Examples of Non-rational Varieties of Adjoint Groups

Philippe Gille

Mathématiques, URA-CNRS 752, Bât. 425, Université de Paris-Sud, 91405, Orsay, France

Communicated by Michel Van den Bergh

Received September 18, 1996

Let $k$ be a field of characteristic $\neq 2$ and $k_0$ a separable closure of $k$. We say that an algebraic variety $X/k$ is stably $k$-rational if there exist two affine spaces $\mathbb{A}^m_k, \mathbb{A}^n_k$ and a $k$-rational map $\mathbb{A}^m_k \times X \rightarrow \mathbb{A}^n_k$. Merkurjev [9] gave a criterion of stable $k$-rationality for the adjoint classical groups with absolute rank $\leq 3$, which covers the case of the variety $\text{PSO}(q)$ for any quadratic form $q/k$ of rank $\leq 6$. This criterion gives examples of field $k$ and quadratic form $q$ of rank 6 with non-trivial signed discriminant such that the variety $\text{PSO}(q)$ is not stably $k$-rational. The main result of this paper is the following:

**Theorem.** There exist a field $k$ of characteristic 0 with cohomological dimension 3 and a quadratic form $q/k$ with rank 8 and trivial signed discriminant such that the variety $\text{PSO}(q)$ is not stably $k$-rational.

This is the first example of the quadratic form with trivial signed discriminant such that the variety $\text{PSO}(q)$ is not stably $k$-rational and since [9], the 8-dimension is minimal. This example is an adjoint group which is an inner form of the split adjoint group of type $D_4$ [22] and it is the first example of an adjoint semisimple group which is an inner form and which is not a stably $k$-rational variety. In Section 3, we give another proof of the theorem with $\text{cd}(k) = 6$ which is more elementary because we don't use the Index Reduction Theory.

I thank J.-P. Tignol for answering my question about multiquadratic extensions (cf. Proposition 3) and the referee for pointing out a mistake in the first version of the paper.

**Notations.** We denote by $G_m = \text{Spec}(\mathbb{Z}[t, 1/t])$, $\mathbb{A}^n = \text{Spec}(\mathbb{Z}[t_1, t_2, \ldots, t_n])$ and for any scheme $X$, we denote by $G_m, X = G_m \times_{\text{Spec}(\mathbb{Z})} X$ and $\mathbb{A}^n_X = \mathbb{A}^n \times_{\text{Spec}(\mathbb{Z})} X$ the affine space of rank $n$ on $X$ ($n \in \mathbb{N}$). Let
$X' \to X$ be a finite locally free morphism of schemes. We can write [4] the
exact sequence of $X$-tori $1 \to R_{X'/X}^1 G_m \to R_{X'/X} G_m \to G_{m,X} \to 1$
where $R_{X'/X}^1 G_m$ is the restriction from $X'$ to $X$ of the $X'$-torus $G_{m,X}$.

Let $X$ be a $k$-variety geometrically irreducible. We say that $X$ is a
$k$-rational variety if there exist an affine space $\mathbb{A}^n_k$ and a $k$-birational map
$X \cong \mathbb{A}^n_k$. We say that $X$ is a stably rational $k$-variety if there exist two
affine spaces $\mathbb{A}^m_k, \mathbb{A}^n_k$ and a $k$-birational map $\mathbb{A}^m_k \times X \cong \mathbb{A}^n_k$. One defines
the norm group of $X$ which is denoted $N_X(k)$ as the subgroup of $k^\times$
generated by the $N_L(k)$ for any finite field extension $L/k$ such that
$X(L)$ is not empty.

If $A/k$ is a central simple algebra, there exists a division algebra $T/k$
and an integer $r$ (Wedderburn’s theorem) such that $A \cong M_r(T)$ and
the integer $r$ and $T$ are well defined. Then we denote $\text{ind}_{k}(D) = \sqrt{\dim_{k}(T)} \in \mathbb{Z}$
and $\deg(A) = \sqrt{\dim_{k}(A)} \in \mathbb{Z}$. If $A/k$, $B/k$ are two
central simple algebras, we say that $A$ and $B$ are similar and we denote
$A \sim B$ if there exist some integers $m$, $n$ such that
$M_m(A) \cong M_n(B)$. If $a, b \in k^\times$, we denote by $(a, b)_k$
the standard quaternion algebra. We assume that all quadratic forms will be regular. If $q/k$, $q'/k$ are two
quadratic forms, we denote by $q \perp q'$ their orthogonal sum, by $q \otimes q'$
their tensor product, and by $\text{rk}(q)$ the rank of $q$. We denote by $C(q)$ the
Clifford algebra of $q$ and by $C_0(q)$ the even Clifford algebra of $q$. We
denote by $W(k)$ the Witt ring of the field $k$, by $I(k)$ the fundamental ideal
generated by forms with even rank, and by $\text{disc} : I(k) \to k^\times/k^\times^2$
the morphism of signed discriminant. We will identify often a quadratic form $q$
and its class $[q] \in W(k)$. If $q$ is a $k$-quadratic form and $E/k$ a field
extension, we denote by $q_E$ the quadratic form extended to $E$.

If $(a_1, \ldots, a_n)$ is a family of elements of $k^\times$, we denote by $\langle a_1, a_2, \ldots, a_n \rangle$
the quadratic form $\sum_{i=1}^{n} a_i X_i^2$ and by $\langle \langle a_1, a_2, \ldots, a_n \rangle \rangle$
the $n$-fold Pfister form $\langle 1, a_1 \rangle \otimes \langle 1, a_2 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$. We denote by $\mathbb{H} = \langle 1, -1 \rangle$
the standard hyperbolic form of rank 2.

We recall that a central simple algebra $D/k$ is a $k$-biquaternionic algebra
if $D/k$ is isomorphic to an algebra $(a, b) \otimes_k (c, d)$ with $a, b, c, d \in k^\times$ [7]. We can associate to this isomorphism the Albert form $\langle a, b, -ab, -c, -d, cd \rangle$. Although this Albert form $\langle a, b, -ab, -c, -d, cd \rangle$ is not canonical, its similarity class is well defined and depends only on $D$. We said that a quadratic form $\psi$ is an Albert form for $D$ if $\psi$ is similar to
$\langle a, b, -ab, -c, -d, cd \rangle$. We recall that $D$ is a division algebra iff the
form $\langle a, b, -ab, -c, -d, cd \rangle$ is anisotropic, and that a $k$-form $\varphi$
with rank 6 and a trivial signed discriminant is an Albert form for some central
simple algebra which is similar to $C(\varphi)$.

If $q$ is a quadratic form with even rank, we denote by $\text{SO}(q)$ (resp.
$\text{PSO}(q)$) the special orthogonal group of $q$ (resp. projective special ortho-
By $G(q)$ the group of similarity factors of $q$, i.e., $G(q) = \{ \alpha \in k^* \mid \alpha q \equiv q \}$. It is well known that $G(1, -a) = N_{k(\sqrt{a})/k}(k(\sqrt{a})^*)$. If $a \in k^*$, we will denote sometimes $N_q(a) = N_{k(\sqrt{a})/k}(k(\sqrt{a})^*)$.

We denote by $cd(k)$ the cohomological dimension of a field $k$ [19] and by $u(k)$ the $u$-invariant of $k$, i.e., the supremum in $\mathbb{N} \cup \{\infty\}$ of the dimensions of anisotropic $k$-quadratic forms. If $P \subset k^*$ is a subset of $k^*$, we denote by $\mathbb{Z}\langle P \rangle$ the subgroup of $k^*$ generated by $P$.

1. PRELIMINARIES

1.1. Norm Groups and $R$-Equivalence [9, 10]

For any quadratic space $(q, V)$ of even rank $n$, we denote by $\text{hyp}(q)$ the subgroup of $k^*$ generated by the $N_{L/k}(L^*)$ for any finite field extension $L/k$ such that $q_L$ is hyperbolic. This condition can be written in another way. Indeed, let $X_q$ be the variety of totally isotropic subspaces of $V$ with dimension $n/2$. It is known that $X_q$ is a $k$-projective smooth variety which has a $k$-rational point iff $q = 0 \in W(k)$, i.e., $q$ is an hyperbolic form. Then we have $\text{hyp}(q) = N_{X_q}(k)$. This invariant is connected with the study of $R$-equivalence on the group $\text{PSO}(q)$. Recall the definition of $R$-equivalence.

