TORSORS OVER AFFINE CURVES, PCMI, JULY 2021// PRELIMINARY VERSION

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Abstract.

Contents

Date: July 12, 2021.

1. Introduction

The theory of fibrations and principal fibrations is ubiquous in Topology and Differential Geometry In 1955, Grothendieck investigated a general theory of fibrations focusing on functoriality issues [21]. In 1958, Grothendieck and Serre extended the setting of G–bundles in algebraic geometry by means of the étale topology [33].

For simplicity we shall present this theory over rings or equivalently over affine schemes. The general framework is close to that and can be found in other references [10, 25, 6].

We shall focuss on the case of affine smooth curve over a field, starting with vector bundles and quadratic vector bundles.

2. The Swan-Serre correspondence

This is the correspondence between projective finite modules of finite rank and vector bundles, it arises from the case of a paracompact topological space [37].

We explicit it in the setting of affine schemes following the book of Görtz-Wedhorn [18, ch. 11] up to slightly different conventions.

2.1. **Vector group schemes.** Let R be a ring (commutative, unital).

(a) Let M be an R–module. We denote by $\mathbf{V}(M)$ the affine R–scheme defined by $\mathbf{V}(M) = \operatorname{Spec}\big(\operatorname{Sym}^\bullet(M)\big)$; it is affine over R and represents the R–functor $S \mapsto \operatorname{Hom}_S(M \otimes_R S, S) = \operatorname{Hom}_R(M, S)$ [11, 9.4.9].

It is called the *vector group scheme* attached to M, this construction commutes with arbitrary base change of rings $R \to R'$.

Proposition 2.1. [32, I.4.6.1] *The functor $M \to \mathbf{V}(M)$ induces an antiequivalence of categories between the category of R–modules and that of vector group schemes over R. An inverse functor is $\mathfrak{G} \mapsto \mathfrak{G}(R)$.*

(b) We assume now that M is locally free of finite rank and denote by $M^\vee$ its dual. In this case $\operatorname{Sym}^\bullet(M)$ is of finite presentation (ibid, 9.4.11). Also the R–functor $S \mapsto M \otimes_R S$ is representable by the affine R–scheme $\mathbf{V}(M^\vee)$ which is also denoted by $\mathbf{W}(M)$ [32, I.4.6].

Remark 2.2. Romagny has shown that the finite locally freeness condition on M is a necessary condition for the representability of $\mathbf{W}(M)$ by a group scheme [29, th. 5.4.5].

Let $r \geq 0$ be an integer.

Definition 2.3. *A vector bundle of rank r over $\operatorname{Spec}(R)$ is an affine R-scheme X such that there exists a partition $1 = f_1 + \cdots + f_n$ and isomorphisms $\phi_i : \mathbf{V}((R_{f_i})^r) \xrightarrow{\sim} X \times_R R_{f_i}$ such that $\phi_i^{-1}\phi_j : \mathbf{V}((R_{f_i f_j})^r) \xrightarrow{\sim} \mathbf{V}(R^r_{f_i f_j})$ is a linear automorphism of $\mathbf{V}((R_{f_i f_j})^r)$ for $i, j = 1, \ldots, n$.*

Theorem 2.4. *(Swan-Serre's correspondence) The above functor $M \mapsto \mathbf{V}(M)$ induces an equivalence of categories between the groupoid of locally free R–modules of rank r and the groupoid of vector bundles over* $\mathrm{Spec}(R)$ *of rank r.*

