Classification of torsors over Laurent polynomials

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(joint work with Vladimir Chernousov, Arturo Pianzola)

Abstract. I will report on joint work with V. Chernousov and A. Pianzola [6]. Given a linear algebraic group G defined over a field k of characteristic zero, G-torsors over Laurent polynomial rings naturally occur in infinite dimensional Lie theory (e.g., the classification and the proof of conjugacy of Cartan subalgebras of Extended Affine Lie Algebras [5, 7]). We explain that one can associate to such a G-torsor another G-torsor, called its loop form, and how that construction clarifies the classification problem of all G-torsors. At the end we will discuss analogies with Brion's theory of homogeneous torsors over abelian varieties.

Let k be a field of characteristic zero and let k_s be a separable closure of k. Let G be a linear algebraic group defined over k. We are interested in the classification of G-torsors over the ring $R_n = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ of Laurent polynomials ($n \geq 1$), that is of principal G–bundles over the torus $(\mathbb{G}_m)^n = \mathrm{Spec}(R_n)$. The G-torsors are classified by the étale cohomology pointed set $H^1(R_n, G)$. Our goal is to compute $H^1(R_n, G)$, far from an easy task. In the case of the linear group GL_d, we have that $H^1(R_n, \mathrm{GL}_d) = 1$; this set classifies finitely generated projective R_n-modules of rank d and those modules are free according to Quillen-Suslin-Swan's theorem. The motivation comes from the example of the algebraic group $\mathrm{Aut}(\mathfrak{g})$ where $\mathfrak{g}$ is a finite dimensional split simple Lie k-algebra. In this case, the yoga of forms shows that the set $H^1(R_n, \mathrm{Aut}(\mathfrak{g}))$ classifies isomorphism classes of R_n-forms of $\mathfrak{g}$, that is of Lie R_n-algebras $\mathcal{L}$ with the property that there exists a flat cover S/R_n such that $\mathcal{L} \otimes_{R_n} S \cong \mathfrak{g} \otimes_k S$.

Examples. (a) If $n = 1$ and $k = \mathbb{C}$, the centreless core of an affine Kac-Moody algebra is a R_1-algebra which is R_1-form of some $\mathfrak{g}$.
(b) More generally, for $n \geq 1$ and $k = \mathbb{C}$, the centreless core of an extended affine Lie algebra is a R_n-algebra which is R_n-form of some $\mathfrak{g}$ provided it is finitely generated over its centroid [1, th. 3.3.1].

In both cases, the R_n-Lie algebras occurring carry a grading and are of some are examples of so-called (multi)loop algebras. For defining the so-called loop G-torsors, we introduce the ring R_n^{sc} which is the universal cover of R_n (in the sense of SGA 1). It plays the role of the Galois closure for a field. We have $R_n = \mathrm{limind}_m k_s[t_1^{\frac{1}{m}}, \dots, t_n^{\frac{1}{m}}]$, where m runs over the positive integers. The fundamental group of R_n is $\pi_1(R_n) = Aut_{R_n}(R_n^{sc})$. It is a profinite group which is isomorphic to the projective limit of the $\mu_m(k_s)^n \times \mathrm{Gal}(k_s/k)$. There is a natural isomorphism

$$(*) \qquad H^1(\pi_1(R_n), G(R_n^{sc})) \cong \ker\Big(H^1(R_n, G) \to H^1(R_n^{sc}, G)\Big)$$

where the right hand side is the (continuous) group non-abelian cohomology set of $\pi_1(R_n)$ with coefficients in the $\pi_1(R_n)$-group $G(R_n^{sc})$. Note that the projection map $\pi_1(R_n) \to \mathcal{G}al(k_s/k)$ gives rise to an action of $\pi_1(R_n)$ on $G(k_s)$.

Definition. A G-torsor E is a *loop* G-torsor if its class belongs to the image of the map

$$H^1(\pi_1(R_n), G(k_s)) \to H^1(\pi_1(R_n), G(R_n^{sc})) \hookrightarrow H^1(R_n, G).$$

We denote by $H^1_{loop}(R_n, G) \subseteq H^1(R_n, G)$ the subset of classes of loop torsors.