Let $G/k$ be a connected linear algebraic group. We recall that two rational points $g_0, g_1 \in G(k)$ are directly $R$-equivalent if there exists $g(t) \in G(k(t))$ such that $g(0) = g_0$ and $g(1) = g_1$ and that the $R$-equivalence is the equivalence relation generated by this elementary relation. It is known [3] that the group $G(k)/R$ is trivial if the variety of the group $G/k$ is stably $k$-rational. Merkurjev gave a formula which computes $G(k)/R$ for the adjoint classical groups. In the case of a group $\text{PSO}(q)$, we have

$$\text{PSO}(q)(k)/R \to G(q)/\text{hyp}(q).k^{\times^2}.$$ 

Moreover, the invariant $G(k)/R$ (on suitable extensions of $k$) allows us to determine if an adjoint semisimple classical group with absolute rank $\leq 3$ is (or is not) a stably $k$-rational variety. More precisely, in the case of $\text{PSO}(q)$ with a quadratic form $q$ of rank $\leq 6$. Merkurjev's criterion is the following:

**Theorem 1** [9]. Let $q/k$ be a quadratic form with rank $2m$ ($m = 2$ or $3$) and signed discriminant $(d) \in k^*/k^{\times^2}$.

(a) If $d \in k^{\times^2}$, the variety $\text{PSO}(q)$ is $k$-rational and one has $G(q)/\text{hyp}(q).k^{\times^2} = 1$. 


If \( d \neq k^{\times 2} \), we denote by \( L = k(\sqrt{d}) \) and \( C_q(q) \) the even Clifford algebra of \( q \) which is a central simple algebra over \( L \). One has the following alternative:

(i) If \( \text{ind}_k(C_q(q)) = 1 \) or 2, then the variety \( \text{PSO}(q) \) is stably \( k \)-rational and \( G(q) / \text{hyp}(q) . k^{\times 2} = 1 \).

(ii) If \( \text{Ind}_k(C_q(q)) = 4 \), then there exists a field extension \( E / k \) such that \( G(q_E) / \text{hyp}(q_E) . E^{\times 2} \neq 1 \) and the variety \( \text{PSO}(q) \) is not stably \( k \)-rational.

Case (ii) can appear only if \( \text{rk}(q) = 6 \). The proof of the theorem uses in a crucial way the Index Reduction theory (cf. [11, 18, 21]).

Remark 1. If \( k \) is a field (\( \text{car}(k) \neq 2 \)) with cohomological dimension 1, it is well known that any group \( \text{PSO}(q) \) is a quasi-split group and a \( k \)-rational variety. For illustrating case (ii) of the theorem, it is necessary to assume \( \text{cd}(k) \geq 2 \). We will show that \( \text{cd}(k) = 2 \) is sufficient.

The construction by Merkurjev [12] for any integer \( n \) \((n \geq 2)\) of a field with \( u \)-invariants (cf. Notations) equal to \( 2n \) from a division algebra \( D / k \) is functorial in \( k \). More precisely, if \( D / k \) is isomorphic to \( Q_1 \otimes_k Q_2 \cdots \otimes_k Q_{n-1} \) where the \( Q_i \)'s are quaternion algebras, one associates a field \( F(k, D) \) with cohomological dimension 2 satisfying \( \text{ind}(D_{F(k, D)}) = 2^{n-1} \) and \( u(F(k, D)) = 2n \). Moreover, if \( k' / k \) is a field extension satisfying \( \text{ind}(D_{k'}) = \text{ind}(D_k) \), one has a natural embedding \( F(k, D) \hookrightarrow F(k', D_k) \).

Let us apply this remark. We fix a field \( k \) of characteristic zero, \( D/k \) a division algebra which is a tensor product of 2 quaternion algebras, and a proper quadratic field extension \( k' = k(\sqrt{d}) \) satisfying \( \text{ind}(D_k) = \text{ind}(D_{k'}) = 4 \). For example, we can take \( k = \mathbb{Q}(X_1, X_2, \ldots, X_{2n-1}) \), \( Q_i = (X_{2i}, X_{2i+1})_k \) for \( i = 1, \ldots, n-1 \) and \( k' = k(\sqrt{X_1}) \). Then we denote \( F = F(k, D) \) and \( F' = F(k', D_k) \). One has a natural embedding \( F \hookrightarrow F' \) and since \( \text{ind}(D_F) = 4 \), one has \( \text{ind}(D_{F'(\sqrt{d})}) = 4 \). Denote \( L = F(\sqrt{d}) \). Let us fix an Albert form \( \psi \) for \( D \) which represents \(-1\) and let us define the \( k \)-form \( q \) with rank 6 and signed discriminant \( d \) by \( \langle 1, -d \rangle \perp \psi = q \perp H \). Then \( C_q(L) \sim D_L \), \( \text{cd}(F) = 2 \), and \( q_F \) is an example of the quadratic form of case (ii) such that the variety \( \text{PSO}(q) \) is not stably \( F \)-rational.

1.2. Norm Group of a Family of Quadratic Forms

For any family of quadratic forms \( \langle q_i \rangle_{i=1}^m \) with even rank, we denote by \( \text{hyp}(q_1, q_2, \ldots, q_m) \) the subgroup of \( k^{\times} \) generated by the \( N_{L/k}(L^{\times}) \) such that the forms \( q_i|L \) are hyperbolic \((i = 1, \ldots, m)\). Let \( X_i \) be the variety of totally isotropic subspaces of \( q_i \) with dimension \( \text{dim}(q_i)/2 \). Then
by definition, we have
\[ \text{hyp}(q_1, q_2, \ldots, q_m) = N_{X_1 \times X_2 \times \cdots \times X_m}(k) \subset k^\times. \]

**Lemma 1.** Let \((q_i/k)_{i=1, \ldots, m}\) be a family of quadratic forms with even rank and \(q/k\) a quadratic form with even rank.

(a) \(G(q \perp H) = G(q)\).

(a') \(\text{hyp}(q_1, q_2, \ldots, q_m, q \perp H) = \text{hyp}(q_1, q_2, \ldots, q_m)\).

(b) \(\text{hyp}(q_1, q_2, \ldots, q_m) \subset \bigcap_{i=1, \ldots, m} \text{hyp}(q_i)\).

(c) If \(L/k\) is a finite field extension, one has
\[ N_{L/k}(\text{hyp}(q_1, q_2, \ldots, q_m, L)) \subset \text{hyp}(q_1, q_2, \ldots, q_m). \]

(d) Let \(L/k\) be a finite splitting field extension for the forms \((q_i)_{i=1, \ldots, m}\). Then
\[ N_{L/k}(\text{hyp}(q_L)) \subset \text{hyp}(q_1, q_2, \ldots, q_m, q). \]

(e) (respectively [5, 9]). Denote \((d) = \text{disc}_\pm(q) \in k^\times/k^\times_2\). Then
\[ G(q) = G(\langle 1, -d \rangle) \cap G(q \perp \langle 1, -d \rangle) \]
and
\[ \text{hyp}(q_1, q_2, \ldots, q_m, q) = \text{hyp}(q_1, q_2, \ldots, q_m, (1, -d), q \perp \langle 1, -d \rangle) = N_{k(\sqrt{d})/k}(\text{hyp}(q_1, q_2, \ldots, q_m, k(\sqrt{d}), q_{k(\sqrt{d})})). \]

(f) Let \(\tilde{G}\) be a subgroup of the profinite Galois group \(Gal(k_\sigma/k)\) and \(\tilde{k} = k_\sigma^G\). Then
\[ \text{hyp}(q_{1, \tilde{k}}, q_{2, \tilde{k}}, \ldots, q_{m, \tilde{k}}) = \bigcup_{k' \subset \tilde{k}} \text{hyp}(q_{1, k'}, q_{2, k'}, \ldots, q_{m, k'}), \]
where the union is taken on the extensions \(k' \subset \tilde{k}\) of finite degree over \(k\).

