Automata and algebraic extensions of free groups

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Universitat Politècnica de Catalunya

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Outline

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Definitions and notation

- $A = \{a_1, \ldots, a_n\}$ is a finite alphabet (n letters).
- $A^{\pm 1} = A \cup A^{-1} = \{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\}$.
- Usually, $A = \{a, b, c\}$.
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- $F_A = (A^{\pm 1})^* / \sim$ is the free group on A (words on $A^{\pm 1}$ modulo $\sim$).
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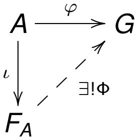
- The universal property: given a group G and a mapping $\varphi\colon A \to G$, there exists a unique group homomorphism $\Phi\colon F_A \to G$ such that the diagram



  commutes (where ι is the inclusion map).

- Every group is a quotient of a free group

$$G = \langle a_1, \ldots, a_n \,|\, r_1, \ldots, r_m \rangle = F_A / \ll r_1, \ldots, r_m \gg .$$

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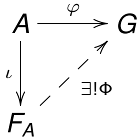

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Comparison with linear algebra

vector spaces	free groups
• K^n f.d. K-vector space	• F_n f.g. free group
• Every f.d. K-vector space is like this,	• Every group G is a quotient of a free group,
• $K^n \simeq K^m \Leftrightarrow n = m$,	• $F_n \simeq F_m \Leftrightarrow n = m$,
• –	• (Nielsen-Schreier) Every subgroup of a free group is free,
• Steinitz Lemma,	• Not true,
• $F \leqslant E \Rightarrow \dim F \leqslant \dim E$,	• Very false: $F_{\aleph_0} \leqslant F_2$.
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A Stallings automata is a finite A-labeled oriented graph with a distinguished vertex, (X, v), such that:

1- *X is connected,*

2- *no vertex of degree 1 except possibly v (X is a core-graph),*

3- *no two edges with the same label go out of (or in to) the same vertex.*

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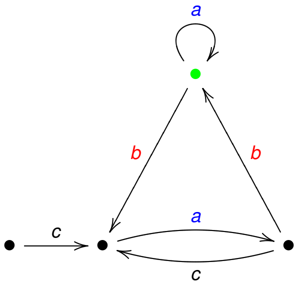

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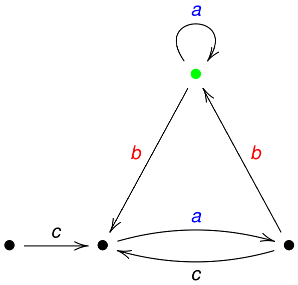

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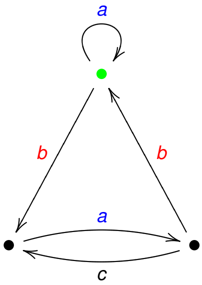

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Stallings (building on previous works) gave a bijection between finitely generated subgroups of F_A and Stallings automata:

$$\{\text{f.g. subgroups of } F_A\} \longleftrightarrow \{\text{Stallings automata}\},$$

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Reading the subgroup from the automata

Definition

To any given (Stallings) automaton (X, v), we associate its fundamental group:

$$\pi(X, v) = \{ \text{labels of closed paths at } v\} \leqslant F_A,$$

clearly, a subgroup of F_A.

$\pi(X, \bullet) = \{1, a, a^{-1}, bab, bc^{-1}b, babab^{-1}cb^{-1}, \ldots\}$

$\pi(X, \bullet) \not\ni bc^{-1}bcaa$

Membership problem in $\pi(X, \bullet)$ is solvable.

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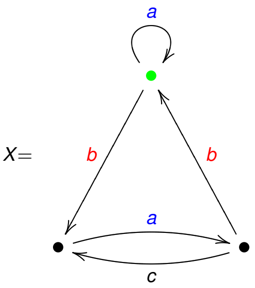

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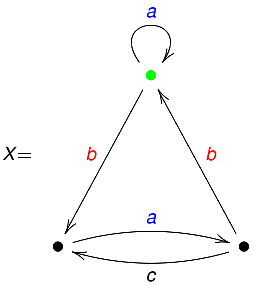

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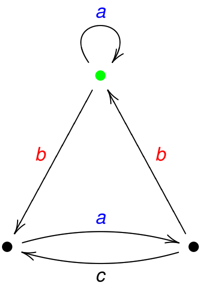
$X=$ 

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Membership problem in $\pi(X, \bullet)$ is solvable.

