THE BOREL-DE SIEBENTHAL'S THEOREM

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This is the following.

0.1. **Theorem.** *Let F be a field of characteristic $\neq 2,3$. Let G/F be a reductive group and let H/F be a reductive subgroup of maximal rank. Then $H = Z_G(Z(H))^0$.*

The original theorem [2] is about compact Lie groups and the specialists know for a long time that is generalizes as stated. Our goal is to present here a proof of that result. The following is well known in the case of subtori [3, 15.3.2].

0.2. **Lemma.** *Let G/F be a reductive group and let T/F be a maximal torus. Let $S \subset T$ be a subgroup.*

(1) *The F–group $Z_G(S)$ is smooth and $Z_G(S)^0$ is reductive.*
(2) *If T is split, let U_α be the root groups associated to $\Phi(G,T)$. Then $Z_G(S)^0$ is generated by T and the root subgroups U_α for the $\alpha \in \Phi(G,T) \subset T^*$ mapping to 0 in S^*.*

Proof. (1) Since S is of multiplicative type, the group $Z_G(S)$ is smooth [4, XI.5.2]. We show that G/k is reductive at the end of the proof.

(2) We can assume that F is algebraically closed. The F-group $Z_G(S)^0$ is smooth so is a linear algebraic group. The group $Z_G(S)^0$ is generated by T and root subgroups U_α of G [1, 13.20]. for α running over $R := \Phi(G,T) \cap \ker(T^* \to S^*)$. Similarly, the radical M of $Z_G(S)^0$ is normalized by T, hence it is generated by $(M \cap T)^0$ and the root subgroups U_α of M. We claim that $M = (M \cap T)^0$. For sake of contradiction, assume that U_α is a root subgroup of M. Its conjugate $U_{-\alpha}$ in $Z_G(S)^0$ is then as well a root subgroup of $Z_G(S)^0$, hence M contain a semisimple group of rank one, which contradicts the solvability of M. Thus M is a torus and we conclude that $Z_G(S)^0$ is reductive. □

We first look at the behaviour of Theorem 0.1 under central extensions.

0.3. **Lemma.** *Under the hypothesis of the theorem, let S be a central subgroup (of multiplicative type) of G and denote by $f : G \to G/S$ the quotient map.*

(1) *$S \subset H$ and H/S is a reductive subgroup of maximal rank of G/S.*
(2) *If $H/S = Z_{G/S}(Z(H/S))^0$, then $H = Z_G(Z(H))^0$.*

Proof. (1) follows from the fact that the center of G is included in all maximal tori of G.

(2) We have an exact sequence of algebraic groups

$$1 \to S \to Z_G(Z(H)) \to Z_{G/S}(Z(H/S))$$

Since $S \subset H \subset Z_G(Z(H))^0$, we have the following exact diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S & \longrightarrow & Z_G(Z(H))^0 & \longrightarrow & Z_{G/S}(Z(H/S))^0 & & \\ & & || & & \cup & & \cup & & \\ 1 & \longrightarrow & S & \longrightarrow & H & \longrightarrow & H/S & \longrightarrow & 1. \end{array}$$

If $H/S = Z_{G/S}(Z(H/S))^0$, it follows by diagram chase that $H = Z_G(Z(H))^0$. □

We can now proceed to the proof of Theorem 0.1.

Proof. Reduction to the case H semisimple: Let $S = Z(H)^0$ be the connected center of H Then $H \subset Z_G(S)$ and H/S is a semisimple subgroup of G/S. If the result is known in the semisimple case, we have $H/S = Z_{Z_G(S)/S}(Z(H/S))^0$. Lemma 0.3 shows that $H = Z_{Z_G(S)}(Z(H))^0$, hence $H = Z_G(Z(H))^0$.

Furthermore Lemma 0.3 applied to $Z(G)$ permits to assume that G is semisimple adjoint. We can assume moreover that k is algebraically closed. We consider a maximal (split) torus T of H.

Case H maximal proper semisimple group of G: We choose compatible orderings on the roots systems $\Phi(H,T) \subsetneq \Phi(G,T) = T^*$. Since F is of characteristic $\neq 2, \neq 3$, $\Phi(H,T)$ is a closed subsystem of $\Phi(G,T)$ [4, XXIII.6.6]. Let A be the root lattice of H, i.e. the sublattice of T^* generated by $\Phi(H,T)$. The center $Z(H)$ of H is a diagonalisable group whose character group is $Z(H)^* = T^*/A$.

To show that $H = Z_G(Z(H))^0$, we note first that $Z_G(Z(H))^0$ is reductive by Lemma 0.2. Since $Z_G(Z(H))^0$ contains the semisimple group H, $Z_G(Z(H))^0$ is necessarily semisimple. We claim that $Z(H) \neq 1$. For sake of contradiction, assume that $A = T^*$, i.e. that H is adjoint. Then our basis for $\Phi(H,T)$ would be a basis for $\Phi(G,T)$, contradiction. So $Z(H) \neq 1$ and $Z_G(Z(H))^0$ is a proper subgroup of G. Since H is a maximal semisimple subgroup, we conclude that $H = Z_G(Z(H))^0$.
General case: By dimension reasons, there is a chain of semisimple groups

$$H = H_0 \subsetneq H_1 \cdots \subsetneq H_{n-1} \subsetneq H_n = G$$

such that H_i is maximal in H_{i+1} for $i = 0, ..., n-1$. By induction on n, we can assume that

$$\begin{aligned} H &= Z_{H_{n-1}}(Z(H))^0 \\ &= \Big(Z_G(Z(H)) \cap H_{n-1}\Big)^0 \\ &= \Big(Z_G(Z(H)) \cap Z_G\big(Z(H_{n-1})\big)\Big)^0 \qquad \text{[maximal case]} \\ &= Z_G(Z(H))^0 \qquad [\, Z(H_{n-1}) \subset Z(H)\,]. \end{aligned}$$

Thus $H = Z_G(Z(H))^0$ as desired. □

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References

[1] A. Borel, *Linear Algebraic Groups (Second enlarged edition)*, Graduate text in Mathematics **126** (1991), Springer.

[2] A. Borel et J. de Siebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Commentarii Math. Helv. **22** (1949), 200-221.

[3] T.A. Springer, *Linear Algebraic Groups*, second edition (1998), Birkäuser.

[4] *Séminaire de Géométrie algébrique de l'I.H.E.S., 1963-1964, schémas en groupes, dirigé par M. Demazure et A. Grothendieck*, Lecture Notes in Math. 151-153. Springer (1970).

UMR 8552 du CNRS, DMA, Ecole Normale Supérieure, F-75005 Paris, France
E-mail address: Philippe.Gille@ens.fr