NOTES ON BRUHAT-TITS THEORY

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Of course, it has no sense to present quickly the full monument which is the Bruhat-Tits theory (references [7] to [12]). We shall limit ourself to very special and nice cases in this survey[1].

1. Introduction, case of linear groups

Let G be a semisimple Lie group. Elie Cartan has proven that G admits a unique (up to conjugacy) maximal compact subgroup K. Furthermore K is algebraic in the sense that it is given by polynomial equations. The basic example is that of $G = \mathrm{SL}_n(\mathbf{R})$, where K is the special orthogonal group $\mathrm{SO}_n(\mathbf{R})$. There are several variations of the proof of the conjugacy theorem.

(1) Consider the quotient $\mathrm{SL}_n(\mathbf{R})/\mathrm{SO}_n(\mathbf{R})$. It is a symmetric space of negative curvature (i.e. sectional curvature ≤ 0) and is equipped with an isometric action of $\mathrm{SL}_n(\mathbf{R})$. Since it is of negative curvature, every compact group K of $\mathrm{SL}_n(\mathbf{R})$ has a fixed point on X [18, §I.13, VI.2]. In other words, K is a subgroup of a conjugate of $\mathrm{SO}_n(\mathbf{R})$.

(2) If K is a compact subgroup of $\mathrm{SL}_n(\mathbf{R})$, the (left invariant) Haar mesure on K permits to form the mean value

$$Q(x) := \int_K q(g^{-1}.x)\, dg$$

of a given positive definite quadratic form q on $\mathbf{R}^n$. Then Q is still a positive definite quadratic form which is K–invariant, hence $K \subset O(Q)$.

One important fact is that $\mathrm{SO}_n(\mathbf{R})$ occurs as the real points of a semisimple algebraic group. This is a general fact about compact subgroups of $\mathrm{GL}_n(\mathbf{C})$ [23, §3.7]. Actually the two ways are close since they have to do with metrics. The point is that we can see the quotient space $\mathrm{SL}_n(\mathbf{R})/\mathrm{SO}_n(\mathbf{R})$ as the space of euclidean metrics on $\mathbf{R}^n$ up to scalars namely $Sym_{n,>0}(\mathbf{R})/\sim$, the quotient of the space of positive definite matrices of size n by $\mathbf{R}_+^\times$. We have a bicontinuous map

$$\mathrm{SL}_n(\mathbf{R})/\mathrm{SO}_n(\mathbf{R}) \longrightarrow Sym_{n,>0}(\mathbf{R})/\sim, \;\; g \mapsto g\,{}^t g.$$

[1] `Version of June 24, 2009.`

Example 1.1. If $n = 2$, the space $Sym_{2,>0}(\mathbf{R})/\sim$ is nothing but the hyperbolic space of dimension 2, namely $\mathbf{H}^2 = \{z \in \mathbf{C} \mid\ \mid z \mid < 1\}$ [17]. A positive definite matrix Q reads

$$Q = P diag(\lambda_+, \lambda_-)\, {}^tP$$

where $P = R_\theta$ is a matrix rotation and λ_+ (resp. λ_-) stands for for the biggest (resp. lowest) eigenvalue of Q. We apply h to $\frac{\lambda_-}{\lambda_+} e^{2i\theta}$. This defines an isomorphism $Sym_{2,>0}(\mathbf{R})/ \xrightarrow{\sim} \mathbf{H}^2$.

Furthermore the distance between the classes of two positive quadratic forms $[q]$ and $[q']$ is

$$d([q],[q']) = \frac{1}{2}\left(\frac{\mathrm{Sup}_{x\neq 0}(q'(x)/q(x))}{\mathrm{Inf}_{x\neq 0}(q'(x)/q(x))}\right).$$

The first issue was to investigate the analogies in the case of p–adic fields. It was achieved by Iwahori and Matsumoto for split semisimple simply connected group groups over p–adic fields [19]. The Bruhat-Tits framework is larger : we are given a field K equipped with a discrete valuation $\omega : K^\times \to \mathbf{Z}$ assumed to be henselian (complete for example). We denote by O its valuation ring, π an uniformizing parameter and by $k = O/\pi$ its residue field [2]. It deals with a reductive group G/K over K. By means of a faithfull representation[3] $G \hookrightarrow \mathrm{GL}_n$, we get a topology on $G(K) \subset \mathrm{GL}_n(K) \subset K^{n^2}$ by taking the induced topology of K^{n^2} on $G(K)$. We are then interested in bounded subgroups of $G(K)$ and especially in the maximal ones. Let us start with the following basic examples.