**Remark 2.** The main result of this paper is based on examples of quadratic forms for which the inclusion (b) is strict. For (e), Merkurjev’s Theorem 1 shows that the inclusion \(N_{k(\sqrt{d})/k}(G(q_{k(\sqrt{d})})) \subset G(q)\) is strict in general. For a quadratic form with rank 6 and signed discriminant \(d\), one has indeed \(G(q)/\text{hyp}(q)k^\times_2 = G(q)/N_{k(\sqrt{d})/k}(\text{hyp}(q_{k(\sqrt{d})}))k^\times_2 = G(q)/N_{k(\sqrt{d})/k}(G(q_{k(\sqrt{d})}))k^\times_2\) and this group is not trivial in general.
Proof. The assertion (a) is a straightforward result of Witt’s theorem. The assertions (b), (c), and (d) are direct consequences of the definition and of the functoriality of norm maps for a tower of field extensions. Let us show the assertion (e). It is clear that we can assume \( d \in k^\times \setminus k^{\times 2} \).

First, the inclusion \( G(\langle 1, -d \rangle) \cap G(q \downarrow \langle 1, -d \rangle) \subseteq G(q) \) is obvious. Conversely, if \( a \in G(q) \), one has \( \langle 1, -a \rangle \otimes q = 0 \in W(k) \) and since \( q = \langle 1, -d \rangle \mod I^2(k) \), one has \( \langle 1, -a \rangle \otimes \langle 1, -d \rangle = 0 \mod I^3(k) \) and it is known [17, p. 88, Theorem 14.3] that \( \langle 1, -a \rangle \otimes \langle 1, -d \rangle = 0 \in W(k) \).

Hence \( \langle 1, -a \rangle \otimes (q \downarrow \langle 1, -d \rangle) = 0 \in W(k) \) and \( a \in G(\langle 1, -d \rangle) \cap G(q \downarrow \langle 1, -d \rangle) \).

The second formula of (e) is simpler and results from the following fact: any field extension \( L/k \) such that \( q_L \) is hyperbolic satisfies \( d \in L^{\times 2} \) and then contains a subfield isomorphic to \( k(\sqrt{d}) \).

(f) This identity is formal. There exists a variety \( X/k \) such that \( \text{hyp}(q_{k,1}, q_{2,k}, \ldots, q_{m,k}) = N_X(k) \) and it is not difficult to show that \( N_Y(k) = \bigcup_{k < m} N_Y(k') \) for any variety \( Y/k \). Then one has the formula. \( \blacksquare \)

Let us give an application of Scharlau’s transfer map [17, Sect. 5] which will be useful for showing Proposition 1.

Lemma 2. Let \( q, q' \) be \( k \)-quadratic forms and \( k' = k(x)/k \) a finite field extension with degree \( [k':k] \). Assume that \( q = \langle 1, x \rangle \otimes q' \in W(k') \).

(a) If \( [k':k] \) is even, then one has \( \langle 1, -N_{k'/k}(x) \rangle \otimes q = \langle 1, -N_{k'/k}(x) \rangle \otimes q = 0 \in W(k) \), i.e., \( N_{k'/k}(x) \in G(q) \cap G(q') \).

(b) If \( [k':k] \) is odd, then one has \( q = \langle 1, N_{k'/k}(x) \rangle \otimes q' \in W(k) \).

Proof. Denote \( r = [k':k] \). In the two cases, we apply Scharlau’s transfer \( s_{a}: W(k(x)) \to W(k) \) associated with the linear form \( s: k(x) \to k \) defined by \( s(1) = 1, s(x) = s(x^2) = \cdots = s(x^{r-1}) = 0 \). One has a projection formula \( s_{a}(\phi \otimes \psi) = \phi \otimes s_{a}(\psi) \) for any \( \phi \in W(k), \psi \in W(k') \) which reduces the calculation to \( s_{a}(\langle 1 \rangle) \) and \( s_{a}(\langle x \rangle) \).

(a) If \( r \) is even, one has \( s_{a}(\langle 1 \rangle) = \langle 1, -N_{k'/k}(x) \rangle \) and \( s_{a}(\langle x \rangle) = 0 \). Applying \( s_{a} \) to \( q \), one has \( \langle 1, -N_{k'/k}(x) \rangle \otimes q = \langle 1, -N_{k'/k}(x) \rangle \otimes q' \in W(k) \). Moreover, since \( \langle x \rangle \otimes q = \langle 1, x \rangle \otimes q' \in W(k') \), it follows \( \langle 1, N_{k'/k}(x) \rangle \otimes q' = 0 \in W(k) \) and \( \langle 1, -N_{k'/k}(x) \rangle \otimes q = 0 \in W(k) \).

(b) If \( r \) is odd, one has \( s_{a}(\langle 1 \rangle) = \langle 1 \rangle \) and \( s_{a}(\langle x \rangle) = \langle N_{k'/k}(x) \rangle \).

Applying \( s_{a} \) to \( q \), we obtain \( q = \langle 1, N_{k'/k}(x) \rangle \otimes q' \in W(k) \). \( \blacksquare \)

1.3. Milnor’s Residue Maps (cf. [17, p. 207])
We denote by $K = k((t))$ the field of formal series with valuation ring $O = k[[t]]$. Recall that there exists an exact sequence of groups

$$0 \to W(k) \overset{i}{\to} W(K) \overset{\partial_t}{\to} W(k) \to 0.$$  

The map $i$ is the restriction of $k$ to $K$ and let us describe the map $\partial_t$. A $K$-quadratic form $q$ can be diagonalized in $\langle u_1, \ldots, u_m, w_1, \ldots, w_n \rangle$ where $u_j, v_j \in O^\times$. Then $\partial_t(q) = \langle \overline{v}_1, \ldots, \overline{v}_n \rangle$ where $\overline{v}_j \in k^t = (O/t)^\times$. Let us give an application for similarity factors.

**Lemma 3.** Let $\gamma$ be a $k$-quadratic form.

(a) If $\gamma$ is not hyperbolic, then $G(\gamma_K) = G(\gamma)K^{\times 2}$.

(b) One has $G(\langle t \rangle \otimes \gamma) = \mathbb{Z}\langle t \rangle.G(\gamma)K^{\times 2}$.

**Proof.** (a) The inclusion $G(\gamma).K^{\times 2} \subset G(\gamma_K)$ is obvious. Conversely, let $x$ be in $G(\gamma_K)$. Then $x = t^{d^2}a^2\alpha$ with $a \in K^\times$, $\alpha \in k^\times$, and $d = 0$ or $1$. If $d = 1$, one has $0 = \partial(\langle 1, -x \rangle \otimes \gamma) = \langle -\alpha \rangle \otimes \gamma \in W(k)$ then $\gamma$ is hyperbolic and $d = 0$. Hence $\alpha \in G(\gamma_K) \cap k^\times$. It follows $0 = \langle 1, -\alpha \rangle \otimes \gamma_K = i(\langle 1, -\alpha \rangle \otimes \gamma)$. Hence $0 = \langle 1, -\alpha \rangle \otimes \gamma \in W(k), \alpha \in G(\gamma)$, and $x \in G(\gamma).K^{\times 2}$.

(b) If the form $\gamma$ is hyperbolic, then the assertion is obvious. We can assume that $\gamma$ is not hyperbolic. The inclusion $\mathbb{Z}\langle t \rangle.G(\gamma).K^{\times 2} \subset G(\varphi_K)$ is obvious. Conversely, let $x$ be in $G(\varphi_K)$. Then $x = t^{d^2}a^2\alpha$ with $a \in O^\times$ and $\alpha \in G(\varphi_k) \cap k^\times$. Applying the residue map $\partial : W(K) \to W(k)$, it yields $0 = \partial(\langle 1, -\alpha \rangle \otimes \varphi) = \partial(\langle 1, -\alpha \rangle \otimes (\langle 1, t \rangle \otimes \gamma)) = \langle 1, -\alpha \rangle \otimes \gamma \in W(k)$. Hence $\alpha \in G(\gamma)$ and $x \in \mathbb{Z}\langle t \rangle.G(\gamma).K^{\times 2}$.

