standard arguments, f is bounded from below by the solution f_0 of $f_0'' = K_0 f_0$ which has the same value and same derivative at x_0. To check that f is not in $\mathcal{L}^2(dx)$, it is therefore enough to see that f_0^2 may not be integrable near 0. But the solutions of $f_0'' = K_0 f_0$ are linear combinations of x^{α_1} and x^{α_2} where α_1 and α_2 are solutions of $\alpha(\alpha - 1) = c$. Since $f_0'(x_0) < 0$, f_0 behaves near the origin like βx^{α_1}, with $\beta \in \mathbf{R}$ and $\alpha_1 = (1 - \sqrt{1+4c})/2$. Then, $f_0 \notin \mathcal{L}^2(dx)$ iff $2\alpha_1 \leq -1$ that is $c \geqslant \frac{3}{4}$ Then the proposition is established. $\rhd$

- Page 102. Proposition 2.6.1 should be replaced by the following.

Proposition 2.6.1 Let $\mathbf{L}f = f'' + cf'$ be a Sturm-Liouville operator on $(-1,+1)$. Assume that c is smooth in $(-1,+1)$ and that there exist $C_1, C_2 > 0$ such that for every $x \in (-1,+1)$,

$$c'(x) + \frac{c^2(x)}{2} \geqslant \frac{3}{4} \max\big((1+x)^{-2}, (1-x)^{-2}\big) - C_2.$$

Then $\mathbf{L}$ is essentially self-adjoint.

- Page 102, line 7, instead of $\min(\alpha_-, \alpha_+) > 2$ read

$$\min(\alpha_-, \alpha_+) \geqslant \frac{2+\sqrt{10}}{2}.$$

- Page 108, line 7, the sentence should be replaced by the folowing :

  that is $L = \Delta - 2x \cdot \nabla$, the Ornstein-Uhlenbeck operator, up to a scaling.

- Page 129, line -1: instead of

$$\Gamma(f)(x) = \lim_{k\to\infty} \Big(\frac{1}{2t_k} P_{t_k}(f^2)(x) - P_{t_k}(f)(x)^2\Big)$$

  read

$$\Gamma(f)(x) = \lim_{k\to\infty} \frac{1}{2t_k}\Big(P_{t_k}(f^2)(x) - P_{t_k}(f)^2(x)\Big)$$

  thanks to Michał Strzelecki.

- Page 152, line -7: Item (iii) has to be understood as follows, for any functions $f_1, \cdots, f_k \in \mathcal{A}$ and $\Psi : \mathbf{R}^k \to \mathbf{R}$ a smooth function ($\mathcal{C}^\infty$), then $\Psi(f_1, \cdots, f_k) \in \mathcal{A}$.

- Page 156, line 17: instead of $f - g$ read $f + g$ (three times), thanks to Michał Strzelecki.

- Page 158, line 13: instead of

$$H(f)(g,h) = \frac{1}{2}\Big[\Gamma\big(g, \Gamma(f,h)\big) + \Gamma\big(h, \Gamma(f,g)\big) - \Gamma\big(f, \Gamma(g,h)\Big].$$

  read (add a))

$$H(f)(g,h) = \frac{1}{2}\Big[\Gamma\big(g, \Gamma(f,h)\big) + \Gamma\big(h, \Gamma(f,g)\big) - \Gamma\big(f, \Gamma(g,h)\big)\Big].$$

- Page 170, line -6: remove $L^*(f)$ at the beginning of the formula.

- Page 200, line -8 to the end of the page. Replace the paragraph by the following:

  The second set *(ii)* of inequalities, without any boundary condition, appears as a consequence of *(iii)* by symmetrization and periodization (for $f : [0,1] \to \mathbf{R}$ arbitrary, define $g : [-1,+1] \to \mathbf{R}$ by $g(x) = f(x)$ for $x \in [0,+1]$, $g(x) = f(-x)$ for $x \in [-1,0]$, and apply *(iii)* to g on the interval $[-1,+1]$ after re-scaling).