Lemma 1.2. *(1) If $G = \mathbb{G}_m^n$, then $(O^\times)^n$ is the unique maximal bounded subgroup of $G(K) = (K^\times)^n$.*

(2) If $\mathfrak{G} = \mathrm{GL}_n$, then $\mathrm{GL}_n(O)$ is the unique maximal bounded subgroup (up to conjugacy) of $\mathrm{GL}_n(K)$.

(3) If $\mathfrak{G} = \mathrm{SL}_n$, then $\mathrm{SL}_n(K)$ admits n maximal bounded subgroups up to conjugacy, namely the $g_i \mathrm{SL}_n(R) g_i^{-1}$ where $g_i = diag(\pi^i, 1, \cdots, 1)$ for $i = 0, ..., n-1$.

Proof. (1) The valuation induces an exact sequence $1 \to (O^\times)^n \to (K^\times)^n \to \mathbf{Z}^n \to 0$. Since a bounded subgroup of $(K^\times)^n$ maps to 0 in $\mathbf{Z}^n$, $(O^\times)^n$ is the unique maximal bounded subgroup of $(K^\times)^n$.

(2) Let $\Gamma \subset \mathrm{GL}_n(K)$ be a bounded subgroup. Consider the R–submodule M of K^n which is generated by $g.R^n$ for g running over Γ. Then M spans the K–vector space K^n and M is bounded, hence M is a lattice. So there exists $g \in \mathrm{GL}_n(K)$ such that $g(M) = R^n$, thus $\Gamma \subset g^{-1}\mathrm{GL}_n(R)g$.

(3) We leave this as an exercise to the reader.

□

The Bruhat-Tits (extended) building of $\mathrm{GL}_n(K)$ is the space of norms of Goldman-Iwahori. We recall the definition of an additive norm of $V = K^n$: it is a function $\alpha : V \to \mathbf{R} \cup \{\infty\}$ satisfying

[2]There is no need at this stage to assume that the residue field k is perfect.

[3]Of course, the topology is independent of that choice, see [38, app. III].

- $\alpha(x+y) \geq \mathrm{Inf}\{\alpha(x), \alpha(y)\}$ for all $x, y \in V$;
- $\alpha(\lambda\, x) = \omega(\lambda) + \alpha(x)$ for all $\lambda \in K$, $x \in V$;
- $\alpha(x) = \infty$ if and only if $x = 0$.

Note that $|x| = exp(-\alpha(x))$ is a ultrametric norm. We denote by $\mathcal{B}(\mathrm{GL}_{n,K})$ the space of additive norms. It is a reunion of "apartments" namely the $\mathcal{A}(e_1, ..., e_n) \cong \mathbf{R}^n$ for $e = (e_i)$ running over the basis of K^n consisting in the additive norms

$$\alpha_{e,c}\Big(\sum \lambda_i e_i\Big) = \mathrm{Inf}\Big\{\omega(\lambda_i) + c_i\Big\}$$

for $c = (c_i) \in \mathbf{R}^n$. Note that

$$\Big\{x \in K^n \mid \alpha_{e,0}(x) \geq 0\Big\} = O\, e_1 \oplus \cdots \oplus O\, e_n$$

is a lattice. Conversely, we have

$$\alpha_{e,0}(x) = \mathrm{Sup}\Big\{n \in \mathbf{Z} \mid x \in \pi^n(O\, e_1 \oplus \cdots \oplus O\, e_n)\Big\}.$$

These kind of additive norms correspond exactly to the maximal bounded subgroups of $\mathrm{GL}_n(K)$, we shall see later that those are the vertices for the simplical structure of $\mathcal{B}(\mathrm{GL}_{n,K})$.

Remark 1.3. This is the prototype of euclidean buildings. Note that two additive norms belong to a common apartment as noticed by A. Weil [39, §II.2]. In odd characteristic, this permits to define buildings of classical groups [12] [3] [24]. For a recent analytic viewpoint on Bruhat-Tits theory, see the recent preprint [27] of Rémy-Thuillier-Werner.

Remark 1.4. Except for the case $n = 2$, the Goldman-Iwahori metric is not the metric of the Bruhat-Tits building but defines the same topology.

2. Bruhat-Tits building of Chevalley groups

The plan is to try to explain how we can guess after Iwahori-Matsumoto what are the maximal bounded subgroups in the case of a split group. This permits to construct the Bruhat-Tits building and to show indeed that the guess was correct. In other words, the strategy is the same than for real groups.