2. PROOF OF THE MAIN RESULT

The main result is a direct consequence of the following proposition and Merkurjev's Theorem 1.

**Proposition 1.** Let $k$ be a field of characteristic zero. Let $(q_i)_{i=1,\ldots,m}$ be a family of $k$-quadratic forms, $a \in k^\times \setminus k^{\times 2}$ and $\psi/k$ a quadratic form satisfying the following condition

(C) For any $b \in k^\times$, the form $\langle \langle -a, b \rangle \rangle \perp \psi$ is not hyperbolic.

We denote by $K = k((t))$ the field of formal series power with valuation ring $O = k[[t]]$ and

$q = \langle \langle -a, t \rangle \rangle \perp \psi$.

Then

$G(q_K) = (G(\langle 1, -a \rangle) \cap G(\psi))K^{\times 2}$.
and
\[ \text{hyp}((q_i, k), q_k).K^x = \text{hyp}((q_i), (1, -a), \psi).K^x. \]

**Proof.** First, we observe that the condition (C) implies that the form \( \psi \) is not hyperbolic.

1st Step. The **first equality.** The inclusion \((G((1, -a)) \cap G(\psi)).K^x \subset G(q_k)\) is obvious. Conversely, let \( x \in G(q_k) \). Then \( x = t \beta^2 b \) with \( \beta \in K^x \), \( d = 0 \) or \( 1 \), and \( b \in k^x \). If \( d = 1 \), applying the residue map \( \partial : W(K) \to W(k) \), one has \( 0 = \partial(\langle 1, -bt \rangle \otimes q) = \partial(\langle -bt, t, -a \rangle) \perp \perp \langle 1, -bt \rangle \otimes \psi = (\langle -b, -a \rangle) \perp \perp \langle -b \rangle \otimes \psi \in W(k) \). Since \( -b \in G(\langle -b, -a \rangle \perp \perp \psi) \), it yields \( \langle -b, -a \rangle \perp \perp \psi = 0 \in W(k) \), which is a contradiction for the hypothesis \( \text{(C)} \).

It follows that \( d = 0 \) and \( b \in G(q_k) \cap k^x \). Applying again the map \( \partial \), one can see easily that \( b \in G(\langle 1, -a \rangle) \) and since \( q = \langle \langle t \rangle \rangle \otimes \langle 1, -a \rangle \perp \perp \psi \), one has \( b \in G(\langle 1, -a \rangle) \cap G(\psi) \) and \( x \in G(\langle 1, -a \rangle) \cap G(\psi).K^x \).

2nd Step. Reduction to the case where the base field \( k \) has no proper odd extension. For the second equality, we will show that we can assume that the base field \( k \) has no proper odd extension. First, let us check that the condition \( \text{(C)} \) stays when we extend the scalars with an odd field extension. If \( k' / k \) is a finite odd extension and if there exists \( b' \in k'^x \) such that \( \langle -a, b' \rangle \perp \psi = 0 \in W(k') \), since \([k': k(b')]\) is odd, Springer's theorem for odd extensions [17, p. 62] yields \( \langle -a, b' \rangle \perp \psi = 0 \in W(k') \) and Lemma 2 implies \( \langle -a, N_{k(b')/k}(b') \rangle \perp \psi = 0 \in W(k) \), which is a contradiction for the hypothesis \( \text{(C)} \).

Let \( G \subset G(\text{al}(k_x/k)) \) be a 2-Sylow subgroup of the profinite Galois group \( G(\text{al}(k_x/k)) \), \( \bar{k} = k_x \bar{G} \), and \( \bar{k} = K \otimes_k \bar{k} \) and let us assume that
\[ \text{hyp}((q_i, \bar{k}), q_{\bar{k}}).\bar{k}^x = \text{hyp}((q_i, \bar{k}), (1, -a)_{\bar{k}}, \psi_{\bar{k}}).\bar{k}^x. \]

Due to Lemma 1(f), one has
\[ \text{hyp}((q_i, \bar{k}), (1, -a)_{\bar{k}}, \psi_{\bar{k}}).\bar{k}^x = \bigcup_{k' \subset \bar{k}} \text{hyp}((q_i, k'), (1, -a)_{k'}, \psi_{k'}).k_{\bar{k}}^x, \]

where the reunion is taken on the subextensions \( k' \subset \bar{k} \) finite over \( k \). Now, we can show the equality
\[ \text{hyp}((q_i, k), q_k).K^x = \text{hyp}((q_i), (1, -a), \psi).K^x, \]

where the inclusion \( \supset \) is obvious. For the inverse inclusion, let \( x \) be in
\[ \text{hyp}((q_i, K)_{i=1, \ldots, m}, q_K).K^{x^2}. \]

Since the inclusion
\[ \text{hyp}((q_i, K)_{i=1, \ldots, m}, q_K).K^{x^2} \subset \text{hyp}((q_i, \tilde{K}).K^{x^2}, \]

there exists a finite odd extension \( k'/k \) such that
\[ x \in \text{hyp}((q_i, k'), (1, -a), \psi_k).K^{x^2}. \]

Hensel’s lemma allows us to assume that \( x \in k^{x^2} \). If \( [k':k] = 2p + 1 \), one has \( \mathcal{N}_{k'/k}(x) = x^{2^{p-1}} \) and Lemma 1(c) yields \( x \in \text{Hyp}((q_i), (1, -a), \psi).K^{x^2}. \)

3rd Step. The Second Equality. We can assume that the field \( k \) has no proper odd extensions. The inclusion
\[ \text{hyp}((q_i), (1, -a), \psi).K^{x^2} \subset \text{hyp}((q_i, K)_{i=1, \ldots, m}, q_K).K^{x^2} \]

is obvious. For the inverse inclusion, we have to show for any finite extension \( L/K \) splitting \( q \) and the \( q_i \)'s that \( \mathcal{N}_{L/k}(L^{x^2}) \subset \text{hyp}((q_i), (1, -a), \psi).K^{x^2} \). Let \( L/K \) be such a finite extension with valuation ring \( O_L \), residue field \( k' \), ramification index \( e \), and residual index \( f \). Let us denote by \( K'/K \) the maximal non-ramified extension of \( K \) with valuation ring \( O' \). Since \( k \) has characteristic zero, the field \( K' \) is \( k \)-isomorphic to \( k'((i)) \). Therefore we can assume that \( K' = k'((i)) \).

\[
\begin{array}{c|c|c}
L & e & f \\
\hline
K' & & \\
K & & \\
\end{array}
\]

We recall that there exists an uniformizing parameter \( \pi \) of \( L/K \) such that \( \pi^{e-1} \in k' \). If \( \pi \) is an uniformizing parameter of \( L \), then \( \pi^{e-1} \) has valuation 1 and since \( O^x/K^{x^e} = k^{x^e}/k^{x^e} \), there exists \( a \in O^x \) such that \( (a, \pi)^{-1} \in k' \). Therefore we can take an uniformizing parameter \( \pi \) of \( L \) such that \( \pi^e = ut \) with \( u \in k' \). With Hensel’s lemma, we can compute easily the norm group \( \mathcal{N}_{L/k}(L^{x^2}) \) up to \( U_1 = \text{Ker}(O^x \to k^x) \), which is sufficient because one has \( U_1 \subset K^{x^2} \).

Lemma 4. \( \mathcal{N}_{L/k}(L^{x^2}) = \mathbb{Z}\left\langle \mathcal{N}_{k'/k}(1+u)^{e-1}(N_{k'/k}(k^{x^e}))^e \right\rangle \mod U_1. \)
In order to use the hypothesis \( q_L \) hyperbolic, we write the functoriality of Milnor's residue maps for the extensions \( K \subset K' \subset L \).

\[
\begin{array}{c}
0 \longrightarrow W(k) \overset{i}{\longrightarrow} W(K) \overset{\partial_i}{\longrightarrow} W(k) \longrightarrow 0 \\
\end{array}
\]

\[
\begin{array}{c}
0 \longrightarrow W(k') \overset{i}{\longrightarrow} W(K') \overset{\partial_i}{\longrightarrow} W(k') \longrightarrow 0 \\
0 \longrightarrow W(k') \overset{j}{\longrightarrow} W(L) \overset{\partial_j}{\longrightarrow} W(k') \longrightarrow 0,
\end{array}
\]

where \( \rho = 0 \) if \( e \) is even and \( \rho = \text{id}_{W(k')} \) if \( e \) is odd. Since \( L/K \) splits the \( q_i \)'s, the diagram shows that the \( q_i \)'s are hyperbolic forms.