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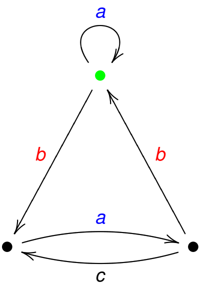

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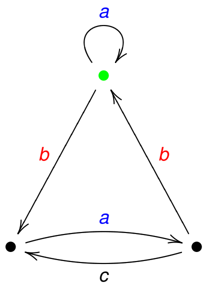

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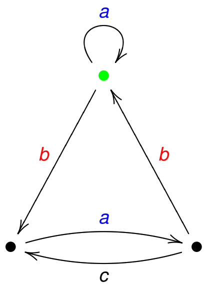
$X=$ 

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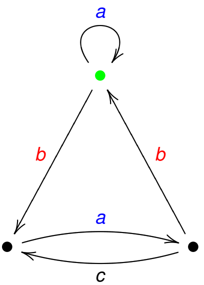

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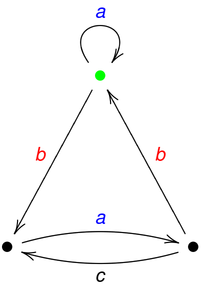

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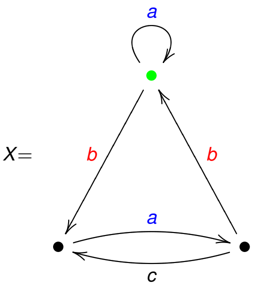

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A basis for $\pi(X, v)$

Proposition

For every Stallings automaton (X, v), the group $\pi(X, v)$ is free of rank $rk(\pi(X, v)) = 1 - |VX| + |EX|$.

Proof:

- Take a maximal tree T in X.
- Write $T[p, q]$ for the geodesic (i.e. the unique reduced path) in T from p to q.
- For every $e \in EX - ET$, $x_e = label(T[v, \iota e] \cdot e \cdot T[\tau e, v])$ belongs to $\pi(X, v)$.
- Not difficult to see that $\{x_e \mid e \in EX - ET\}$ is a basis for $\pi(X, v)$.
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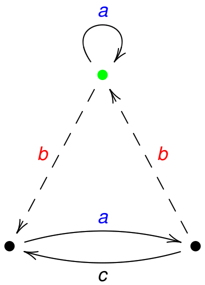
Example



$H = \langle \quad \rangle$

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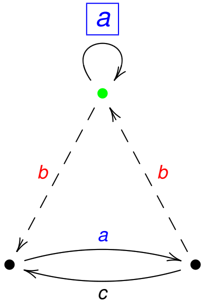

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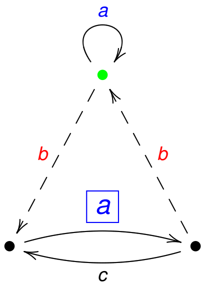
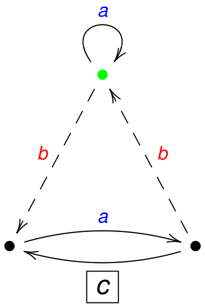


$$H = \langle a, \; bab, \quad \rangle$$

Example



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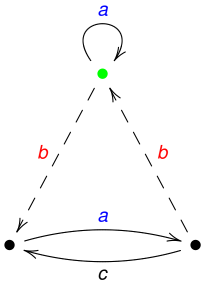
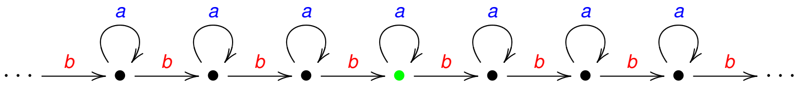


$$H = \langle a,\ bab,\ b^{-1}cb^{-1} \rangle, \quad rk(H) = 1 - 3 + 5 = 3.$$

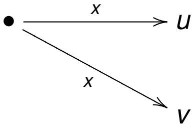
Example-2



$$F_{\aleph_0} \simeq H = \langle \ldots, b^{-2}ab^{2}, b^{-1}ab, a, bab^{-1}, b^{2}ab^{-2}, \ldots \rangle \leqslant F_2.$$

Constructing the automata from the subgroup

In any automaton containing the following situation, for $x \in A^{\pm 1}$,



we can fold and identify vertices u and v to obtain

This operation, $(X, v) \rightsquigarrow (X', v)$, is called a Stallings folding.

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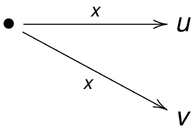

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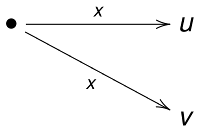

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Lemma (Stallings)

If $(X, v) \rightsquigarrow (X', v')$ is a Stallings folding then $\pi(X, v) = \pi(X', v')$.