2.1. **The standard apartment.** Let $G/\mathbf{Z}$ be an almost simple simply connected Chevalley group equipped with a pinning. Recall this is the following data.

- a maximal $\mathbf{Z}-$ split torus $T/\mathbf{Z}$ of G,
- an irreducible and reduced root system $\Phi = \Phi(T, G) \subset \widehat{T} \otimes_{\mathbf{Z}} \mathbf{R}$ (where $\widehat{T} = \mathrm{Hom}_{\mathbf{Z}-gr}(T, \mathbb{G}_{m,\mathbf{Z}})$ stands for the cocharacter group of T) equipped with a basis Δ which defines the set of positive roots Φ^+,
- A family of morphisms $(U_\alpha : \mathbb{G}_{a,\mathbf{Z}} \to G)_{\alpha \in \Phi}$ and a Borel subgroup B/K of G such that for each ordering $\Phi^+ = (\alpha_i)_{i=1,..,q}$, the product law on G

induces an isomorphism of **Z**–schemes

$$T \times \prod_{i=1,..,q} \mathbb{G}_a \xrightarrow{id\times \prod\limits_{i=1,..,q} U_{\alpha_i}} B.$$

We are interested in bounded groups of $G(K)$ which contain the maximal subgroup $T(K)_b$ of $T(K)$. We shall see later that the subgroup $G(O)$ is a maximal bounded subgroup of $G(K)$ which plays an important role. Since a building is done by apartments, let us define the standard apartment

$$\mathcal{A} = \phi + \widehat{T}^0 \otimes_{\mathbf{Z}} \mathbf{R}$$

This is an affine space which is defined by means of a given point ϕ and its underlying vector space $V = \widehat{T}^0 \otimes_{\mathbf{Z}} \mathbf{R}$, the coroot system vector space.

An affine coroot $a = (\alpha, n)$ with $\alpha \in \Phi(G,T)$ and $n \in \mathbf{Z}$ is the affine function

$$V \to \mathbf{R}, \;\; v \mapsto a(v) = \langle \alpha, v\rangle + n.$$

We denote by α_0 the opposite of the highest root of Φ, which reads as follows

$$\alpha_0 + \sum_{\alpha\in\Delta} c_\alpha \alpha = 0.$$

Then the set of affine roots

$$\Delta_a = \{(\alpha,0)\}_{\alpha\in\Delta} \cup \{(\alpha_0,1)\},$$

stands for the set of vertices of the extended Dynkin diagram of Δ.

2.2. **Certain bounded subgroups.** For a subset $\Omega \subset \mathcal{A}$, define the integer

$$n_\Omega(\alpha) = \mathrm{Sup}\Big\{ [(\alpha, v)], \; v \in \Omega \Big\}$$

for each $\alpha \in \Phi$. We consider the subgroup P_Ω of $G(K)$ which is generated by

$$T(K)_b, U_\alpha(\pi^{n_\Omega(\alpha)} O) \quad (\alpha \in \Phi).$$

It is easy to check that P_Ω is bounded[4]. Note that $P_\phi = G(O)$.

2.3. **The Tits system.** Since $T(K) = \widehat{T}^0 \otimes_{\mathbf{Z}} K^\times$, the valuation $\omega : K^\times \to \mathbf{Z}$ induces a map $\nu : T(K) \to \widehat{T}^0$.

Define $N = N_G(T)$ and $W = N_G(T)/T$.

The affine space $\mathcal{A}$ is equipped with an action of $N(K) = T(K).W$ by

$$n_w.(\phi + v) = \phi + \, w.\, v,$$

for $n_w \in N(\mathbf{Z})$ a lift of $w \in W$ and

$$\tau.(\phi + v) = \phi + v - \nu(g) \;\; (g \in T(K)).$$

The kernel of this action is $T(O)$ and we define the affine Weyl group

$$W_a := N(K)/T(O) \xrightarrow{\sim} \widehat{T}^0 \rtimes W.$$

[4]A nice way to ckeck that is to notice that P_Ω stabilizes the lattice

$$\mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \pi^{n_\Omega(\alpha)} \mathfrak{u}_\alpha$$

inside the Lie algebra of G. Note that we deals with a Lie algebra over O, this is the first appearance of the algebraic structure linked to P_Ω.