  Finally *(i)* is a consequence of *(iii)* by anti-symmetrization and periodization. For $f : [0,1] \to \mathbf{R}$ such that $f(0) = f(1) = 0$, define $g : [-1,+1] \to \mathbf{R}$ by $g(x) = f(x)$ for $x \in [0,+1]$, $g(x) = -f(-x)$ for $x \in [-1,0]$. Then

$$\int_{[0,1]} f^2 dx = \int_{[-1,1]} g^2 \frac{dx}{2} - \Big(\int_{[-1,1]} g\frac{dx}{2}\Big)^2 \leq \frac{1}{\pi^2}\int_{[-1,1]} g'^2 \frac{dx}{2} = \frac{1}{\pi^2}\int_{[0,1]} f'^2 dx,$$

  where *(iii)* has been applied to the probability measure $1_{[-1,1]}\frac{dx}{2}$ with the optimal constant $1/\pi^2$. The function $f(x) = \sin(\pi x)$ is an optimal function from a direct computation.

- Page 201, line -7: instead of
$$\int_K (f_\ell^2 - \frac{1}{\mu(K)} \int_K f_\ell d\mu)^2 d\mu,$$
read
$$\int_K (f_\ell - \frac{1}{\mu(K)} \int_K f_\ell d\mu)^2 d\mu,$$
(thanks to Arnak Dalalyan).

- Page 205, Proposition 4.6.4: instead of
$$C_{K\cup L} \leq \frac{\mu(K\cap L)}{\mu(K\cup L)} \max(C_K, C_L),$$
read
$$C_{K\cup L} \leq 2\frac{\mu(K\cap L)}{\mu(K\cup L)} \max(C_K, C_L),$$
(thanks to Michał Strzeleck).

- Page 211, line 5: instead of $\Gamma(P_t f) = O(t^{-1/2})$ read $\sqrt{\Gamma(P_t f)} = O(t^{-1/2})$.

- Page 240, line -1: instead of $s = \int_E f d\mu$ read $s = f$, thanks to Michał Strzelecki.

- Page 249, Proposition 5.2.7: instead of $\mathbb{E}_1, \mathbb{E}_2$ read E_1, E_2.

- Page 251, formula (5.3.2), read $(q-1)^{k/2}$ instead of $(q-1)^k$, thanks to Max Fathi.

- Page 263, line 11: instead of $\Lambda^{q-1}(s)$ in the LHS, read
$$\frac{q^2}{q'}\Lambda^{q-1}(s)\Lambda'(s),$$
moreover the function q is decreasing, thanks to Michał Strzelecki.

- Page 267, line -7: instead of
$$\frac{d(x,y)}{2t}$$
in the RHS, read
$$\frac{d(x,y)}{2\sqrt{t}},$$
thanks to Michał Strzelecki.

- Page 298, line -8: instead of
$$P_t(f\log f) - P_t f \log P_t f \leq t\Delta P_t f + \frac{n}{2}\log(1 - \frac{2t}{n}\frac{P_t(f\Delta(\log f))}{P_t f}),$$
read
$$P_t(f\log f) - P_t f \log P_t f \leq t\Delta P_t f + \frac{n}{2}P_t f\log(1 - \frac{2t}{n}\frac{P_t(f\Delta(\log f))}{P_t f}).$$

- Page 298. The proof of Theorem 6.7.3 can be simplified as follows.

  Let f be a nonnegative function and let, as usual, for $s \in [0,t]$
$$\Lambda(s) = P_s(P_{t-s}f \log P_{t-s}f).$$
  As already observed,
$$\Lambda'(s) = P_s(P_{t-s}f\Gamma(\log P_{t-s}f)),$$
$$\Lambda''(s) = 2P_s(P_{t-s}f\Gamma_2(\log P_{t-s}f))$$
  and the $CD(0,n)$ condition yields the inequality (6.7.6) page 300,
$$\Lambda''(s) \geqslant \frac{2}{nP_tf}[LP_tf - \Lambda'(s)]^2.$$
  Now, letting $\varphi(s) = \Lambda(s) - sLP_tf$, the previous inequality can be reformulated as,
$$\varphi''(s) \geqslant \frac{2}{nP_tf}(\varphi'(s))^2, \quad s \in [0,t].$$
  In other words, the map
$$[0,t] \ni s \mapsto \exp\left(-\frac{2}{nP_tf}\varphi(s)\right)$$
  is concave.