Proof. See [18, prop. 11.7] for the general case. We check first that the functor is well-defined. If M is locally free of rank r, there exists a partition $1 = f_1 + \dots + f_n$ and trivializations $\psi_i : (R_{f_i})^r \xrightarrow{\sim} M_{f_i}$. It follows that the maps $(\psi_i)^{-1}(\psi_j^*) : (R_{f_i f_j})^r \xrightarrow{\sim} (R_{f_i f_j})^r$ is a linear isomorphism for $i, j = 1, \dots, n$. By applying the functor $\mathbf{V}$, we get that $\mathbf{V}(M)$ is a vector bundle of rank r and the trivializations are the $\psi_i : (\psi_i^{-1})^* : \mathbf{V}((R_{f_i})^r) \xrightarrow{\sim} \mathbf{V}(M) \times_R R_{f_i}$ So $\mathbf{V}$ is well-defined and is fully faithful. To check it is essentially surjective, it is enough to observe that the inverse functor $\mathfrak{G} \to \mathfrak{G}(R)$ of $\mathbf{V}$ applies a vector bundle of rank r to a locally free R–module of rank r. □

Examples 2.1.1. (a) Given a smooth map of affine schemes $X = \mathrm{Spec}(S) \to Y = \mathrm{Spec}(R)$ of relative dimension $r \geq 1$, the tangent bundle $T_{X/Y} = \mathbf{V}(\Omega^1_{S/R})$ is a vector bundle over $\mathrm{Spec}(S)$ of dimension r [12, 16.5.12].

(b) The tangent bundle of the real sphere $Z = \mathrm{Spec}\Big(\mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)\Big)$ is an example of vector bundle of dimension 2 which is not trivial. It can be proven by differential topology (hairy ball theorem) but there are also algebraic proofs, see for instance [38]. A consequence is that Z cannot be equipped with a structure of real algebraic group.

2.2. **Linear groups.** Let M be a locally free R–module of finite rank. We consider the R–algebra $\mathrm{End}_R(M) = M^\vee \otimes_R M$. It is locally free R–module of finite rank so that we can consider the vector R–group scheme $\mathbf{V}\big(\mathrm{End}_R(M)\big)$ which is an R–functor in associative and unital algebras [11, 9.6.2]. Now we consider the R–functor $S \mapsto \mathrm{Aut}_S(M \otimes_R S)$. It is representable by an open R–subscheme of $\mathbf{W}\big(\mathrm{End}_R(M)\big)$ which is denoted by $\mathrm{GL}(M)$ (*loc. cit.*, 9.6.4). We bear in mind that the action of the group scheme $\mathrm{GL}(M)$ on $\mathbf{W}(M)$ (resp. $\mathbf{V}(M)$) is a left (resp. right) action.

In particular, we denote by $\mathrm{GL}_r = \mathrm{Aut}(R^r)$.

Remark 2.5. For R noetherian, Nitsure has shown that the finite locally freeness condition on M is a necessary condition for the representability of $\mathrm{GL}(M)$ by a group scheme [28].

(c) If $\mathcal{B}$ is a locally free $\mathcal{O}_S$–algebra of finite rank, we recall that the functor of invertible elements of $\mathcal{B}$ is representable by an affine S-group scheme which is a principal open subset of $\mathbf{W}(\mathcal{B})$. It is denoted by $\mathrm{GL}_1(\mathcal{B})$ [6, 2.4.2.1].

2.3. **Cocycles.** Let M be a locally free R–module of rank r. We consider a partition $1 = f_1 + \dots + f_n$ and isomorphisms $\phi_i : (R_{f_i})^r \xrightarrow{\sim} M \times_R R_{f_i}$. Then the $R_{f_i f_j}$–isomorphism $\phi_i^{-1}\phi_j : (R_{f_i f_j})^r \xrightarrow{\sim} (R_{f_i f_j})^r$ is linear so defines an element $g_{i,j} \in \mathrm{GL}_r(R_{f_i f_j})$. More precisely we have $(\phi_i^{-1}\phi_j)(v) = g_{i,j} \,.\, v$ for each $v \in (R_{f_i f_j})^r$ (in other words, $(R_{f_i f_j})^r$ is seen as column vectors).

Lemma 2.6. *The element $g = (g_{i,j})$ is a 1–cocycle, that is, satisfies the relation*

$$g_{i,j}\, g_{j,k} = g_{i,k} \in \mathrm{GL}_r(R_{f_i f_j f_k})$$

for all $i, j, k = 1, \dots, n$.