(i) 1st Case. \( e \) Is Even. Lemma 4 shows that \( N_{L/K}(L^{\infty}) \subset \mathbb{Z}(N_{k'/k}(-u)\langle a \rangle)K^{\times 2} \). It is sufficient to show that \( f \) is even and that \( N_{k'/k}(-u) \in \text{hyp}(q_i, \langle 1, -a \rangle, \psi)k^{\times 2} \). One has \( q_L = \langle \langle u, -a \rangle \rangle \perp \psi = \langle \langle u^2, -a \rangle \rangle \perp \psi = j(\langle \langle u, -a \rangle \rangle) \perp \psi \). Then \( q_L = j(\langle \langle u, -a \rangle \rangle) \perp \psi \) and since \( q_L = 0 \in W(L) \), it follows

\[
0 = \langle \langle u, -a \rangle \rangle \perp \psi \in W(k').
\]

The hypothesis (C) implies that \( f = [k': k] = 2^2 > 1 \) and \( f \) is even. It remains to show that \( N_{k'/k}(-u) \in \text{hyp}(q_i, \langle 1, -a \rangle, \psi)k^{\times 2} \). If \( [k': k(u)] = 2^2 > 1 \), one has \( N_{k'/k}(-u) \in k^{\times 2} \) and there is nothing to do. We can assume that \( k' = k(u) \). Let us denote \( k_1 = k(u^2) \subset k' = k(u) \) which is a quadratic extension and let us consider the following diagram of quadratic extensions:

\[
\begin{array}{ccc}
k' & \overset{k'(\sqrt{a})}{\longrightarrow} & k_1(\sqrt{a}) \\
\downarrow & & \downarrow \\
k_1 & \overset{k(\sqrt{a})}{\longrightarrow} & k_2
\end{array}
\]

Lemma 2 applied to the extension \( k'/k_1 = k_1(u)/k_1 \) and the identity \( \langle 1, u \rangle \otimes \langle 1, -a \rangle k_1 = \langle -1 \rangle \otimes \psi_k \) yields

\[
N_{k'/k}(u) \in G(\langle 1, a \rangle k_1) \cap G(\psi_{k_1}).
\]

Then \( N_{k'/k}(u) = N_{k'/k_1}(u) \subset N_{k_1/k_1}(u) \). On the other hand, since \( k_1(u) = k' \) and \( k_1(\sqrt{a}) \) are two quadratic extensions of \( k_1 \), it is
known (Lemma 1.4 of [7]) that
\[ N_{k'/k}(k^\times) \cap N_{k'(\sqrt{-a})/k}(k_1(\sqrt{a})^\times) = N_{k'(\sqrt{-a})/k}(k_1(\sqrt{a})^\times).k_1^{\times^2}. \]

The extension \( k'(\sqrt{a}) \) splits the forms \( (1, -a), \psi_k = (-1, -u) \) and \( (1, -a)_{k'} \), and the \( q_i \)'s. Therefore one has \( N_{k'/k}(\{-u\}) \subseteq \text{hyp}(q_1, k_1), (1, -a)_{k'}, \psi_k).k_1^{\times^2} \). Applying Lemma 1(c) to the extension \( k_1/k \), it follows that \( N_{k'/k}(u) = N_{k_1/k}(N_{k'/k}(\{-u\}) \subseteq \text{hyp}(q_1), (1, -a), \psi).k_1^{\times^2} \). We showed this case.

(ii) 2nd Case. \( e \) Is Odd. With the diagram of Milnor's residue maps, we see that the form \( (1, -a)_{k'} = \delta_2(q_1) \) is hyperbolic. Moreover, \( 0 = d_2 = j((1, -a)_{k'}, \psi). \) Then the form \( \psi_{k'} \) is hyperbolic. Since \( \psi \) is not hyperbolic, according to Springer's theorem for odd extensions [17, p. 62], the integer \( f = [k' : k] \) is even. Hence, one has \( N_{k'/k}(L^\times) \subseteq N_{k'/k}(k^\times).K^{\times^2} \). The forms \( q_i, \psi_{k'}, \psi_k \) and \( (1, -a)_{k'} \) are hyperbolic and then it yields \( N_{k'/k}(k^\times) \subseteq \text{hyp}(q_i, \psi, (1, -a)) \) and \( N_{k'/k}(L^\times) \subseteq \text{hyp}(q_i, (1, -a), \psi).K^{\times^2} \).

**Theorem 2.** Let \( a \in k^\times \setminus k^{\times^2} \) and \( D/k \) be a biquaternion algebra and let \( \psi/k \) be an Albert form associated with \( D \) which represents \(-1\). Denote \( K = k((t)) \). Let us define the \( k \)-form \( q_0 \) and the \( K \)-form \( q \) by
\[ (1, -a) \perp \psi = q_0 \perp \mathbb{H} \]
and
\[ \langle \langle -a, t \rangle \rangle \perp \psi = q \perp \mathbb{H}. \]
One has \( \text{rk}(q_0) = 6, \text{disc}(q_0) = (a), \text{rk}(q_0) = 8, \) and \( \text{disc}(q) = 1. \)

(a) If \( \text{ind}(D_{k(\sqrt{-a})}) = 1 \), there exists a natural isomorphism
\[ G(q_0)/\text{hyp}(q_0).k^{\times^2} \sim G(q_K)/\text{hyp}(q_K).K^{\times^2}. \]

(b) If \( \text{ind}(D_{k(\sqrt{-a})}) = 4 \), there exists a field extension \( E/k \) such that \( G(q_{K \otimes E})/\text{hyp}(q_{K \otimes E})(K \otimes E)^{\times^2} \neq 1 \) and such that the variety \( \text{PSO}(q) \) is not stably \( K \)-rational.

**Proof.** Due to Lemma 1(a), we can do the proof with \( q_0 = (1, -a) \perp \psi \) and \( q = \langle \langle -a, t \rangle \rangle \perp \psi. \)

(a) We will apply the preceding proposition to the form \( q \) and we have to check hypothesis (C). Let \( b \in k^\times \) be such that \( \langle \langle -a, b \rangle \rangle \perp \psi = 0 \in W(k) \). Then the form \( \psi_{k(\sqrt{-a})} \) is hyperbolic and the algebra \( D_{k(\sqrt{-a})} \) is split, which is a contradiction for the hypothesis \( \text{ind}(D_{k(\sqrt{-a})}) > 1 \). The
hypothesis (C) is checked and the proposition yields

\[ G(q_K) = (G(\langle 1, -a \rangle) \cap G(\psi)) \cdot K^{x^2} \]

and

\[ \text{hyp}(q_K) \cdot K^{x^2} = \text{hyp}(\langle 1, -a \rangle, \psi) \cdot K^{x^2}. \]

Due to Lemma 1(e), one has \( G(q_0) = G(\langle 1, -a \rangle) \cap G(\psi) \) and \( \text{hyp}(q_0) \cdot k^{x^2} = N_{k(\langle \psi \rangle)/k}(\text{hyp}(\psi_{k(\langle \psi \rangle)})) \cdot k^{x^2} \). Then we have an isomorphism

\[ G(q_0)/\text{hyp}(q_0) \cdot k^{x^2} \xrightarrow{\sim} G(q_K)/\text{hyp}(q_K) \cdot K^{x^2}. \]

(b) Since \( \text{ind}(D_{k(\langle \psi \rangle)}) = 4 \), Theorem 1 shows the existence of a field extension \( E/k \) such that \( a \notin E^{x^2} \), \( \text{ind}(D_{E(\langle \psi \rangle)}) = 4 \), and \( G(q_{0,E})/\text{hyp}(q_{0,E}) \cdot E^{x^2} \neq 1 \). Hence \( G(q_{K \otimes E})/\text{hyp}(q_{K \otimes E})(K \otimes_k E)^{x^2} \neq 1 \) and the variety \( \text{PSO}(q_K) \) is not stably \( K \)-rational.