Remarks (a) The map $H^1(\pi_1(R_n), G(k_s)) \to H^1(\pi_1(R_n), G(R_n^{sc}))$ has no reason to be injective nor surjective. In case $n = 1$, it is true that all G-torsors are loop but for $n \geq 2$, there are exotic (=not loop) G-torsors for example for $G = \mathrm{PGL}_d$ [11, §3].
(b) The acyclicity theorem states that $H^1(R_n^{sc}, G) = 1$ [12] so that we have a bijection $H^1(\pi_1(R_n), G(R_n^{sc})) \cong H^1(R_n, G)$. In other words, the cohomology set $H^1(R_n, G)$ can be computed by means of Galois cohomology cocycles.

Example. Let q be a regular quadratic form over k and consider the orthogonal group $O(q)$. Then $H^1(R_n, O(q))$ classifies regular quadratic R_n-forms of rank $\dim(q)$. By analogy, we can call loop quadratic forms the quadratic forms whose underlying cohomology class is loop. Loop quadratic forms are those of the following form

$$\bigoplus_{I \subseteq \{1,\dots,n\}}^{\perp} q_I \otimes \langle t_I \rangle$$

where q_I is a regular quadratic k–form and $t_I = \prod_{i \in I} t_i$. In dimension 4 and $n = 2$, there are exotic quadratic forms (Ojanguren-Sridharan's construction), and those are not diagonalizable.

Our main result of [6] states that there is a map $H^1(R_n, G) \to H^1_{loop}(R_n, G)$, $\gamma \mapsto \gamma^{loop}$ such that for each class γ, then γ and γ^{loop} coincide locally for the Zariski topology. Such a map is unique and we note that $\gamma^{loop} = (\gamma^{loop})^{loop}$.

For quadratic forms, this implies that we can associate to a regular R_n-quadratic form q a unique diagonalizable R_n-quadratic form q^{loop} such that q and q^{loop} are locally isometric with respect to the Zariski topology.

Remarks (a) We denote by $F_n = k((t_1))\dots((t_n))$ the field of iterated Laurent series. The retraction is defined by using that the composite map

$$H^1_{loop}(R_n, G) \hookrightarrow H^1(R_n, G) \to H^1(F_n, G)$$

is an isomorphism. This map is then not easy to manipulate and the reason to work with F_n is the use of Bruhat-Tits theory. The crucial step is to show that γ and γ^{loop} coincide rationally, something that is accomplished by using the technique of unramified Galois cohomology developed by Colliot-Thélène and Sansuc [9, §3]. To conclude that γ and γ^{loop} coincide locally for the Zariski topology requires Fedorov-Panin's theorem [10] (former Grothendieck-Serre's injectivity conjecture).
(b) The result shows that the classification of G-torsors requires two steps. The first one is the classification of loop torsors and the second one is to compute the Zariski topology cohomology set $H^1_{Zar}(R_n, {}^E G)$ for each loop G-torsor E where ${}^E G$ stands for the twisted R_n-group scheme.

If k is algebraically closed, the first step has been done completely for $n = 2$ [13] and the second step in classical cases (in rank large enough) by Steinmetz-Ziketch [14]. For $n \geq 3$, there are partial results for step one and not much is known concerning the second question beyond the fact that $H^1_{Zar}(R_n, G) = 1$ [12].

From now on, we assume that k is algebraically closed and that G is reductive. The notion of loop torsors is not specific to Laurent polynomials and can be defined for an arbitrary geometrically connected k-variety [13, §3.1].

It turns out that M. Brion investigated loop torsors in the case of an abelian variety X [2]. He proved that a G-torsor E over X is loop if and only if it is homogeneous, that is its class in $H^1(X, G)$ is invariant by the action of the group $X(k)$ induced by the left translations (*ibid*, Th. 1.1). A natural question is whether such a characterization (or a suitable variant) holds over R_n which is the coordinate ring of the torus $(\mathbb{G}_m)^n$. Conversely, one can ask whether the isotriviality statement over R_n has some counterpart over the abelian variety X.

Question. Let E be a G-torsor over X. Does there exists a finite étale cover $X' \to X$ such that the G-torsor $E \times_X X'$ over X' is locally trivial for the Zariski topology.

Remarks (a) There is a similar question raised by Colliot-Thélène and Iyer for homogeneous projective G-schemes over X [8, Question 3.4].

(b) In the case of PGL_d, the question admits a positive answer due to Campana-Peternell [3, Lemma 7.4].

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