Given a f.g. subgroup $H = \langle w_1, \ldots w_m \rangle \leqslant F_A$ (we assume w_i are reduced words), do the following:

1- *Draw the flower automaton,*

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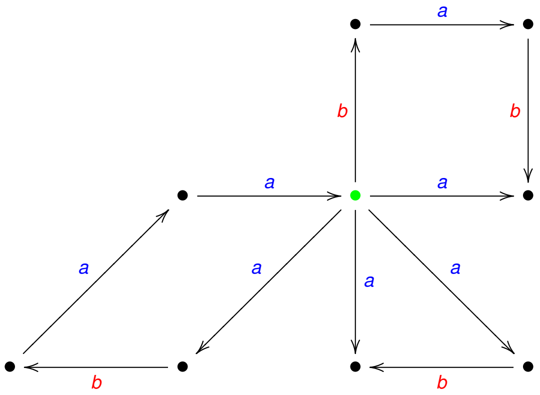

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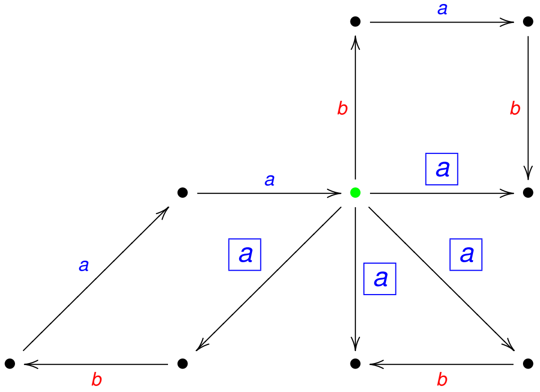

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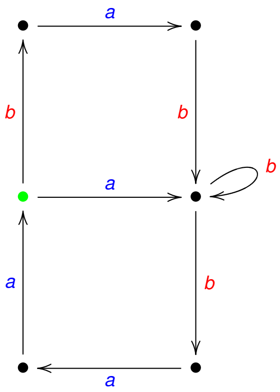

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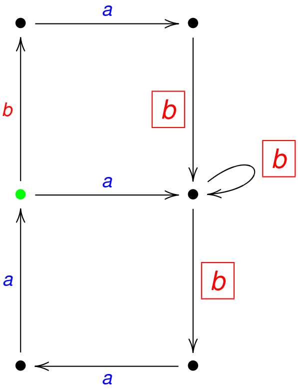

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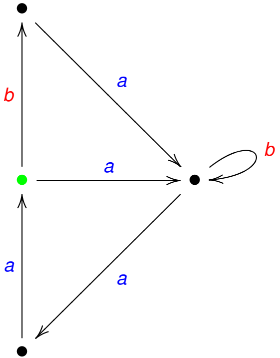

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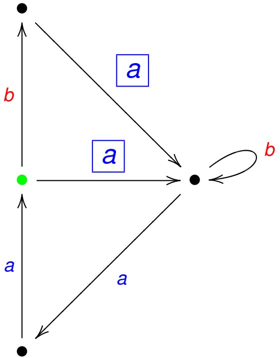

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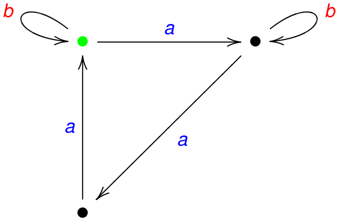

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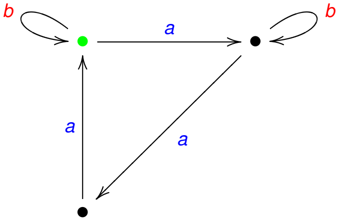

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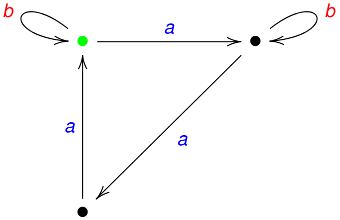

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By Stallings Lemma, $\pi(\Gamma(H), \bullet) = \langle baba^{-1}, aba^{-1}, aba^2 \rangle$
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Independence from the process

Proposition

The automaton $\Gamma(H)$ does not depend on the sequence of foldings

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The automaton $\Gamma(H)$ does not depend on the generators of H.

Proofs can be made completely graphical and are not difficult.

This gives a very useful bijection:

$$\begin{array}{rcl} \{\text{f.g. subgroups of } F_A\} & \longleftrightarrow & \{\text{Stallings automata}\} \\ H & \rightarrow & \Gamma(H) \\ \pi(X, v) & \leftarrow & (X, v) \end{array}$$

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Theorem (Nielsen-Schreier)

Every subgroup of F_A is free.

Proof:

- Let $H = \langle w_1, \ldots, w_p \rangle \leqslant_{f.g.} F_A$.
- By the bijection, we know that $H = \pi(\Gamma(H))$.
- By the previous observation, H is free.
- Everything extends easily to the infinitely generated case (considering infinite graphs). $\square$

- The original proof (1920's) was combinatorial and much more technical.

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(Membership)

Does w belong to $H = \langle w_1, \ldots, w_m \rangle$?

- Construct $\Gamma(H)$,
- Check whether w is readable as a closed path in $\Gamma(H)$ (at the basepoint).

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- Check whether w is readable as a closed path in $\Gamma(H)$ (at the basepoint).

(Containment)

Given $H = \langle w_1, \ldots, w_m \rangle$ and $K = \langle v_1, \ldots, v_n \rangle$, is $H \leqslant K$?

- Construct $\Gamma(K)$,
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Basis & conjugacy

(Computing a basis)

Given $H = \langle w_1, \ldots, w_m \rangle$, find a basis for H.

