Kneser-Tits problem for trialitarian groups and bounded generation by restricted elements

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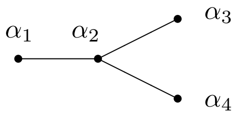
1. Introduction

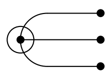
Let F be a field. Let F_s/F be a separable closure of F and denote by Γ_F the Galois group of F_s/F. We consider a semisimple group G/F of absolute type D_4 [9], whose root system can be depicted as



The automorphism group of this Dynkin diagram is S_3, hence G defines a class in $\mathrm{H}^1(F, S_3) = \mathrm{Hom}_{ct}(\Gamma_F, S_3)/S_3$, namely an isomorphism class of cubic étale algebras [4, §18]. If this cubic étale algebra, say K/F, is a field, we say that G is trialitarian. The following result answers the Kneser-Tits problem for those groups.

Theorem 1.1. [3, §6.1] *Let G/F be a semisimple simply connected trialitarian group. If G is isotropic, then the (abstract) group $G(F)$ is simple.*

Since $Z(G) = \ker(R_{K/F}(\mu_2) \to \mu_2)$, note that $Z(G)(F) = 1$. If G is quasi-split (for example in the case of finite fields), this is a special case of Chevalley's theorem [1]. By Tits tables for indices, the only other case to consider is that with Tits index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In the number field case, this has been proven by G. Prasad and M.S. Raghunathan [6]. Our goal is to explain how this result follows from a general statement and how it applies together with Prasad's approach to a nice understanding of generators for the rational points of the anisotropic kernel of G.

2. Invariance under transcendental extensions

Assume for convenience that F is infinite. Let G/F be a semisimple connected group which is absolutely almost simple and isotropic. We denote by $G^+(F)$ the (normal) subgroup of $G(F)$ which is generated by the $R_u(P)(F)$ for P running over the F–parabolic subgroups of G. Tits showed that any proper normal subgroup of $G^+(F)$ is central [8] [10]. So for proving that $G(F)/Z(G)(F)$ is simple, the plan is to show the triviality of the Whitehead group

$$W(F,G) = G(F)/G^+(F).$$

This is the Kneser-Tits problem. Note that by Platonov's work, $W(F,G)$ can be non-trivial, e.g. for special linear groups of central simple algebras [5].

Theorem 2.1. [3, §5.3] *The map $W(F,G) \to W(F(t),G)$ is an isomorphism.*

Corollary 2.2. *If G/F is a F-rational variety, then $W(F,G)=1$.*

Let us sketch the proof of the Corollary. The idea is to consider the generic element $\xi \in G(F(G))$. Since $F(G)$ is purely transcendental over F, it follows that $\xi \in G(F)\cdot G^+(F(G))$. Since $G^+(F)$ is Zariski dense in G, we can see by specialization that $\xi \in G^+(F(G))$. Therefore there exists an dense open subset U of G such that $U(F) \subset G^+(F)$. But $U(F)\cdot U(F) = G(F)$, thus $W(F,G)=1$.

Assume now that G/F is trialitarian. Since Chernousov and Platonov have shown that such a group is an F-rational variety [2, §8], we conclude that $W(F,G) = 1$.

3. Bounded generation by restricted elements

We assume that $\mathrm{char}(F) \neq 2$ and for convenience that F is perfect and infinite. In [6], Prasad gives an explicit description of $W(F,G)$ in terms of the the Tits algebra of G, which is the Allen algebra $M_2(D)$ for D a quaternion division algebra over K satisfying $\mathrm{cor}_{K/F}[D] = 0 \in \mathrm{Br}(F)$, where K is a cubic étale extension of F. We have

$$W(F,G) = U/\langle R\rangle,$$

where U is the group of elements of the quaternion algebra D/K whose reduced norm is in $F^\times$, and R is the set of elements $x \neq 0$ for which both the reduced norm and the reduced trace are in F. Combined with Theorem 1.1, we get the

Corollary 3.1. $\langle R\rangle = U$.

This leaves open the question of bounding the number of generators from R required to express every element of U.

One may consider the same question when K is a cubic étale extension which is not a field, namely, $K = F \times L$ for L a quadratic field extension of F, or $K = F \times F \times F$, and D is an Azumaya algebra over K. In the former case, $D = D_1 \times D_2$ where D_1 is a quaternion algebra over F and D_2 a quaternion algebra over L, with $\mathrm{cor}_{L/F} D_2 \sim D_1$. In the latter, $D = D_1 \times D_2 \times D_3$, where D_i

($i = 1, 2, 3$) are quaternion algebras over F, and $D_1 \otimes_F D_2 \otimes_F D_3 \sim F$. The sets V and R can be defined in the same manner as above.

This is not an artificial generalization: extending scalars from F to $\tilde{F} = K$, the algebra becomes $\tilde{D} = D \otimes_F K$ which is an Azumaya algebra over $\tilde{K} = K \otimes_F K$, and $\tilde{K}$ is a cubic étale extension of $\tilde{F}$, which is not a field.

Theorem 3.2 ([7, §2])**.** *When K is not a field, every element of U is a product of at most 3 elements of R.*

On the other hand, by means of generic counterexamples, one can show that 3 is the best possible:

Proposition 3.3 ([7, Cor. 4.0.4])**.** *Let $F = \mathbb{Q}(\eta, \lambda)$, $K = F \times F \times F$, and $D = (\alpha, \eta + 1)_F \times (\alpha, \lambda)_F \times (\alpha, (\eta + 1)\lambda)_F$, where $\alpha = \eta^2 - 4$. Let x_i, y_i ($i = 1, 2, 3$) be standard generators for the i'th component.*

Then the element $v = ((\eta + x_1)(\eta + 2 + 2y_1), \eta(1 + x_2), 2\eta) \in D_1 \times D_2 \times D_3$ is in V, but not in $R \cdot R$. In particular $V \not\subseteq R \cdot R$.

Another explicit counterexample [7, Cor. 4.0.4] shows that $V \not\subset R \cdot R$ when $K = F \times L$. By means of extending scalars [7, §5], it also follows that $V \not\subset R \cdot R$ when K is a field.

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