Kneser-Tits problem for trialitarian groups and bounded generation
by restricted elements
Philippe Gille and Uzi Vishne

1. Introduction

Let $F$ be a field. Let $F_s/F$ be a separable closure of $F$ and denote by $\Gamma_F$ the Galois group of $F_s/F$. We consider a semisimple group $G/F$ of absolute type $D_4$ [9], whose root system can be depicted as

\[ \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4 \]

The automorphism group of this Dynkin diagram is $S_3$, hence $G$ defines a class in $H^1(F, S_3) = \Hom_{set}(\Gamma_F, S_3)/S_3$, namely an isomorphism class of cubic étale algebras [4, §18]. If this cubic étale algebra, say $K/F$, is a field, we say that $G$ is trialitarian. The following result answers the Kneser-Tits problem for those groups.

Theorem 1.1. [3, §6.1] Let $G/F$ be a semisimple simply connected trialitarian group. If $G$ is isotropic, then the (abstract) group $G(F)$ is simple.

Since $Z(G) = \ker(R_{K/F} \mu_2) \rightarrow \mu_2)$, note that $Z(G)(F) = 1$. If $G$ is quasi-split (for example in the case of finite fields), this is a special case of Chevalley’s theorem [1]. By Tits tables for indices, the only other case to consider is that with Tits index

\[ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \]

In the number field case, this has been proven by G. Prasad and M.S. Raghunathan [6]. Our goal is to explain how this result follows from a general statement and how it applies together with Prasad’s approach to a nice understanding of generators for the rational points of the anisotropic kernel of $G$. 

\[ \text{In the number field case, this has been proven by G. Prasad and M.S. Raghunathan [6]. Our goal is to explain how this result follows from a general statement and how it applies together with Prasad’s approach to a nice understanding of generators for the rational points of the anisotropic kernel of } G. \]
2. Invariance under transcendental extensions

Assume for convenience that $F$ is infinite. Let $G/F$ be a semisimple connected group which is absolutely almost simple and isotropic. We denote by $G^+(F)$ the (normal) subgroup of $G(F)$ which is generated by the $R_u(P)(F)$ for $P$ running over the $F$-parabolic subgroups of $G$. Tits showed that any proper normal subgroup of $G^+(F)$ is central [8] [10]. So for proving that $G(F)/Z(G)(F)$ is simple, the plan is to show the triviality of the Whitehead group

$$W(F,G) = G(F)/G^+(F).$$

This is the Kneser-Tits problem. Note that by Platonov’s work, $W(F,G)$ can be non-trivial, e.g. for special linear groups of central simple algebras [5].

**Theorem 2.1.** [3, §5.3] The map $W(F,G) \to W(F(t),G)$ is an isomorphism.

**Corollary 2.2.** If $G/F$ is a $F$-rational variety, then $W(F,G) = 1$.

Let us sketch the proof of the Corollary. The idea is to consider the generic element $\xi \in G(F(G))$. Since $F(G)$ is purely transcendental over $F$, it follows that $\xi \in G(F) \cdot G^+(F(G))$. Since $G^+(F)$ is Zariski dense in $G$, we can see by specialization that $\xi \in G^+(F(G))$. Therefore there exists an dense open subset $U$ of $G$ such that $U(F) \subset G^+(F)$. But $U(F) \cdot U(F) = G(F)$, thus $W(F,G) = 1$.

Assume now that $G/F$ is trialitarian. Since Chernousov and Platonov have shown that such a group is an $F$-rational variety [2, §8], we conclude that $W(F,G) = 1$.

3. Bounded generation by restricted elements

We assume that char($F$) $\neq 2$ and for convenience that $F$ is perfect and infinite. In [6], Prasad gives an explicit description of $W(F,G)$ in terms of the the Tits algebra of $G$, which is the Allen algebra $M_2(D)$ for $D$ a quaternion division algebra over $K$ satisfying $\text{cor}_{K/F}[D] = 0 \in \text{Br}(F)$, where $K$ is a cubic étale extension of $F$. We have

$$W(F,G) = U/\langle R \rangle,$$

where $U$ is the group of elements of the quaternion algebra $D/K$ whose reduced norm is in $F^*$, and $R$ is the set of elements $x \neq 0$ for which both the reduced norm and the reduced trace are in $F$. Combined with Theorem 1.1, we get the

**Corollary 3.1.** $\langle R \rangle = U$.

This leaves open the question of bounding the number of generators from $R$ required to express every element of $U$.

One may consider the same question when $K$ is a cubic étale extension which is not a field, namely, $K = F \times L$ for $L$ a quadratic field extension of $F$, or $K = F \times F \times F$, and $D$ is an Azumaya algebra over $K$. In the former case, $D = D_1 \times D_2$ where $D_1$ is a quaternion algebra over $F$ and $D_2$ a quaternion algebra over $L$, with $\text{cor}_{L/F}D_2 \sim D_1$. In the latter, $D = D_1 \times D_2 \times D_3$, where $D_i$
$(i = 1, 2, 3)$ are quaternion algebras over $F$, and $D_1 \otimes_F D_2 \otimes_F D_3 \sim F$. The sets $V$ and $R$ can be defined in the same manner as above.

This is not an artificial generalization: extending scalars from $F$ to $\tilde{F} = K$, the algebra becomes $\tilde{D} = D \otimes_F K$ which is an Azumaya algebra over $\tilde{K} = K \otimes_F K$, and $\tilde{K}$ is a cubic étale extension of $\tilde{F}$, which is not a field.

**Theorem 3.2** ([7, §2]). When $K$ is not a field, every element of $U$ is a product of at most 3 elements of $R$.

On the other hand, by means of generic counterexamples, one can show that 3 is the best possible:

**Proposition 3.3** ([7, Cor. 4.0.4]). Let $F = \mathbb{Q}(\eta, \lambda), K = F \times F \times F$, and $D = (\alpha, \eta + 1)F \times (\alpha, \lambda)F \times (\alpha, (\eta + 1)\lambda)F$, where $\alpha = \eta^2 - 4$. Let $x_i, y_i$ ($i = 1, 2, 3$) be standard generators for the $i$'th component.

Then the element $v = ((\eta + x_1)(\eta + 2 + 2y_1), \eta(1 + x_2), 2\eta) \in D_1 \times D_2 \times D_3$ is in $V$, but not in $R \cdot R$. In particular $V \not\subseteq R \cdot R$.

Another explicit counterexample [7, Cor. 4.0.4] shows that $V \not\subseteq R \cdot R$ when $K = F \times L$. By means of extending scalars [7, §5], it also follows that $V \not\subseteq R \cdot R$ when $K$ is a field.

**References**