Remark 1 yields a field \( k \) with cohomological dimension 2, a quadratic field extension \( L = k(\sqrt{a}) \), and an Albert form \( \psi \) which represents \(-1\) and satisfies \( \text{ind}(C_{\psi}(\psi_L)) = 4 \). We showed the result claimed in the introduction.

**Theorem 3.** There exist a field \( k \) of characteristic 0, with cohomological dimension 3 and a quadratic form \( q \) with rank 8 and signed discriminant 1 such that the variety \( \text{PSO}(q)/k \) is not stably \( k \)-rational.

Due to Theorem 1, the dimension 8 is minimal for such an example with trivial signed discriminant. On the other hand, we don't know if there exists such an example with \( \text{cd}(k) = 2 \). The method used here brings nothing if \( \text{cd}(k) = 2 \). In this case, due to the Merkurjev–Suslin theorem, the Galois symbol yields an isomorphism \( I^2(k') \rightarrow Br(k') \) for any finite extension \( k'/k \) (cf. [1]). For any quadratic form \( q \) with trivial signed discriminant and Clifford algebra \( C(q)/k \), one has \( k^{x^2} = \text{Nrd}(C(q)^{x^2}) \cdot k^{x^2} = \text{hyp}(q) \cdot k^{x^2} = G(q) \) and the invariant \( G(q)/\text{hyp}(q) \cdot k^{x^2} \) is trivial on \( k \).

We have to underline that we used the Index Reduction Theory through [9] for giving proof of our result. We shall see that with cohomological dimension 6 instead 3, we can show the same result without the Index Reduction Theory and thus we can produce explicit elementary examples of non-rational adjoint groups built from an iteration of Proposition 1 with a field of iterated formal power series. This method contains some analogies with Platonov's counterexample [15] to the Kneser–Tits conjecture, showing the existence of simply connected semisimple groups defined over a field \( k \), which are not \( k \)-rational varieties.
3. SUMS OF QUATERNIONIC FORMS

First, we introduce an invariant related to the multiquadratic extensions. This invariant will be used for computing some group PSO(q)/R.

**Definition 1.** Let \( A = (a_i)_{i=1,\ldots,m} \) be a family of elements of \( k^\times \). Denote \( k_i = k[i(i^2 - a_i)] \) for \( i = 1, \ldots, m \) and \( M = k_1 \otimes k_2 \otimes \cdots \otimes k_m \). One defines the group

\[
\Lambda(A/k) = \left( \prod_{i=1,\ldots,m} N_{k_i/k}(a_i) \right)/N_{M/k}(M^\times).k^\times^2.
\]

**Proposition 2.** Let \( k = (a_i)_{i=1,\ldots,m} \) be a family of elements of \( k^\times \) and \( M/k \) as in the definition. The following assertions hold.

(a) \( \text{hyp}((1 - a_i)_{i=1,\ldots,m}).k^\times^2 = N_{M/k}(M^\times).k^\times^2 \).

(b) Let \( T \) be the \( k \)-torus defined by the equations

\[
N_{k_1/k}(y_1) = N_{k_2/k}(y_2) = \cdots = N_{k_m/k}(y_m) \neq 0.
\]

Then, we have a natural isomorphism \( T(k)/R \simeq \Lambda(A/k) \).

**Proof.** We denote by \( G = \text{Gal}(M/k) \) the Galois group of \( k_1, k_2, \ldots, k_m/k \) and by \( G_i \subset G \) the subgroup which fixes \( \sqrt{a_i} \) (\( i = 1, \ldots, m \)). One can assume that \( a_i \neq k^\times_2 \) for \( i = 1, \ldots, m \) and let us denote by \( \sigma_i \) the generator of \( G/G_i = \text{Gal}(k_i/k) \). One has an injective morphism \( j: T \subset \prod_{i=1,\ldots,m} R_{k_i/k}\mathbb{G}_m \) and a morphism \( q = N_{k_1/k} \circ j: T \to \mathbb{G}_m \) whose kernel is denoted by \( T' = \prod_{i=1,\ldots,m} R_{k_i/k}\mathbb{G}_m \). We define a surjective morphism of \( k \)-tori

\[
p: R_{M/k}\mathbb{G}_m \times \mathbb{G}_m \times \prod_{i=1,\ldots,m} R_{k_i/k}\mathbb{G}_m \to T \subset \prod_{i=1,\ldots,m} R_{k_i/k}\mathbb{G}_m,
\]

where

\[
[p(y, x, y_1, \ldots, y_m)]_i = N_{k_i/k}(y).x.y_i/\sigma_i(y) \quad \text{for} \quad i = 1, \ldots, m.
\]

Let us denote by \( E = R_{M/k}\mathbb{G}_m \times \mathbb{G}_m \times \prod_{i=1,\ldots,m} R_{k_i/k}\mathbb{G}_m \), by \( S = \ker(p) \) the torus kernel of \( p \), and by \( \hat{S}^0 \) the Galois module of cocharacters of \( S \), i.e., \( \hat{S}^0 = \text{Hom}_{k^\times}(\mathbb{G}_m, S) \). The following lemma is easy to show.

**Lemma 5.** \( H^3(H, \hat{S}^0) = 0 \) for any subgroup \( H \subset G \).

In other words, the morphism \( p \) defines an exact sequence of \( k \)-tori

\[
1 \to S \to E \xrightarrow{p} T \to 1,
\]
which is a flasque resolution of the torus $T$ (cf. [3]) and then the boundary map $\partial: T(k) \to H^1(k, S)$ induces an isomorphism $T(k)/R = H^1(k, S)$. Since $H^1(k, E) = 1$ [20, chap. X], one has an isomorphism $T(k)/p(E(k)) = H^1(k, S)$. We consider the following commutative exact diagram

$$
\begin{array}{c}
T'(k) \\
\downarrow \\
E(k) \xrightarrow{p} T(k) \xrightarrow{q} T(k)/R \longrightarrow 1.
\end{array}
$$

Since the torus $T' = \prod_{i=1,\ldots,m} R_{k_i/k} \mathbb{G}_m$ is a rational variety, the map $T(k) \to T(k)/R$ factorizes by $p$ and then one has an isomorphism $T(k)/R = T(k)/p(E(k)) = q(T(k))/q \circ p(E(k)) = \Lambda(A/k)$.

Remark 3. Following [8], if $k$ is a number field, the invariant $\Lambda(A/k)$ is always trivial, and Colliot-Thélène and Sansuc showed that the group $T(k)/R$ is finite for any torus defined over a field of finite type over the prime field [3]. Therefore, if the field $k$ is of finite type over the prime field, the group $\Lambda(A/k)$ is finite.

We know that $n = 2$ yields $\Lambda(A/k) = 1$ [7, Lemma 1.4]. We can show this with the proposition (b). The torus $T$ is indeed an open subset of a quadric having a rational point which is a rational variety, hence $1 = T(k)/R = \Lambda(A/k)$.

For $n = 3$, we can deduce the non-triviality of the invariant $\Lambda$ of Proposition 2.4 of [8]. More precisely, one has the following nice result of Tignol which connects the invariant $N_1$ of a triquadratic extensions and $\Lambda$.