Proof. Over $R_{i,j,k}$ we have $\phi_i^{-1}\phi_k = (\phi_i^{-1}\phi_j) \circ (\phi_j^{-1}\phi_k) = L_{g_{i,j}} \circ L_{g_{j,k}} = L_{g_{i,j}g_{j,k}}$. □

If we replace the ϕ_i's by the $\phi'_i = \phi_i \circ g_i$ for $g_i \in \mathrm{GL}_r(R_{f_i})$, we get $g'_{i,j} = g_i^{-1} g_{i,j} g_j$ and we say that $(g'_{i,j})$ is cohomologous to $(g_{i,j})$.

We denote by $\mathcal{U} = (\mathrm{Spec}(R_{f_i})_{i=1,..,n}$ the affine cover of $\mathrm{Spec}(R)$, by $Z^1(\mathcal{U}/R, \mathrm{GL}_r)$ the set of 1-cocycles and by $H^1(\mathcal{U}/R, \mathrm{GL}_r) = Z^1(\mathcal{U}/R, \mathrm{GL}_r)/\sim$ the set of 1–cocycles modulo the cohomology relation. The set $H^1(\mathcal{U}/R, \mathrm{GL}_r)$ is called the pointed set of Čech cohomology with respect to $\mathcal{U}$.

Summarizing we attached to the vector bundle $\mathbf{V}(M)$ of rank r a class $\gamma(M) \in H^1(\mathcal{U}/R, \mathrm{GL}_r)$.

Conversely by Zariski glueing, we can attach to a cocycle $(g_{i,j})$ a vector bundle $\mathbf{V}_g$ over R of rank r equipped with trivializations $\phi_i : \mathbf{V}(R^r_{f_i}) \xrightarrow{\sim} \mathbf{V}_g \times_R R_{f_i}$ such that $\phi_i^{-1}\phi_j = g_{i,j}$.

Lemma 2.7. *The pointed set $H^1(\mathcal{U}/R, \mathrm{GL}_r)$ classifies the isomorphism classes of vector bundles of rank r over $\mathrm{Spec}(R)$ which are trivialized by $\mathcal{U}$.*

For the proof, see [18, 11.15]. We can pass the limit of this construction over all affine open subsets of X. We define the pointed set $\check{H}^1_{Zar}(R, \mathrm{GL}_r) = \varinjlim_{\mathcal{U}} H^1(\mathcal{U}/R, \mathrm{GL}_r)$ of Čech non–abelian cohomology of GL_n with respect to the Zariski topology of $\mathrm{Spec}(R)$. By passage to the limit, Lemma 2.7 implies that $\check{H}^1_{Zar}(R, \mathrm{GL}_r)$ classifies the isomorphism classes of vector bundles of rank r over $\mathrm{Spec}(R)$.

2.4. **Functoriality.** The principle is that nice constructions for vector bundles arise from homomorphisms of group schemes. Given a map $f : \mathrm{GL}_r \to \mathrm{GL}_s$, we can attach to a vector bundle $\mathbf{V}_g$ of rank r the vector bundle $\mathbf{V}_{f(g)}$ of rank s. This extends to a functor $X \mapsto f_*(X)$ from vector bundles of rank r to vector bundles to rank s.

We consider now the three following cases.

(a) *Direct sum.* If $r = r_1 + r_2$, we consider the map $f : \mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2} \to \mathrm{GL}_r$, $(A_1, A_2) \mapsto A_1 \oplus A_2$. We have then $f_*(\mathbf{V}_1, \mathbf{V}_2) = \mathbf{V}_1 \oplus \mathbf{V}_2$.

Of course, it can be done with $r = r_1 + \cdots + r_l$, in particular we have in the case $r = 1 + \cdots + 1$ the diagonal map $(\mathbb{G}_m)^r \to \mathrm{GL}_r$ which leads to decomposable vector bundles, that is, direct sum of rank one vector bundles.