- Construct $\Gamma(H)$,
- Choose a maximal tree,
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(Conjugacy)

Given $H = \langle w_1, \ldots, w_m \rangle$ and $K = \langle v_1, \ldots, v_n \rangle$, are they conjugate (i.e. $H^x = K$ for some $x \in F_A$) ?

- Construct $\Gamma(H)$ and $\Gamma(K)$,
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- Every path between the two basepoints spells a valid x.

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Outline

1. The friendly and unfriendly free group
2. The bijection between subgroups and automata
3. Several algebraic applications
   - First results
   - Finite index subgroups
   - Intersections
4. Algebraic extensions and Takahasi's theorem
   - Takahasi's theorem
   - Computing the set of algebraic extensions
   - The algebraic closure
   - Pro-$\mathcal{V}$ closures
   - Other closures

Finite index subgroups

(Finite index)

Given $H = \langle w_1, \ldots, w_m \rangle$, is $H \leqslant_{f.i.} F_A$? If yes, find a set of coset representatives.

→ For $u \in V\Gamma(H)$, choose p (the label of) a path from • to u; then,

$$\{\text{labels of paths from } \bullet \text{ to } u\} = \pi(\Gamma(H), \bullet) \cdot p = H \cdot p$$

is a coset of F_A/H.

→ F_A/H is in bijection with the set of vertices of the "extended $\Gamma(H)$"

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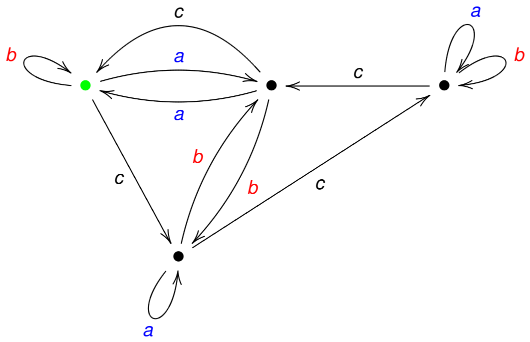
$\Gamma(H) =$ 

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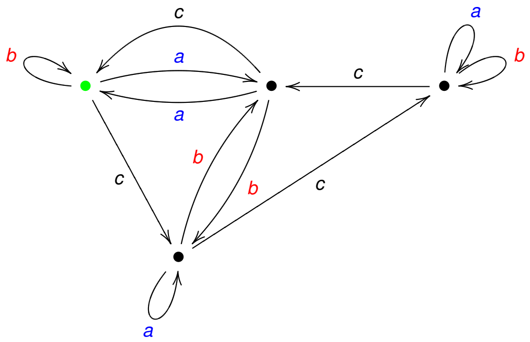

$$F_3 = H \sqcup Hc \sqcup Ha \sqcup Hac^{-1}.$$

More on finite index

(Schreier index formula)

If $H \leqslant_{f.i.} F_A$ is of index $[F : H]$, then $r(H) = 1 + [F : H] \cdot (r(F_A) - 1)$.

Proof:

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Every f.g. subgroup $H \leqslant_{fg} F_A$ is a free factor of a finite index one, $H \leqslant_{ff} H * L \leqslant_{f.i.} F_A$.

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Example

$$H = \langle b, cbc, c^2bc^{-2} \rangle$$

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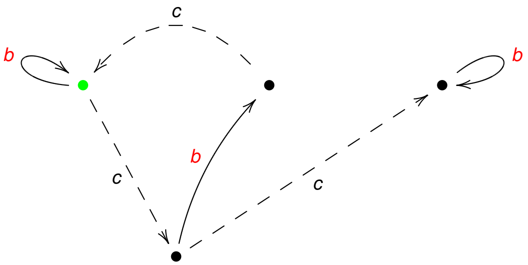

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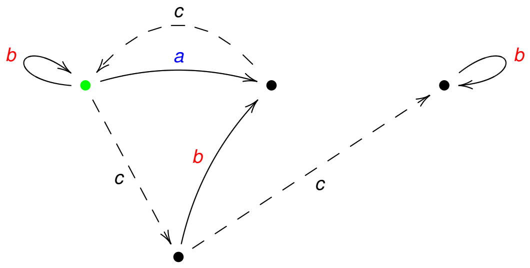

$$H \leqslant_{ff} H * \langle ac \rangle$$

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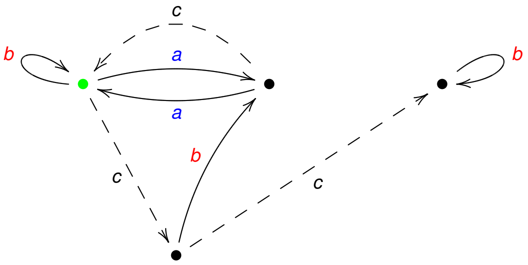

$$H \leqslant_{ff} H * \langle ac, c^{-1}a \rangle$$

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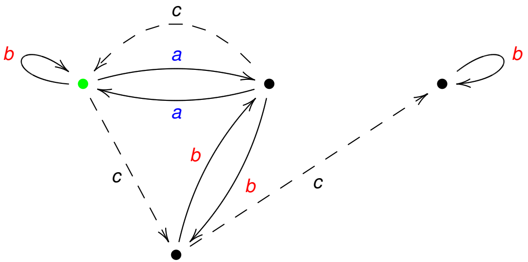