We define the chamber

$$C = \Big\{ \ \phi + v \in \mathfrak{A} \ \mid \ \langle \alpha, v \rangle > 0 \ \forall \alpha \in \Delta \ \ \langle \alpha_0, v \rangle + 1 < 0 \Big\}.$$

The closure of the chamber is a simplex which is a simplicial fundamental domain for the action of the group W_a on $\mathcal{A}$. To a point c of C, we can attach its underlying facet $F_c \subset \overline{C}$, it is defined as the interior of the smallest facet of $\overline{C}$ which contains C. By transport of structure, this permits to define the facet F_x attached to an arbitrary point $x \in \mathcal{A}$.

Let $(\alpha^*)_{\alpha \in \Delta}$ be the dual basis Δ of $\widehat{T} \otimes_{\mathbf{Z}} \mathbf{R}$. The extremal points of C are $\theta_{\alpha_0} = 0$ and the $\theta_\alpha = \frac{\alpha^*}{c_\alpha}$ for α running over Δ, so are given by Δ_a.

Definition 2.1. *The type of a facet F_c is the set of the extremal points of $\overline{F}_c$. This defines the type of an arbitrary facet of $\mathcal{A}$.*

The expected maximal bounded subgroups of $G(K)$ are the subgroups P_{θ_α} for $\alpha \in \Delta_e$.

The space V comes equipped with a scalar product which is W–invariant.

Consider the set $S = (r_\alpha)_{\alpha \in \Delta_a}$ of orthogonal reflexions of $\mathcal{A}$ with respect to the walls $\phi + \ker(\alpha)$ for $\alpha \in \Delta$ and $\phi + \alpha_0^{-1}(-1)$.

The subgroup P_C will play the role of the Borel subgroup in the classical theory; it is called an Iwahori subgroup and is nothing but

$$P_C = \Big\{ \, g \in G(O) \ \mid \ \overline{g} \in B(k) \, \Big\}$$

where $\overline{g}$ stands for the image of $g \in G(k)$. We have $N(K) \cap P_C = T(O)$ and $N(K)/T(O) = W_a$.

Theorem 2.2. *The quadruple $(G(K), P_C, N(K), S)$ is a Tits system, namely satisfies the following rules:*

(T1) The set $P_C \cup N(K)$ generates $G(K)$ and $P_C \cap N(K) = T(O)$ is a normal subgroup of $N(K)$.

(T2) The set S generates W_a and consists of elements of order 2.

(T3) For each $s \in S$ and for each $w \in W_a$, we have

$$s \, P_C \, w \subset P_C \, w \, P_C \ \cup \ P_C \, sw \, P_C$$

(T4) For each $s \in S$, we have $s \, P_C \, s \neq P_C$.

The only serious point is (3). There are several consequences of that result by taking into account the theory of Tits systems.

(1) The Bruhat decomposition :

$$G(K) = \bigsqcup_{w \in W_a} P_C \, w \, P_C \, .$$

(2) Parahoric subgroups : if $X \subset S$, define $W_{a,X}$ as the (finite) subgroup of W_a which is generated by X. Then $P_X \, W_{a,X} \, P_C$ is a bounded subgroup of $G(K)$. Furthermore, the parahoric subgroups P_X and $P_{X'}$ are conjugated under $G(K)$ if and only if $X = X'$.

If $x \in \overline{C}$, we know that the fixator $W_{a,x}$ is generated by $W_{a,x} \cap S$. Hence $P_x = P_C \, W_{a,x} \, P_C$ is a subgroup of $G(K)$, and is actually nothing but the subgroup P_x defined before.

Remark 2.3. The reason why the theory is slightly simpler in the semisimple simply connected case is that the action of W_a on $\mathfrak{A}$ preserves the type and furthermore that the fixators and the stabilizers are the same. The same properties occur for the action of $G(K)$ on $\mathcal{B}$.

2.4. **The building.** We define the Bruhat-Tits building $\mathcal{B} = \mathcal{B}(G_K)$ by the standard procedure

$$\mathcal{B} = \big(G(K) \times \overline{C} \big) / \sim$$

where $(g, x) \sim (g', x')$ if $x = x'$ and $g^{-1} \, g' \in P_x$. The group $G(K)$ acts on $\mathcal{B}$ by

$$g.(h.x) = (gh, x).$$

The building is a simplicial space and by construction $1 \times \overline{C}$ is a simplicial fundamental domain for the action of $G(K)$ on $\mathcal{B}$.