(b) *Tensor product.* If $r = r_1 r_2$, we consider the map $f : \mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2} \to \mathrm{GL}_r$, $(A_1, A_2) \mapsto A_1 \otimes A_2$ (called the Knonecker product). We have then $f_*(\mathbf{V}_1, \mathbf{V}_2) = \mathbf{V}_1 \otimes \mathbf{V}_2$.

(c) *Determinant.* We put $\det(\mathbf{V}) = \det_*(\mathbf{V})$, this is the determinant bundle.

2.5. **The case of a Dedekind ring.** Let R be a Dedekind ring, that is, a noetherian domain such that the localization at each maximal ideal is a discrete valuation ring. The next result is a classical fact of commutative algebra, see [20, II.4, th. 13].

Theorem 2.8. *A locally free R–module of rank $r \geq 1$ is isomorphic to $R^{r-1} \oplus I$ for I an invertible R–module which is unique up to isomorphism.*

Since I is the determinant of $R^{r-1} \oplus I$, the last assertion is clear. Our goal is to provide a geometric proof of this statement.

Firstly it states that vector bundles over R are decomposable and secondly that vector bundles over R are classified by their determinant. We limit ourself to prove the following corollary.

Corollary 2.9. *A locally free R–module of rank $r \geq 1$ is trivial if and only its determinant is trivial.*

Proof. We are given a vector bundle $\mathbf{V}(M)$. It trivializes over an open affine subset $\mathrm{Spec}(R_f)$ and we put $\Sigma = \mathrm{Spec}(R) \setminus \mathrm{Spec}(R_f) = \{\mathbf{p}_1, \dots, \mathbf{p}_c\}$ where the $\mathbf{p}_j$'s are maximal ideals of R. Let $\widehat{R}_{\mathbf{p}_j}$ be the completion of the DVR $R_{\mathbf{p}_i}$ and denote by $\widehat{K}_{\mathbf{p}_j} = K \otimes_R \widehat{R}_{f_i}$ its fraction field.

According to Nakayama lemma, the $\widehat{R}_{\mathbf{p}_i}$–module $M \otimes_R \widehat{R}_{\mathbf{p}_i}$ is free so we pick a trivialization $\phi_i : (\widehat{R}_{\mathbf{p}_i})^r \xrightarrow{\sim} M \times_R \widehat{R}_{\mathbf{p}_i}$.

On the other hand, let $\phi_f : (R_f)^r \xrightarrow{\sim} M \times_R R_f$ a trivialization. The linear map $\phi_f^{-1}\widehat{\phi}_i : (\widehat{K}_{\mathbf{p}_i})^r \to (\widehat{K}_{\mathbf{p}_i})^r$ gives rise to an element $g_i \in \mathrm{GL}_r(\widehat{K}_{\mathbf{p}_i})$. Taking into account the choices, we attached to M an element of the double coset

$$c_\Sigma(R, \mathrm{GL}_r) := \mathrm{GL}_r(R_f) \backslash \prod_{j=1,\dots,c} \mathrm{GL}_r(\widehat{K}_{\mathbf{p}_i}) / \mathrm{GL}_r(\widehat{R}_{\mathbf{p}_i}).$$

Claim 2.10. *The map*

$$\ker\Big(H^1(R, \mathrm{GL}_r) \to H^1(R_f, \mathrm{GL}_r)\Big) \to c_\Sigma(R, \mathrm{GL}_r)$$

is injective.

For the sequel we need only to know that it has trivial kernel. Indeed if (g_i) belongs in the kernel, it means that we can adjust the trivializations in order to get $g_i = 1$ for $i = 1, \dots, c$. We claim that the isomorphism $\phi_f : M_f \xrightarrow{\sim} (R_f)^r$ extends (uniquely) to an isomorphism $M \xrightarrow{\sim} R^r$. Since the map $\phi_f : M_f^r \xrightarrow{\sim} (R_f)^r$ extended over $\widehat{K}_{\mathbf{p}_i}$ extends to $\widehat{R}_{\mathbf{p}_i}$, it means that there are no denominators involved so that the map extends ϕ_f to an R–linear mapping $\psi : M^r \to R^r$. For the same reason $(\phi_f)^{-1}$ extends as well and we conclude that ϕ_f extends to an R–linear isomorphism $\psi : M^r \xrightarrow{\sim} R^r$.