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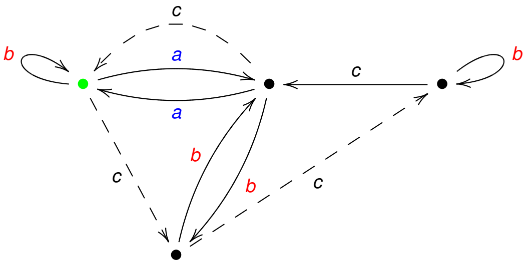

$$H \leqslant_{ff} H * \langle ac, c^{-1}a, c^{-1}bc^{-1}, c^4 \rangle$$

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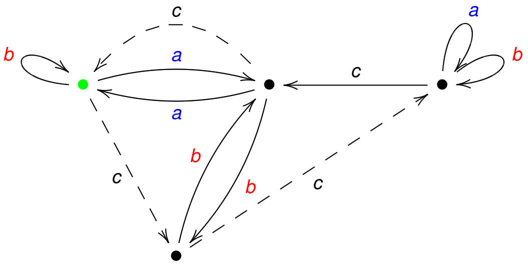

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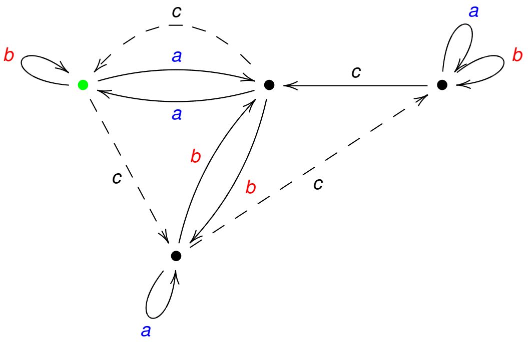

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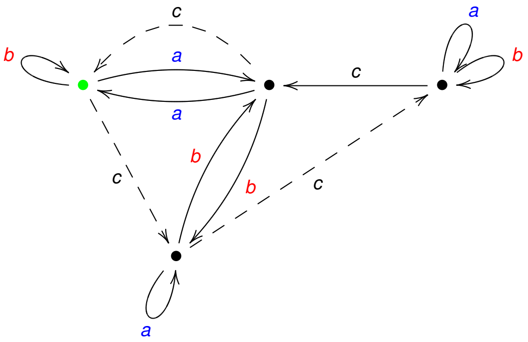

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Outline

Pull-back of automata

Definition

The pull-back of two Stallings automata, (X, v) and (Y, w), is the cartesian product $(X \times Y, (v, w))$ (respecting labels). This is not in general connected, neither without degree 1 vertices, but it is folded.

Theorem ((H. Neumann)-Stallings)

For every f.g. subgroups $H, K \leqslant_{fg} F_A$, $\Gamma(H \cap K)$ coincides with the connected component of $\Gamma(H) \times \Gamma(K)$ containing the basepoint, after trimming.

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Let $H = \langle a, b^2, bab \rangle$ and $K = \langle b^2, ba^2 \rangle$ be subgroups of F_2.
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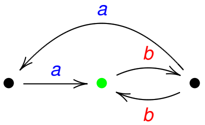

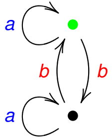

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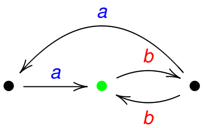

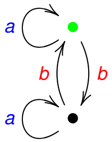

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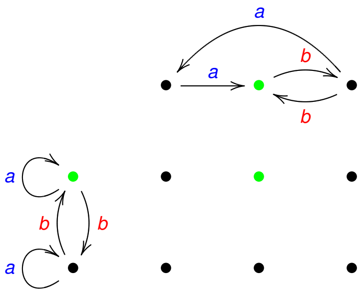

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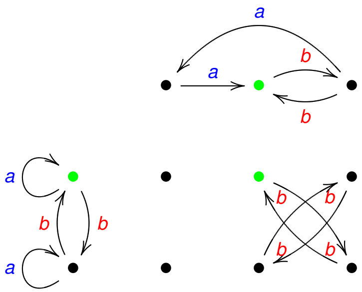

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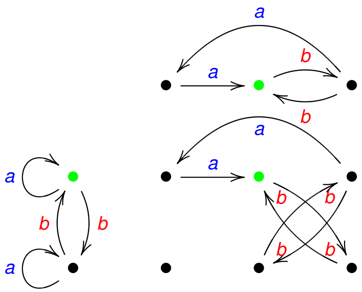