**Proposition 3** (Tignol, unpublished). Let $A = (a, b, c)$ be a family $k^\times$. Denote $M = k(\sqrt[3]{a}, \sqrt[3]{b}, \sqrt[3]{c})$ and $E = k(\sqrt[3]{c})$. Then there exists an isomorphism of groups

$$N_1(a, b, c) = \frac{k^\times \cap N_k(a) \cdot N_k(b)}{(k^\times \cap N_k(a)) \cdot (k^\times \cap N_k(b))} \to \frac{N_k(a) \cap N_k(b) \cap N_k(c)}{N_{M/k}(M^\times) \cdot k^\times \cap N_k(a) \cdot N_k(b) \cdot N_k(c)}.$$

**Proof.** If the extension $E/k$ is not proper, the two groups are trivial. We can assume that $E/k$ is a proper extension and we denote by $h \to \overline{h}$ the action of $\text{Gal}(E/k)$ on $E$. One defines the map between the two quotients with the following map $\theta$. If $f = N_{E(\sqrt[3]{c})/E}(x), N_{E(\sqrt[3]{b})/E}(y) \in k^\times$, we have
we define

$$\theta(f) = \left[ N_{E/\varpi}(x) \right] = \left[ f N_{E/\varpi}(y) \right]$$

$$\in N_k(a) \cap N_k(b) \cap N_k(c) \mod N_{M/k}(M^\times).k^2.$$ 

Let us show that the element $\theta(f)$ is well defined. Indeed, if $f = N_{E(\varpi)/E}(x')N_{E(\varpi)/E}(y')$, one has

$$N_{E(\varpi)/E}(x'y'^{-1}) = N_{E(\varpi)/E}(yy'^{-1})$$

$$\in N_k(a) \cap N_k(b) = N_{M/E}(M^\times).k^2$$

using again Lemma 1.4 of [7]. Hence $N_{E(\varpi)/E}(x') \in N_{M/k}(M^\times).k^2$. On the other hand, if $f \in (k^\times \cap N_k(a))(k^\times \cap N_k(b))$, then we can assume $N_{E(\varpi)/E}(x') \in k^\times$ and hence $N_{E(\varpi)/E}(x') \in k^\times^2$. Denoting again the quotient map by $\theta$, we define a morphism of groups

$$\theta: \frac{k^\times \cap N_k(a) \cdot N_k(b)}{(k^\times \cap N_k(a))(k^\times \cap N_k(b))} \rightarrow \frac{N_k(a) \cap N_k(b) \cap N_k(c)}{N_{M/k}(M^\times).k^2}.$$ 

Let us show the injectivity of $\theta$. If $N_{E(\varpi)/E}(x') = g^2 N_{M/E}(z)$ with $g \in k^\times$, $z \in M^\times$, then

$$N_{E/k}(N_{E(\varpi)/E}(x)) = N_{E/k}(g N_{M/E}(z)).$$ 

Hence by [20, chap. X], $N_{E(\varpi)/E}(x') = g N_{M/E}(z) h^{-1}$ with $h \in E$. Then $N_{E(\varpi)/E}(x) = (ghh^{-1}).N_{M/E}(z))$. One has

$$h^{-2} N_{M/E}(z) \in E^\times. N_{M/E}(M^\times) = N_k(a) \cap N_k(b).$$

Then the preceding equality shows that $ghh \in k^\times \cap N_k(a)$. On the other hand, since $f = N_{E(\varpi)/E}(x')N_{E(\varpi)/E}(y')$, one has

$$f = (ghh^{-1}).N_{M/E}(z) \cdot N_{E(\varpi)/E}(y').$$

The second term is an element of $N_k(b)$ but has to be also an element of $k^\times$, then $f \in (k^\times \cap N_k(a))(k^\times \cap N_k(b))$.

Let us show the surjectivity of $\theta$ for finishing the proof. If $t \in N_k(a) \cap N_k(b) \cap N_k(c)$, we can choose $u \in E^\times$ such that $t = N_{E/k}(u)$. Since $t \in N_k(a)$, one has $u \in k^\times N_k(a)$; in the same way, one has $u \in k^\times N_k(b)$ because $t \in N_k(b)$. Then

$$u = g N_{E(\varpi)/E}(x) = h N_{E(\varpi)/E}(y^{-1}).$$
and $g^{-1}h = N_{E(\overline{\mathbb{F}})/E}(x) \cdot N_{E(\overline{\mathbb{F}})/E}(y) \in k^\times \cap N_{E/E}(a)N_{E}(b)$ has for image by $	heta$, $N_{E(\overline{\mathbb{F}})/k}(x) = g^{-1}N_{E/k}(u) = t \mod k^\times$.

**Remark 4.** We denote by $\mathbb{Q}_2$ the 2-adic completion of $\mathbb{Q}$. If $k \in \mathbb{Q}_2(x)$ (or $\mathbb{Q}(x)$), it is shown in [16, Sect. 5.4] that $N_{k}(x+4, x+1, x) \neq 1$. Then for $A = (x+4, x+1, x)$, the group $\Lambda(A/\mathbb{Q}_2(x))$ is not trivial. Let us give an explicit element of $\Lambda(A/\mathbb{Q}_2(x))$. Due to Theorem 5.1 of [16], we know that the class of 2 in $N_{k}(x+4, x+1, x)$ is not trivial. If $\theta$ denotes the isomorphism $N_{k}(x+4, x+1, x) = \Lambda(A/\mathbb{Q}_2(x))$ given by the proposition, one computes easily $\theta(2) = -x$. Hence the class of $-x$ is not trivial in $\Lambda(A/\mathbb{Q}_2(x))$. There exists an example of non-trivial invariant $\Lambda$ with the base field $\mathbb{C}(t_2, t_1)$ which has cohomological dimension 2 [16].

**Theorem 4.** Assume that the base field $k$ has characteristic 0. Let $m$ be an integer, $m \geq 2$, and $A = (a_i)_{1 \leq i \leq m}$ a family of elements in $k^\times \setminus k^\times$ such that $a_i/a_{i-1} \notin k^\times$ for $i = 2, \ldots, m$. Denote $k_i = k(\sqrt[n]{a_i})$ for $i = 1, \ldots, m$ and $M = k_1 k_2 \cdots k_m$. Let $(c_i)_{1 \leq i \leq m}$ be a family of elements of $k^\times$ and $(X_i)_{1 \leq i \leq m}$ a family of indeterminates on $k$. Denote $F_0 = k$, $F_i = k((X_i))((X_{i-1}))$, $F = F_m$, and

$$
\Phi = \langle c_1 \rangle \otimes \langle -a_1, X_1 \rangle \perp \langle c_2 \rangle \otimes \langle -a_2, X_2 \rangle \cdots \perp \langle c_m \rangle \otimes \langle -a_m, X_m \rangle.
$$

Then one has

$$
G(\Phi_F) = \left( \bigcap_{i=1}^m N_{k}(a_i) \right) F^\times, \quad \text{hyp}(\Phi_F) F^\times = N_{M/k}(M^\times) F^\times,
$$

and

$$
\Lambda(A/k) \sim G(\Phi_F)/\text{hyp}(\Phi_F) F^\times.
$$

In order to apply Proposition 1, we have to check the validity of condition (C).

**Lemma 6.** Let $m, \Phi_1, \ldots$ as in Theorem 4. Denote

$$
\Phi^{m-1} = \langle c_1 \rangle \otimes \langle -a_1, X_1 \rangle \perp \langle c_2 \rangle \otimes \langle -a_2, X_2 \rangle \perp \cdots \perp \langle c_{m-1} \rangle \otimes \langle -a_{m-1}, X_{m-1} \rangle.
$$

Then for any $b \in F_{m-1}^\times$, one has

$$
\langle -a_m, b \rangle \perp \langle c_m^{-1} \rangle \Phi^{m-1} \neq 0 \in W(F_{m-1}).
$$
Proof of the Lemma. We denote by \( v_{X_{m-1}}; F_{m-1}^\times \rightarrow \mathbb{Z} \) the valuation associated to the uniformizing parameter \( X_{m-1} \). We apply the residue map \( \partial_{X_{m-1}}; W(F_{m-1}) \rightarrow W(F_{m-2}) \) to a relation \( \langle \langle - a_m, b \rangle \rangle \perp \langle c_m^{-1} \rangle \otimes \Phi^{m-1} \) 0 \in W(F_{m-1}) \) where \( b \in F_{m-1}^\times \). If \( v_{X_{m-1}}(b) \) is even, then \( \langle c_m^{-1} \rangle \otimes \langle 1, -a_m \rangle = 0 \in W(F_{m-2}) \) and \( a_m \in k^\times \cap F_{m-2}^\times = k^\times 2 \), which is wrong by hypothesis. Then \( v_{X_{m-1}}(b) \) is odd and the map \( \partial_{X_{m-1}} \) yields \( \langle c_m \rangle \otimes \langle 1, -a_m \rangle \perp \langle c_m^{-1} \rangle \otimes \langle 1, -a_m \rangle = 0 \in W(F_{m-2}) \). Taking the signed discriminant, we have \( a_m/a_{m-1} \in k^\times \cap F_{m-2}^\times = k^\times 2 \), which is wrong by hypothesis. We showed the lemma.