We assume now that the determinant of $\mathbf{V}(M)$ is trivial so that (g_i) belongs by functoriality to the kernel of the map $\det_* : c_\Sigma(R, \mathrm{GL}_r) \to c_\Sigma(R, \mathbb{G}_m) = R_f^\times \backslash \prod_{j=1,\dots,c} (\widehat{K}_{\mathbf{p}_i}^\times / \widehat{R}_{\mathbf{p}_i}^\times)$.

Up to change the trivializations we can then assume that $g_i \in \mathrm{SL}_n(\widehat{K}_{\mathbf{p}_i})$ for $i = 1,\dots,c$. Since $\mathrm{SL}_n(\widehat{K}_{\mathbf{p}_i})$ is generated by elementary matrices and since R_f is dense in $\prod_i \widehat{K}_{\mathbf{p}_i}$, it follows that $\mathrm{SL}_r(R_f)$ is dense in $\prod_{i=1,\dots,c} \mathrm{SL}_r(\widehat{K}_{\mathbf{p}_i})$. On the other hand, each group $\mathrm{SL}_n(\widehat{R}_{\mathbf{p}_i})$ is open (actually clopen) in $\mathrm{SL}_r(\widehat{K}_{\mathbf{p}_i})$ so that $c_\Sigma(R, \mathrm{SL}_r) = 1$. The Claim 2.10 enables us to conclude that $\mathbf{V}(M)$ is a trivial vector bundle. □

Remarks 2.11. (a) The general case is close; we need to apply the previous argument to $\mathrm{GL}(R^{r-1} \oplus I)$ for an invertible R–module I.

(b) $c_\Sigma(R, \mathbb{G}_m) = \mathrm{Div}_\Sigma(R)/R_f^\times$ is isomorphic to $\ker\bigl(\mathrm{Pic}(R) \to \mathrm{Pic}(R_f)\bigr)$. This is a general fact, i.e. the map of Claim 2.10 is surjective. It can be seen by using patching techniques; more elementary one can use the fact that $\mathrm{GL}_r(K)^r$ maps onto $\prod_{j=1,\dots,c} \mathrm{GL}_r(\widehat{K}_{\mathbf{p}_i})/\,\mathrm{GL}_r(\widehat{R}_{\mathbf{p}_i})$.

3. Zariski topology is not fine enough

The above definition of non–abelian cohomology extends for an arbitrary group scheme. There are several complementary reasons for try to extend this theory.

3.1. **The example of quadratic bundles.** A quadratic form over an R–module M is a map $q : M \to R$ which satisfies

(i) $q(\lambda x) = \lambda^2 q(x)$ for all $\lambda \in R$, $x \in M$.

(ii) The form $M \times M \to R$, $(x, y) \mapsto b_q(x, y) = q(x+y) - q(x) - q(y)$ is (symmetric) bilinear.

This is stable by arbitrary base change. The form q is *regular* if b_q induces an isomorphism $M \xrightarrow{\sim} M^\vee$. A fundamental example is the hyperbolic form $(V \oplus V^\vee, hyp)$ attached to a locally free R–module of finite rank defined by $hyp(v, \phi) \to \phi(v)$.

We are given a regular quadratic form (M, q) where M is locally free of rank r. It is tempting to make analogies with vector bundles and to use the orthogonal group scheme $O(q, M)$ which a closed subgroup scheme of $\mathrm{GL}(M)$. More precisely, we have

$$O(q, M)(S) = \bigl\{ g \in \mathrm{GL}(M)(S) = \mathrm{Aut}_S(M \otimes_R S), \mid\ q_S \circ g = q_S \bigr\}$$

for each R–ring S. For an open cover $\mathcal{U}$ of R as above we define similarly $Z^1(\mathcal{U}/R, O(q, M))$ and $H^1(\mathcal{U}/R, O(q, M))$ (it makes sense for any R–group scheme). What we get is the following.