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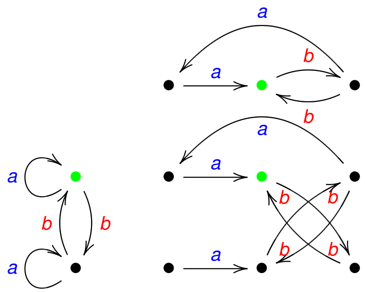

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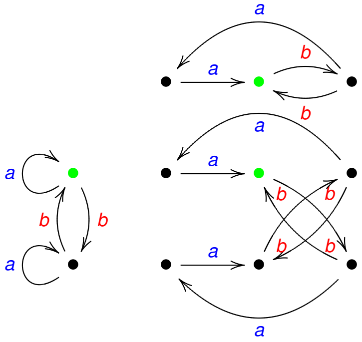
To compute a basis for $H \cap K$:



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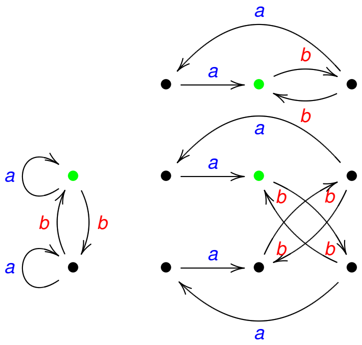

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Theorem (Howson)

The intersection of finitely generated subgroups of F_A is again finitely generated.

But the intersection can have bigger rank: "$3 = 3 \cap 2 \leqslant 2$"

Theorem (H. Neumann)

$\tilde{r}(H \cap K) \leqslant 2\tilde{r}(H)\tilde{r}(K)$, *where* $\tilde{r}(H) = \max\{0, r(H) - 1\}$.

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Status of Hanna Neumann Conjecture

- HNC holds if H (or K) has rank 1 (immediate),
- HNC holds for finite index subgroups (elementary),
- HNC holds if H has rank 2 (Tardös, 1992), (not easy),
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Outline

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Takahasi's Theorem

In linear algebra,

$$F \leqslant E \quad \Rightarrow \quad E = F \oplus L, \text{ for some } L$$

(every basis of F can be extended to a basis of E).

In free groups this is clearly false but ... almost true.

Theorem (Takahasi, 1951)

Every $H \leqslant_{fg} F_A$, has a finite set of extensions, $\{H_0 = H, H_1, \ldots, H_m\}$, all of them finitely generated and computable, satisfying: for every $H \leqslant K \leqslant F_A$, $\exists\ i = 0, \ldots, m$ such that $H \leqslant H_i \leqslant_{ff} H_i * L = K$.

Let us reformulate this in a different way.

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Free and algebraic extensions

Definition

And extension of subgroups $H \leqslant K$, in F_A is called

- *a free extension if H is a free factor of K (i.e. $K = H * L$ for some $L \leqslant F_A$), denoted $H \leqslant_{ff} K$;*
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- $\langle a \rangle \leqslant_{ff} \langle a, b \rangle \leqslant_{ff} \langle a, b, c \rangle$, and $\langle x^r \rangle \leqslant_{alg} \langle x \rangle$, $\forall x \in F_A \; \forall r \in \mathbb{Z}$.
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Theorem (Takahasi, 1951)

For every $H \leqslant_{fg} F_A$, the set of algebraic extensions, denoted $\mathcal{AE}(H)$, is finite.

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Proof:

- Let us (temporarily) attach some “hairs" to $\Gamma(H)$ an denote the resulting (folded) automata by $\tilde{\Gamma}(H)$.
- Given $H \leqslant K$ (both f.g.), we can obtain $\Gamma(K)$ from $\Gamma(H)$ by 1) adding the appropriate hairs, 2) identifying several vertices to $\bullet$, 3) folding; (note that adding extra hairs, the result will be the same if we 4) trim at the end).
- Hence, if $H \leqslant K$ (both f.g.) then $\Gamma(K)$ contains as a subgraph either $\Gamma(H)$ or some quotient of it (i.e. $\Gamma(H)$ after identifying several sets of vertices ($\sim$) and then folding, $\Gamma(H)/\sim$).
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- Given $H \leqslant K$ (both f.g.), we can obtain $\Gamma(K)$ from $\Gamma(H)$ by 1) adding the appropriate hairs, 2) identifying several vertices to $\bullet$, 3) folding; (note that adding extra hairs, the result will be the same if we 4) trim at the end).
- Hence, if $H \leqslant K$ (both f.g.) then $\Gamma(K)$ contains as a subgraph either $\Gamma(H)$ or some quotient of it (i.e. $\Gamma(H)$ after identifying several sets of vertices ($\sim$) and then folding, $\Gamma(H)/\sim$).
- The overgroups of H:
  $\mathcal{O}(H) = \{\pi(\Gamma(H)/\sim, \bullet) \mid \sim \text{ is a partition of } V\Gamma(H)\}$.
- Hence, for every $H \leqslant K$, there exists $L \in \mathcal{O}(H)$ such that $H \leqslant L \leqslant_{ff} K$.
- Thus, $\mathcal{AE}(H) \subseteq \mathcal{O}(H)$ and so, it is finite. □

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Outline

Computing $\mathcal{AE}(H)$

Corollary

$\mathcal{AE}(H)$ is computable.