  Then the two inequalities hold true:
$$-\frac{2}{nP_tf}\varphi'(t)\exp\left(-\frac{2}{nP_tf}\varphi(t)\right) \leq \frac{\exp\left(-\frac{2}{nP_tf}\varphi(t)\right) - \exp\left(-\frac{2}{nP_tf}\varphi(0)\right)}{t} \leq$$
$$-\frac{2}{nP_tf}\varphi'(0)\exp\left(-\frac{2}{nP_tf}\varphi(0)\right).$$
  The first inequality can be written as
$$P_t\left(\frac{\Gamma(f)}{f}\right) - LP_tf + \frac{n}{2t}P_tf \geqslant \frac{n}{2t}P_tf\exp\left(-\frac{2}{nP_tf}(\varphi(0)-\varphi(t))\right),$$
  which is a reformulation of inequality (6.7.4), and the second one can be written as
$$-\frac{\Gamma(P_tf)}{P_tf} + LP_tf + \frac{n}{2t}P_tf \geqslant \frac{n}{2t}P_tf\exp\left(-\frac{2}{nP_tf}(\varphi(t)-\varphi(0))\right),$$
  which is a reformulation of inequality (6.7.5). We recover the Li-Yau inequality since the exponential is positive.

- Page 301, line 11: instead of
$$\Lambda''(s) \geqslant \frac{2[LP_tf - \Lambda'(s)]^2}{nP_tf} + \rho\Lambda'(s),$$
  read
$$\Lambda''(s) \geqslant \frac{2[LP_tf - \Lambda'(s)]^2}{nP_tf} + 2\rho\Lambda'(s).$$

- Page 308, additional information on Theorem 6.8.3. For all the computations explained on page 309, the extremal function f has to satisfy some properties.

  First, from the indentity
$$\int (f^{q-1} - (1+\epsilon)f)ud\mu = C\mathcal{E}(f,u),$$
  we get
$$\int f^{q-1}ud\mu = C\int f\Big(\frac{1+\epsilon}{C}u - Lu\Big)d\mu.$$
  That is, if $R_\lambda(u) = g$ with $\lambda = \frac{1+\epsilon}{C}$, the equality becomes
$$\int (R_\lambda(f^{q-1}) - Cf)gd\mu = 0.$$
  This equalition implies back that
$$f = \frac{1}{C}R_\lambda(f^{q-1})$$
  and then, $f \in \mathcal{D}(L)$.

  It is proved that f is bounded from above and below (by a strictly positive constant). From the equation satisfied by f, we know that Lf is also bounded. To apply the various integration by parts formula, we need to prove that for any constant $a \in \mathbf{R}$, $f^a \in \mathcal{D}(L)$. One way to prove it is to show that $\Gamma(f)$ is a bounded function.

  From the first formula page 312, we have
$$f = \frac{1}{C}R_\lambda(f^{q-1}),$$
  which implies that
$$\sqrt{\Gamma(f)} \leq \frac{1}{C}\int_0^\infty e^{-\lambda t}\sqrt{\Gamma(P_t(f^{q-1}))}dt.$$
  Now, since the model satisfies the $CD(0,\infty)$ condition and f^{q-1} is a bounded function, Inequality 4.7.7 page 211 implies that
$$\sqrt{\Gamma(P_t(f^{q-1}))} \leq \frac{||f^{q-1}||_\infty}{\sqrt{t}},\ t > 0.$$
  The two previous inequalities imply that $\Gamma(f)$ is a bounded function.

- Page 315, formula (6.9.2): instead of $\hat{L}$, read $\hat{L}(f)$.

- Page 317, line 12: instead of $\nabla W(f)$, read $\Gamma(W,f)$.

- Page 318, line 13: instead of μ, read $\mu_{\mathfrak{g}}$.

- Page 321, Proposition 6.9.6 and its proof have to be replaced by the following (see also [1] for a more developed proof).

Proposition 6.9.6 *Let $d\mu = e^{-W}d\mu_{\mathfrak{g}}$ and $\alpha \in \mathbf{R}$, then*

$$S_\alpha(\mu,\Gamma) = \gamma_n(\alpha)[sc_{\mathfrak{g}} - \alpha\Delta_{\mathfrak{g}}W + \beta_n(\alpha)\Gamma(W)]$$

is n-conformal invariant where

$$\beta_n(\alpha) = \frac{\alpha(n-2n_0+2) - 2(n_0-1)}{2(n-n_0)}$$

and

$$\gamma_n(\alpha) = \frac{n-2}{4(n_0-1) - 2\alpha(n-n_0)}.$$

Proof

$\lhd$ It is enough to check that $S_\alpha(\mu,\Gamma)$ satisfies the condition (6.9.1). The measure μ is transformed to $\hat{\mu} = c^{-n}\mu$, and Γ to $\hat{\Gamma} = c^2\Gamma$. From the previous computations, $sc_{\mathfrak{g}}$ becomes

$$\hat{sc}_{\mathfrak{g}} = c^2[sc_{\mathfrak{g}} + (n_0-1)(2\Delta_{\mathfrak{g}}\tau - (n_0-2)\Gamma(\tau))],$$