Lemma 3.1. *The set $H^1_{Zar}(\mathcal{U}/R, O(q, M))$ classifies the isomorphism of regular quadratic forms (q', M') which are locally isomorphic over $\mathcal{U}$ to (q, M).*

This is nice but the point is that regular quadratic forms over R of dimension r have no reason to be locally isomorphic to (M, q) (e.g. this occurs already with $R = \mathbb{R}$, the field of real numbers). So the set $H^1(R, O(q, M))$ is only a piece of what we would like to obtain.

3.2. **Functoriality.** If we have a map $f : G \to H$ of group schemes, we would like to have some control on the map $f_* : H^1_{Zar}(R, G) \to H^1_{Zar}(R, H)$.

A basic example is the Kummer map $f_d : \mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^d$ for an integer d. It gives rise to the multiplication by d mapping on the Picard group $\mathrm{Pic}(R)$. In terms of invertible modules, it corresponds to the map $M \mapsto M^{\otimes d}$.

We would like to understand its kernel and its image. We can already say something about the kernel. Given $[M] \in \ker\big(\mathrm{Pic}(R) \xrightarrow{\times d} \mathrm{Pic}(R)\big)$, then there exists a trivialization $\theta : R \xrightarrow{\sim} M^{\otimes d}$. We define then the commutative group $A_d(R)$ of isomorphism classes of couples (M, θ) where M is an invertible R–module equipped with a trivialization $\theta : R \xrightarrow{\sim} M^{\otimes d}$. We have a forgetful map $A(R) \to \mathrm{Pic}(R)$ and we claim that we have an exact sequence

$$R^\times/(R^\times)^d \xrightarrow{\phi} A_d(R) \to \mathrm{Pic}(R) \xrightarrow{\times d} \mathrm{Pic}(R)$$

with $\phi(a) = [(R, \theta_a)]$ where $\theta_a : R \xrightarrow{\sim} R^{\otimes d} = R$, $x \mapsto ax$. We let this as exercise to the reader. We will see later that we can provide a cohomological meaning to the group $A_d(R)$ (Remark 4.11).

4. General definitions

Grothendieck-Serre's idea is to extend the notion of covers in algebraic geometry. They did it originally with étale covers but it turns out that the flat cover setting is simpler in a first approach (this is that of the book by Demazure-Gabriel [10, §III], there are other variants).

4.1. Čech non-abelian cohomology.

Definition 4.1. *A flat (or fppf= fidèlement plat de présentation finie) cover of R is a finite collection $(S_i)_{i\in I}$ of R–rings satisfying*

(i) S_i is a flat R–algebra of finite presentation for $i = 1, \dots, c$;

(ii) $\mathrm{Spec}(R) = \bigcup_{i\in I} \mathrm{Im}\Big(\mathrm{Spec}(S_i) \to \mathrm{Spec}(R)\Big)$

If we put $S = \prod_{i\in I} S_i$, the conditions rephrase by saying that S is a faithfully flat R–algebra of finite presentation. We can then always deal with a unique ring.

Remark 4.2. *For a partition $1 = f_1 + \cdots + f_n$, then $(R_{f_j})_{j=1,\dots,n}$ is a flat cover of R and so is $R_{f_1} \times \cdots \times R_{f_n}$.*

We define firstly non abelian cohomology. Let S is a faithfully flat R–algebra of finite presentation. We denote by $p_i^* : S \to S \otimes_R S$ the coprojections $(i = 1, 2)$ and similarly $q_i^* : S \to S \otimes_R S \otimes_R S$ $(i = 1, 2, 3)$, $q_{i,j}^* : S \otimes_R S \to S \otimes_R S \otimes_R S$ the partial coprojections $(i < j)$.