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- Compute $\Gamma(H)$,
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Given $H, K \leqslant F_A$, it is algorithmically decidable whether $H \leqslant_{ff} K$.

Proved by:

- Whitehead 1930's (classical and exponential),
- Silva-Weil 2006 (graphical algorithm, faster but still exponential),
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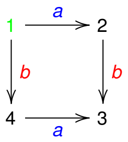

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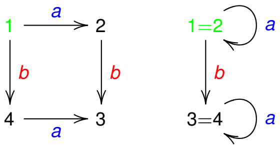

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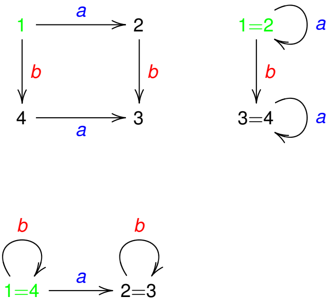

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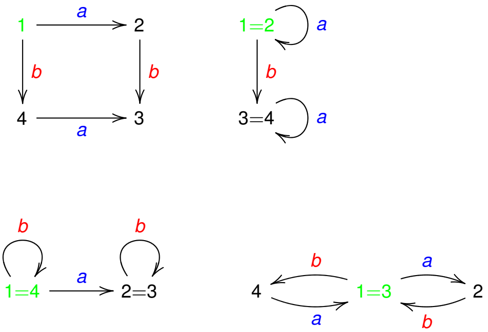

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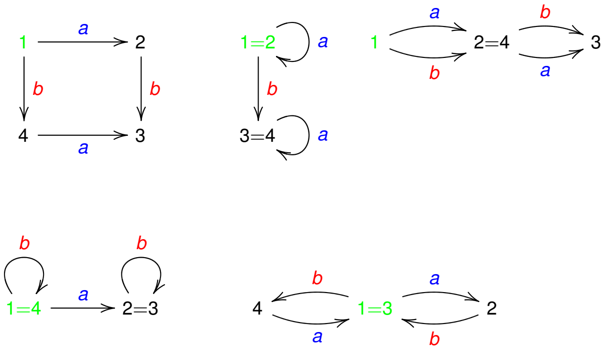

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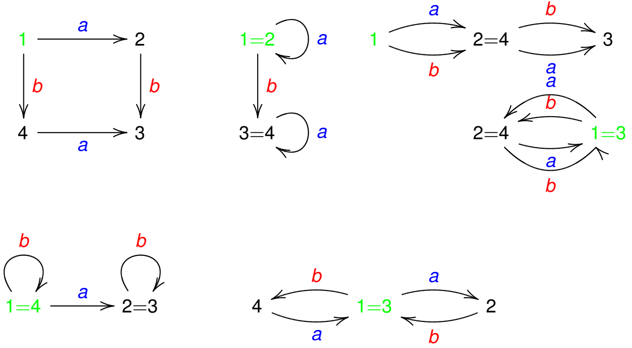

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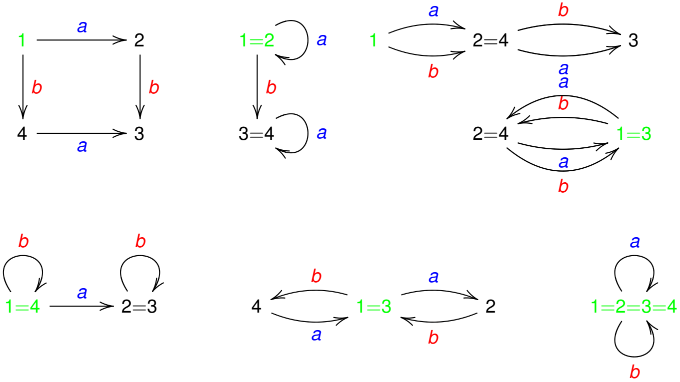

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The algebraic closure

Lemma

If $H \leqslant_{alg} K_1$ and $H \leqslant_{alg} K_2$ then $H \leqslant_{alg} \langle K_1 \cup K_2 \rangle$.

Corollary

Let $H \leqslant_{fg} F_A$. For an intermediate extension $H \leqslant M \leqslant F_A$, TFAE:

(a) *M is the smallest free factor of F_A containing H,*

(b) *M is the biggest algebraic extension of H in $F(A)$,*

(c) *M is a maximal element in $\mathcal{AE}(H)$,*

The unique subgroup M satisfying these conditions is called the algebraic closure of H in F_A, and denoted $Cl_{F_A}(H)$. In particular, $H \leqslant_{alg} Cl_{F_A}(H) \leqslant_{ff} F_A$.