We have

$$\mathcal{A} = (W_a \times \overline{C}) / \sim$$

where $(w, x) \sim (w', x')$ if $x = x'$ and $g^{-1} \, g' \in W_{a,x}$. So we can embed $\mathcal{A}$ in $\mathcal{B}$ by

$$j([(w, x]) = [(n_w, x)]$$

where $n_w \in N(K)$ stands for an arbitrary lifting of w in $N(K)$. We see then $\mathcal{A}$ inside $\mathcal{B}$ and the apartments of $\mathcal{B}$ are then by definition the $g \,.\, \mathcal{A}$ for g running over $G(K)$. This also permits to define the facet of a given point of $\mathcal{B}$ and its type. We review two crucial facts about apartments.

Lemma 2.4. *There is a one-to-one correspondence between the apartments of $\mathcal{B}$ and the maximal K-split tori of G_K.*

Proof. Since maximal K–tori are conjugated, both sets are homogeneous under $G(K)$. The stabilizer of $\mathcal{A}$ in $T(K)$ is $N_G(T)(K) = N(K)$, hence the statement. □

Lemma 2.5. *Two points of $\mathcal{B}$ belong to a common apartment.*

Proof. It is enough to show that for each chamber C' of $\mathcal{B}$, there is an apartment which contains C and C'. We have $C' = g.C$. Using Bruhat decomposition, we have $C' = pn \, C$ with $n \in N(K)$ and $p \in P_C$. Hence $C' = p \, n \, p^{-1} \, p \, C$, thus C and C' are both inside the apartment $p.\mathcal{A}$. □

2.5. **The metric.** The standard apartment is equipped with an euclidean metric, so each apartment $\mathcal{A}'$ of $\mathcal{B}$ is equipped by transport of structure of an euclidean metric $d_{\mathcal{A}'}$. Given two points $x, y \in \mathcal{B}$, a subtle point[5] is to check that $d_{\mathcal{A}'}(x, y)$ is the same for each apartment $\mathcal{A}'$ containing x and y. This permits to define the distance function $d : \mathcal{B} \times \mathcal{B} \to \mathbf{R}_+$.

By the same kind of arguments, it is indeed a distance which makes $\mathcal{B}$ as a complete geodesic space. By construction, $G(K)$ acts isometrically on $\mathcal{B}$.

[5] Involving the retraction to an apartment [8, §2.3].

Furthermore the equality $d(x,z) = d(x,y) + d(y,z)$ implies that x, y, z belong to a common apartment. This is related to the negative curvature property called $CAT(0)$ (see [2, §11]). It means that given three points x, y, z of $\mathcal{B}$, for any $p \in [x,y]$, the distance $d(z,p)$ is lower that for a triangle of the euclidean space with same lengths.

This negative curvature permits to define the circumcenter $c(Y)$ of a bounded subset $Y \subset \mathcal{B}$. This is the unique point of $\mathcal{B}$ where the radius function

$$r(x,Y) := \mathrm{Sup}\Big\{ d(x,y) \mid y \in Y \Big\}$$

takes its minimal value. In other words, $c(Y)$ is the center of the smallest closed ball which contains Y.

Theorem 2.6. *(Bruhat-Tits fixed point theorem) Let Γ be a group acting isometrally on $\mathcal{B}$ such that it stabilizes a non-empty bounded subset of $\mathcal{B}$. Then $\mathcal{B}^\Gamma \neq \emptyset$.*

The fixed point is the circumcenter of the given non-empty bounded subset Y stabilized by G.

2.6. **The maximal bounded subgroups.**

Theorem 2.7. *There are* $\mathrm{rank}(G) + 1$ *$G(K)$-conjugacy classes of maximal bounded subgroups of $G(K)$, namely the parahoric subgroups attached to the extremal points of $\overline{C}$.*

In particular $G(O) = P_\phi$ is a maximal bounded subgroup of $G(K)$.

Proof. By the second consequence of Theorem 2.2, this is enough to show that a given bounded subgroup Γ of $G(K)$ belongs to some P_x, i.e. that Γ has a fixed point on $\mathcal{B}$. But the orbit $\Gamma.\phi$ is bounded, hence the fixed point theorem applies and shows that $\mathcal{B}^\Gamma \neq \emptyset$.

□

2.7. **Functoriality.** The functoriality with respect of groups and field extensions is a complicate topic of Bruhat-Tits theory, see [8, §9.1.19] [28] [21], [26]. What we use in the lectures are the two following easy facts which follow of the construction.

(1) If K'/K is an unramified extension of henselian DVR, we have a natural embedding (and metric) $\mathcal{B}(G_K) \to \mathcal{B}(G_{K'})$.