$W = -\log\frac{d\mu}{d\mu_{\mathfrak{g}}}$ becomes

$$\hat{W} = -\log\frac{d\hat{\mu}}{d\hat{\mu}_{\mathfrak{g}}} = -\log\frac{c^{-n}d\mu}{c^{-n_0}d\mu_{\mathfrak{g}}} = -\log c^{n_0-n}\frac{d\mu}{d\mu_{\mathfrak{g}}} = W + (n-n_0)\tau,$$

and finally, $\Delta_{\mathfrak{g}}$ becomes

$$\hat{\Delta}_{\mathfrak{g}} = c^2[\Delta_{\mathfrak{g}} - (n_0-2)\Gamma(\tau,\cdot)].$$

So,

$$\begin{aligned} S_\alpha(c^{-n}\mu, c^2\Gamma) = c^2\gamma_n(\alpha)\Big[&sc_{\mathfrak{g}} + [2(n_0-1) - \alpha(n-n_0)]\Delta_{\mathfrak{g}}(\tau) \\ &+ [\beta_n(\alpha)(n-n_0)^2 - (n_0-1)(n_0-2) + \alpha(n_0-2)(n-n_0)]\Gamma(\tau) \\ &- \alpha\Delta_{\mathfrak{g}}(W) + [\alpha(n_0-2) + 2\beta_n(\alpha)(n-n_0)]\Gamma(\tau,W) + \beta_n(\alpha)\Gamma(W)\Big]. \end{aligned}$$

It has to be equal to

$$c^2\Big[\gamma_n(\alpha)[sc_{\mathfrak{g}} - \alpha\Delta_{\mathfrak{g}}(W) + \beta_n(\alpha)\Gamma(W)] + \frac{n-2}{2}\Big(\Delta_{\mathfrak{g}}(\tau) - \Gamma(W,\tau) - \frac{n-2}{2}\Gamma(\tau)\Big)\Big].$$

On can check the values of $\gamma_n(\alpha)$ and $\alpha_n(\alpha)$ proposed do the job. $\rhd$

- Page 322, line -18: instead $\nabla\nabla U = -U Id$ read $\nabla\nabla U = -U g_{\mathbf{S}^{n_0}}$ where $g_{\mathbf{S}}$ is the spherical metric.
- Page 338, line -1: instead of $I(u)$, read $I_{\mu,F}(u)$.
- Page 364, line -6: The sentence starting by *In the finite measure case...* is not correct. It has to be replaced by the following one: In the finite measure case, the tight Nash inequality (3.2.3), p. 281, corresponds to a function Φ which is the inverse function of $(1,+\infty) \ni x \mapsto (x^{1+2/n} - x)/C$.

- Page 372, line -5: instead of $e^{-C/t}$, read $e^{-t/C}$.
- Page 373, line -13: instead of $w(x) = p(x)^{1/2}(1+x^2)^{-\beta}$, read $w(x) = p(x)^{-1/2}(1+x^2)^{-\beta}$ (thanks to Persi Diaconis).
- Page 425, Theorem 8.6.3: the set A_{d_t} should be here the d_t-closed neighborhood of A instead of the open one ($A_{d_t} = \{x \in E\,;\, d(x,A) \leq d_t\}$ instead of $A_{d_t} = \{x \in E\,;\, d(x,A) < d_t\}$).
- Page 448, line -7: (the line before formula (9.3.5)) the integration is w.r.t. the measure $u^{1-1/n}dx$ instead of udx (thanks to Emanuel Milman).
- Page 464, formula (9.7.4) should be

  $$W_2^2(P_t f\mu, P_t g\mu) \leq W_2^2(f\mu, g\mu) + 2n(\sqrt{t} - \sqrt{s})^2,$$

  thanks to Luigia Ripani.
- Page 516, in the formula (C.6.5) the last term should be

  $$H(f_i)(f_j, f_l)$$

  instead of

  $$H(f_i)(f_i, f_l)$$

  thanks to François Bolley.
- Page 514, line -2: instead of wrapped product, read, of course, warped products !

References

[1] L. Dupaigne, I. Gentil, S. Zugmeyer. A conformal geometric point of view on the Caffarelli-Kohn-Nirenberg inequality. Preprint 2021.