With this lemma, we can apply Proposition 1. Let us show by induction on \( m \geq 2 \) the equalities

(1) \( G(\Phi_F) = \left( \bigcap_{i=1}^{m} N_k(a_i) \right) F_{m}^\times \)

(2) \( \text{hyp}(q_{j,F}, \Phi_F).F_{m}^\times = \text{hyp}(\langle q_j \rangle, \langle 1, -a_1 \rangle, \ldots, \langle 1, -a_m \rangle).F_{m}^\times \)

for any finite family \( \{q_j\} \) of \( k \)-forms.

\( m = 2 \). Due to Proposition 1 applied to the base field of \( F_2 \) and forms \( \langle \langle a_1, X_1 \rangle \rangle, \psi = \Phi^1 = \langle c_1 \rangle \otimes \langle -a_1, X_1 \rangle \rangle \) and the uniformizing parameter \( X_1 \), one has

\[ G(\Phi_{F_2}) = (G(\langle \langle - a_1, X_1 \rangle \rangle_{F_1}) \cap G(\langle 1, -a_2 \rangle_{F_1})).F_{2}^\times. \]

Applying Lemma 3 with the uniformizing parameter \( X_1 \), it produces

\[ G(\langle 1, -a_2 \rangle_{F_1}) = G(\langle 1, -a_2 \rangle).F_{2}^\times. \]

Since \( k^\times \cap G(\langle 1, -a_2 \rangle).F_{2}^\times = G(\langle 1, -a_2 \rangle) \), one has

\[ G(\Phi_{F_2}) = \left( \bigcap_{i=1,2} G(\langle 1, -a_i \rangle) \right) F_{2}^\times = \left( \bigcap_{i=1,2} N_k(a_i) \right) F_{2}^\times. \]

For the other equality, Proposition 1 shows that

\[ \text{hyp}(\langle q_j, F_1 \rangle, \Phi_{F_2}).F_{2}^\times = \text{hyp}(\langle q_j, F_1 \rangle, \langle c_1 \rangle \otimes \langle -a_1, X_1 \rangle).F_{2}^\times = \text{hyp}(\langle q_j, F_1 \rangle, \langle 1, -a_1 \rangle_{F_1}, \langle 1, -a_2 \rangle_{F_1}).F_{2}^\times = \text{hyp}(\langle q_j \rangle, \langle 1, -a_1 \rangle, \langle 1, -a_2 \rangle).F_{2}^\times \quad (\text{Lemma 3}). \]

\( m \geq 3 \). Let us denote \( \Phi^{m-1} = \langle c_1 \rangle \otimes \langle - a_1, X_1 \rangle \perp \langle c_2 \rangle \otimes \langle - a_2, X_2 \rangle \perp \cdots \perp \langle c_{m-1} \rangle \otimes \langle - a_{m-1}, X_{m-1} \rangle \). Lemma 6 allows
us to apply Proposition 1 with the uniformizing parameter \( X_m \) and it yields
\[
G(\Phi_{F_m}) = (G(\Phi_{F_{m-1}}^{-1}) \cap G(\langle 1, -a_m \rangle_{F_{m-1}})).F_m^{x^2}.
\]
The induction hypothesis yields
\[
G(\Phi_{F_{m-1}}^{-1}) = \left( \bigcap_{i=1, \ldots, m-1} G(\langle 1, -a_i \rangle) \right).F_{m-1}^{x^2}
\]
and with an iteration of Lemma 3(b), one has
\[
G(\langle 1, -a_m \rangle_{F_{m-1}}) = G(\langle 1, -a_m \rangle).F_{m-1}^{x^2}.
\]
Since \( k^x \cap G(\langle 1, -a_m \rangle).F_{m-1}^{x^2} = G(\langle 1, -a_m \rangle) \), we have
\[
G(\Phi_{F_m}) = \left( G(\langle 1, -a_m \rangle) \cap \bigcap_{i=1, \ldots, m-1} G(\langle 1, -a_i \rangle) \right).F_m^{x^2}
\]
\[
= \left( \bigcap_{i=1, \ldots, m} N_k(a_i) \right).F_m^{x^2}.
\]
For the equality (2), Proposition 1 shows that
\[
\text{hyp}(\langle q_j, F_m \rangle, \Phi_{F_m}).F_m^{x^2}
\]
\[
= \text{hyp}(\langle q_j, F_{m-1} \rangle, \Phi_{F_{m-1}}^{-1}, \langle 1, -a_m \rangle_{F_{m-1}}).F_{m-1}^{x^2}
\]
\[
= \text{hyp}(\langle q_j, \langle 1, -a_1 \rangle, \langle 1, -a_2 \rangle, \ldots, \langle 1, -a_m \rangle \rangle).F_{m-1}^{x^2}
\]
due to the induction hypothesis applied with \( m-1 \) and the set of \( k \)-forms \( \langle q_j, \langle 1, -a_m \rangle \rangle \). We showed by induction the two equalities. Taking \( q_j = 0 \) in the equality (2), we have
\[
\text{hyp}(\Phi_{F_m}).F_m^{x^2} = \text{hyp}(\langle 1, -a_1 \rangle, \ldots, \langle 1, -a_m \rangle).F_m^{x^2} = N_{M/k}(M^x).F_m^{x^2}.
\]
Since \( k^x \cap F_{m-1}^{x^2} = k_{m-1}^{x^2} \), it is easy to check that one has an isomorphism
\[
\Lambda( k/k ) \cong G(\Phi_{F_m})/\text{hyp}(\Phi_{F_m}).F_m^{x^2}.
\]
**Application.** Let \( a_1, a_2, a_3 \) be in \( k^x \) such that \( a_1/a_2, a_2/a_3 \notin k^{x^2} \) and let us denote \( F = k((X_1))((X_2))((X_3)) \) and \( M = k(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}) \). Let us apply Theorem 4 to the quadratic form
\[
\Phi = \langle \langle -a_1, X_1 \rangle \rangle \perp \langle -1 \rangle \otimes \langle \langle -a_2, X_2 \rangle \rangle \perp \langle a_1 \rangle \otimes \langle \langle -a_3, X_3 \rangle \rangle
\]
\[
= q \perp \mathbb{H} \perp \mathbb{H}.
\]
The form \( q \) has rank 8 and signed discriminant 1, and with the notations of the theorem above, one has

\[
\Lambda(A/k) \sim G(\Phi_F)/\text{hyp}(\Phi_F).F^{\times 2} \sim G(q_F)/\text{hyp}(q_F).F^{\times 2}
\]

Then, for the field \( F = \mathbb{Q}_2(x)(X_1)(X_2)(X_3) \), which has cohomological dimension 6, or for the field \( F = \mathbb{Q}(x)(X_1)(X_2)(X_3) \), following Remark 4 and taking \( a_1 = x + 4 \), \( a_2 = x + 1 \), \( a_3 = x \), we have \( G(q_F)/\text{hyp}(q_F).F^{\times 2} \neq 1 \) and the variety \( \text{PSO}(q)/F \) is not \( F \)-stably rational. More precisely, in this case we have

\[
\Phi = \langle \langle - (x + 4), X_1 \rangle \rangle \perp \langle -1 \rangle \otimes \langle \langle - (x + 1), X_2 \rangle \rangle
\]

\[
\perp \langle x + 4 \rangle \otimes \langle \langle -x, X_3 \rangle \rangle
\]

\[
= q \perp H \perp H,
\]

and \( -x \) is a similarity factor of \( q_F \) such that \( -x \not\in \text{hyp}(q_F).F^{\times 2} \).

REFERENCES