(2) Let K'/K be a finite Galois extension, then $G(K') \rtimes \mathcal{G}al(K'/K)$ acts on $\mathcal{B}(G_{K'})$.

3. Models

If X/K is an affine scheme, a model is a flat affine O-scheme such $\mathfrak{X}$ such that $\mathbf{X} \times_O K \cong X$. If X/K is an algebraic group, we require that $\mathfrak{X}/O$ is a flat group scheme. For constructing models, it is convenient to assume that O is strictly henselian, i.e. the residue field k is separably closed[6]. We assume this until the end of the section.

[6] In practice, we have then to replace O by its strict henselization O^{sh} and go down by Galois descent for defining the wished group scheme over O.

Theorem 3.1. *Let $\Omega \subset \mathcal{A}$ be a non-empty subset. Then there exists a unique smooth model $\mathfrak{P}_\Omega/O$ of G/K such that $\mathfrak{P}_\Omega(O) = P_\Omega$. Furthermore*

$$O[\mathfrak{P}_\Omega] = \Big\{ f \in K[G] \mid f(P_\Omega) \subset O \Big\}$$

and $\mathfrak{P}_\Omega/O$ is connected.

The group scheme $\mathfrak{P}_\Omega$ is called the canonical Bruhat-Tits smooth model attached to P_Ω. Its unicity is guaranteed by the following general fact.

Lemma 3.2. *([9, §1.7]) Let $\mathfrak{X}/O$ be a smooth scheme of generic fiber X. Then*

$$O[\mathfrak{X}] = \Big\{ f \in K[\mathfrak{X}] \mid f(P_\Omega) \subset O \Big\}.$$

The hard thing is then the existence of an integral model. There are three different constructions, the original construction by Bruhat-Tits [9], the application of Artin-Weil's theorem (see [13, §5]) or Yu's construction [40]. We sketch it in the case of a maximal parahoric subgroup P_x.

Proof. The idea is to define the O–group scheme $\mathfrak{G}$ by

$$O[\mathfrak{G}] = \Big\{ f \in K[G] \mid f(P_x) \subset O \Big\}.$$

It is a flat Hopf algebra over O, so defines indeed a group scheme $\mathfrak{G}/O$ which is a model of G/K. By construction we have $P_x \subset \mathfrak{G}(O)$. Since $\mathfrak{G}$ is a closed subgroup of some $\mathrm{GL}_{n,O}$ [9, §1.4.5], $\mathfrak{G}(O)$ is a bounded subgroup of $G(K)$, hence $P_x = \mathfrak{G}(O)$.

If k is of characteristic zero, $\mathfrak{G}$ is smooth by Cartier theorem and we are done. In positive characteristic, Raynaud's smoothening theorem [13, §3.1] provides a smooth affine model $\tilde{\mathfrak{G}}/O$ such that $\tilde{\mathfrak{G}}(O) = \mathfrak{G}(O)$. □

4. Soulé's theorem

We assume now that $K = k((\frac{1}{t}))$ and we consider the action of the group $\Gamma = G(k[t])$ on $\mathcal{B}(G_K)$. Define $\widehat{T}^0_+ = \big\{ \lambda \in \widehat{T}^0_+ \mid \langle \lambda, \alpha \rangle \geq 0 \;\; \forall \alpha \in \Delta \big\}$.

Theorem 4.1. *The "quartier" $\mathcal{Q} = \phi + \widehat{T}^0_+ \otimes_{\mathbf{Z}} \mathbf{R}$ is a simplicial fundamental domain for the action on Γ on $\mathcal{B}(G_K)$.*

For the SL_2 case, see [30, II.1.6]. For the proof see the original paper [32] or its generalization by Margaux [22]. Since $\mathcal{B}(G_K)$ is connected and simply connected, it follows that Γ is the direct limit of the stabilizers $(\Gamma_x)_{x \in \mathcal{Q}}$ with respect to their intersections. This can be refined as the direct limit of $(\Gamma_I)_{I \subset \Delta}$ with respect to their intersections with

$$\Gamma_I = U_I(k[t]) \rtimes L_I(k)$$

where $P_I = U_I \rtimes L_I$ stands for the standard parabolic subgroup of type I.

Remark 4.2. Using the theory of twin buildings, Abramenko showed an analogous result for the action of $G(k[t, t^{-1}])$ on $\mathcal{B}(G_{k((t))}) \times \mathcal{B}(G_{k((\frac{1}{t}))})$ [1, prop. 5]. This result actually covers Soulé 's theorem.

5. Application to Galois cohomology

6. The non-split case

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