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For an arbitrary extension of f.g. subgroups $F \leqslant K \leqslant F_A$, we can do the same relative to K and get:

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Every extension $H \leqslant K$ of f.g. subgroups of F_A splits, in a unique way, in an algebraic part and a free factor part, $H \leqslant_{alg} Cl_K(H) \leqslant_{ff} K$.

One can define the notions of algebraically closed and algebraically dense subgroups, in a similar way as in field theory.

Some properties are similar, some other are different...

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Varieties of finite groups

Definition

A variety $\mathcal{V}$ of finite groups is a family of finite groups closed under taking subgroups, quotients, and finite direct products. $\mathcal{V}$ is extension-closed if, for every short exact sequence
$1 \to G_1 \to G_2 \to G_3 \to 1, \quad G_1, G_3 \in \mathcal{V}$ *implies* $G_2 \in \mathcal{V}$.

Examples:

- $\mathcal{V} =$ all finite groups, (it is ext. closed),
- $\mathcal{V} =$ the p-groups, where p a prime number (it is ext. closed),
- $\mathcal{V} =$ the P-groups, where P is a set of primes (it is ext. closed),
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Varieties of finite groups

Definition

A variety $\mathcal{V}$ of finite groups is a family of finite groups closed under taking subgroups, quotients, and finite direct products. $\mathcal{V}$ is extension-closed if, for every short exact sequence
$1 \to G_1 \to G_2 \to G_3 \to 1, \quad G_1, G_3 \in \mathcal{V}$ *implies* $G_2 \in \mathcal{V}$.

Examples:

- $\mathcal{V} =$ all finite groups, (it is ext. closed),
- $\mathcal{V} =$ the p-groups, where p a prime number (it is ext. closed),
- $\mathcal{V} =$ the P-groups, where P is a set of primes (it is ext. closed),
- $\mathcal{V} =$ the nilpotent groups (it is not ext. closed),
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The pro-$\mathcal{V}$ topology

Definition

Let $\mathcal{V}$ be a variety of finite groups. We can define the pro-$\mathcal{V}$ topology in F_A in either of the following equivalent ways:

- *the smallest topology for which all morphisms $F_A \to G \in \mathcal{V}$ are continuous,*
- *the topology for which the normal subgroups $N \leqslant F_A$ with $F/N \in \mathcal{V}$ form a basis of neighborhoods of the unit,*
- *the topology induced by the metric $d(x, y) = 2^{-s(x,y)}$, where $s(x, y) = \min\{\#G \mid G \in \mathcal{V}, \exists \varphi \colon F_A \to G \text{ such that } \varphi(x) \neq \varphi(y)\}$.*

This topology is interesting for the study of other aspects of the free group.

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Computing some pro-$\mathcal{V}$ closures

Theorem (Ribes-Zalesskiĭ)

If $\mathcal{V}$ is an extension-closed variety then, in the pro-$\mathcal{V}$ topology, every free factor of a closed subgroup of F_A is again closed.

Corollary

If $\mathcal{V}$ is extension-closed then, for every $H \leqslant_{fg} F_A$, $H \leqslant_{alg} Cl_{\mathcal{V}}(H)$. In particular, $Cl_{\mathcal{V}}(H)$ is again finitely generated.

Proposition

There is an algorithm to compute the

- *pro-p closure,*
- *pro-P closure,*
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of finitely generated subgroups of F_A.

But no algorithm is known for computing pro-solvable closures.

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Outline

The malnormal closure

Definition

A subgroup $H \leqslant F_A$ is called malnormal if, for every $x \in F_A$, $H^x \cap H$ equals either H or 1.

Proposition

- *It is algorithmically decidable wether a given $H \leqslant F_A$ is malnormal.*
- *$H_1, H_2 \leqslant F_A$ malnormal $\Rightarrow H_1 \cap H_2$ malnormal.*
- *$H \leqslant_{ff} K \leqslant F_A$, and K malnormal $\Rightarrow H$ malnormal.*

Corollary

The malnormal closure of H (i.e. the smallest extension $H \leqslant K$ being malnormal) is an algebraic extension of H and it is computable.

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A subgroup $H \leqslant F_A$ is called pure if $x^r \in H$ implies $x \in H$.

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- *It is algorithmically decidable wether a given $H \leqslant F_A$ is pure.*
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The inert closure

Definition

A subgroup $H \leqslant F_A$ is called inert if $r(H \cap K) \leqslant r(K)$ for every $K \leqslant F_A$.

Proposition

- $H_1, H_2 \leqslant F_A$ inert $\Rightarrow H_1 \cap H_2$ inert.
- $H \leqslant_{ff} K \leqslant F_A$, and K inert $\Rightarrow H$ inert.

Question

- Is the inert closure of H (i.e. the smallest extension $H \leqslant K$ being inert) computable ?
- Is it algorithmically decidable wether a given $H \leqslant F_A$ is inert ?

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THANKS