THE BOREL-DE SIEBENTHAL’S THEOREM

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This is the following.

0.1. **Theorem.** Let $F$ be a field of characteristic $\neq 2, 3$. Let $G/F$ be a reductive group and let $H/F$ be a reductive subgroup of maximal rank. Then $H = Z_G(Z(H))^0$.

The original theorem [2] is about compact Lie groups and the specialists know for a long time that is generalizes as stated. Our goal is to present here a proof of that result. The following is well known in the case of subtori [3, 15.3.2].

0.2. **Lemma.** Let $G/F$ be a reductive group and let $T/F$ be a maximal torus. Let $S \subset T$ be a subgroup.

1. The $F$–group $Z_G(S)$ is smooth and $Z_G(S)^0$ is reductive.
2. If $T$ is split, let $U_\alpha$ be the root groups associated to $\Phi(G,T)$. Then $Z_G(S)^0$ is generated by $T$ and the root subgroups $U_\alpha$ for the $\alpha \in \Phi(G,T) \cap T^*$ mapping to $0$ in $S^*$.

**Proof.** (1) Since $S$ is of multiplicative type, the group $Z_G(S)$ is smooth [4, XI.5.2]. We show that $G/k$ is reductive at the end of the proof.

(2) We can assume that $F$ is algebraically closed. The $F$-group $Z_G(S)^0$ is generated by $T$ and root subgroups $U_\alpha$ of $G$ [1, 13.20]. for $\alpha$ running over $R := \Phi(G,T) \cap \ker(T^* \rightarrow S^*)$. Similarly, the radical $M$ of $Z_G(S)^0$ is normalized by $T$, hence it is generated by $(M \cap T)^0$ and the root subgroups $U_\alpha$ of $M$. We claim that $M = (M \cap T)^0$. For sake of contradiction, assume that $U_\alpha$ is a root subgroup of $M$. Its conjugate $U_{-\alpha}$ in $Z_G(S)^0$ is then as well a root subgroup of $Z_G(S)^0$, hence $M$ contain a semisimple group of rank one, which contradicts the solvability of $M$. Thus $M$ is a torus and we conclude that $Z_G(S)^0$ is reductive. $\square$

We first look at the behaviour of Theorem 0.1 under central extensions.

0.3. **Lemma.** Under the hypothesis of the theorem, let $S$ be a central subgroup (of multiplicative type) of $G$ and denote by $f : G \rightarrow G/S$ the quotient map.

1. $S \subset H$ and $H/S$ is a reductive subgroup of maximal rank of $G/S$.
2. If $H/S = Z_G/S(Z(H/S))^0$, then $H = Z_G(Z(H))^0$.

**Proof.** (1) follows from the fact that the center of $G$ is included in all maximal tori of $G$. 

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(2) We have an exact sequence of algebraic groups

\[ 1 \to S \to Z_G(Z(H)) \to Z_{G/S}(Z(H/S)) \]

Since \( S \subset H \subset Z_G(Z(H))^0 \), we have the following exact diagram

\[
\begin{array}{cccccc}
1 & \to & S & \to & Z_G(Z(H))^0 & \to & Z_{G/S}(Z(H/S))^0 \\
| & & | & & \cup & & \\
1 & \to & S & \to & H & \to & H/S \to 1.
\end{array}
\]

If \( H/S = Z_{G/S}(Z(H/S))^0 \), it follows by diagram chase that \( H = Z_G(Z(H))^0 \).

We can now proceed to the proof of Theorem 0.1.

Proof. Reduction to the case \( H \) semisimple: Let \( S = Z(H)^0 \) be the connected center of \( H \). Then \( H \subset Z_G(S) \) and \( H/S \) is a semisimple subgroup of \( G/S \). If the result is known in the semisimple case, we have \( H/S = Z_{Z_G(S)}(Z(H))^0 \). Lemma 0.3 shows that \( H = Z_{Z_G(S)}(Z(H))^0 \), hence \( H = Z_G(Z(H))^0 \).

Furthermore Lemma 0.3 applied to \( Z(G) \) permits to assume that \( G \) is semisimple adjoint. We can assume moreover that \( k \) is algebraically closed. We consider a maximal (split) torus \( T \) of \( H \).

Case \( H \) maximal proper semisimple group of \( G \): We choose compatible orderings on the root systems \( \Phi(H, T) \subset \Phi(G, T) = T^* \). Since \( F \) is of characteristic \( \neq 2, \neq 3 \), \( \Phi(H, T) \) is a closed subsystem of \( \Phi(G, T) \) [4, XXIII.6.6]. Let \( A \) be the root lattice of \( H \), i.e. the sublattice of \( T^* \) generated by \( \Phi(H, T) \). The center \( Z(H) \) of \( H \) is a diagonalisable group whose character group is \( Z(H)^* = T^*/A \).

To show that \( H = Z_G(Z(H))^0 \), we note first that \( Z_G(Z(H))^0 \) is reductive by Lemma 0.2. Since \( Z_G(Z(H))^0 \) contains the semisimple group \( H \), \( Z_G(Z(H))^0 \) is necessarily semisimple. We claim that \( Z(H) \neq 1 \). For sake of contradiction, assume that \( A = T^* \), i.e. that \( H \) is adjoint. Then our basis for \( \Phi(H, T) \) would be a basis for \( \Phi(G, T) \), contradiction. So \( Z(H) \neq 1 \) and \( Z_G(Z(H))^0 \) is a proper subgroup of \( G \). Since \( H \) is a maximal semisimple subgroup, we conclude that \( H = Z_G(Z(H))^0 \).

General case: By dimension reasons, there is a chain of semisimple groups

\[ H = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_{n-1} \subsetneq H_n = G \]

such that \( H_i \) is maximal in \( H_{i+1} \) for \( i = 0, ..., n-1 \). By induction on \( n \), we can assume that
\[ H = Z_{H_{n-1}}(Z(H))^0 \]
\[ = \left( Z_G(Z(H)) \cap H_{n-1} \right)^0 \]
\[ = \left( Z_G(Z(H)) \cap Z_G(Z(H_{n-1})) \right)^0 \quad \text{[maximal case]} \]
\[ = Z_G(Z(H))^0 \quad \text{[} Z(H_{n-1}) \subset Z(H) \].

Thus \( H = Z_G(Z(H))^0 \) as desired. \( \square \)

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References