Let G be an R–group scheme. A 1-*cocycle* for G and S/R is an element $g \in G\big(S \otimes_R S\big)$ satisfying

$$q_{1,2}^*(g)\, q_{2,3}^*(g) = q_{1,3}^*(g) \quad \in \quad G\big(S \otimes_R S \otimes_R S\big).$$

We denote by $\mathrm{Z}^1(S/R, G)$ the pointed set of 1-cocycles of S/R with values in G (it is pointed by the trivial 1–cocycle).

Two such cocycles $g, g' \in G(S)$ are cohomologous if there exists $h \in G(S)$ such that $g = p_1^*(h^{-1})\, g'\, p_2^*(h)$. We denote by $\check{\mathrm{H}}^1(S/R, G) = \mathrm{Z}^1(S/R, G)/\sim$ the pointed set of 1-cocycles up to cohomology equivalence.

Remark 4.3. In the case of a Zariski cover given by a partition of 1, the definition is the same as in §3.1. What is behind is that intersection of open subschemes is a special case of fiber product.

We can pass to the limit on all flat covers of $\mathrm{Spec}(R)$ and define $\check{\mathrm{H}}^1_{fppf}(R, G) = \varinjlim \check{\mathrm{H}}^1(S/R, G)$ [1]. This construction is functorial in R and in the group scheme G.

4.2. **Torsors.** A (right) G–torsor X (with respect to the flat topology) is an R-scheme equipped with a right action of G which satisfies the following properties:

(i) the action map $X \times_R G \to X \times_R X$, $(x, g) \mapsto (x, x.g)$, is an isomorphism;

(ii) There exists a flat cover R'/R such that $X(R') \neq \emptyset$.

The first condition reflects the simply transitivity of the action, we mean that $G(T)$ acts simply transitively on $X(T)$ for all R–rings T.

The second condition is a local triviality condition. An example is $X = G$ with G acting by right translations, it is called the split G–torsor.

If $X(R) \neq \emptyset$, a point $x \in X(R)$ defines an morphism $G \to X$, $\phi_x : g \mapsto x.g$ which is an isomorphism by the simple transitive property; we say that X is trivial and that ϕ_x is a trivialization.

Condition (ii) states that an R–torsor X under G is locally trivial for the flat topology.

A morphism of G–torsors $X \to Y$ is a G–equivariant map; once again the simple transitivity condition shows that such a morphism is an isomorphism. Thus the category of G–torsors under G is a groupoid.

The R–functor of automorphisms of the trivial G–torsor G is representable by G (acting by left translations).

We denote by $H^1_{fppf}(R, G)$ the set of isomorphism classes of G–torsors for the flat topology. If S is a flat cover R, we denote by $H^1_{fppf}(S/R, G)$ the subset of isomorphism classes of G–torsors trivialized over S.

As in the vector bundle case, we shall construct a class map $\gamma : H^1_{fppf}(S/R, G) \to \check{H}^1_{fppf}(S/R, G)$ as follows.

Let X be a G–torsor over R equipped with a trivialization $\phi : G \times_R S \xrightarrow{\sim} X \times_R S$. Over $S \otimes_R S$, we have then two trivializations $p_1^*(\phi) : G \times_R (S \otimes_R S) \xrightarrow{\sim} X \times_R (S \otimes_R S)$ and $p_1^*(\phi)$. It follows that $p_1^*(\phi)^{-1} \circ p_2^*(\phi)$ is an automorphism of the trivial G–torsor over $S \otimes_R S$ so is the left translation by an element $g \in G(S \otimes_R S)$. A computation

[1] There are subtle set-theoretic issues there, see [10, III.1.3] and [40]

shows that g is a 1–cocycle [16, §2.2]; also changing ϕ changes g by a cohomologous cocycle. The class map is then well-defined. Its study involves a glueing technique in the flat setting.

4.3. **Interlude: Faithfully flat descent.** Let T be a faithfully flat extension of the ring R (not necessarily of finite presentation). We put $T^{\otimes d} = T \otimes_R T \cdots \otimes_R T$ (d times). One first important thing is that the Amitsur complex

$$0 \to M \to M \otimes_R T \xrightarrow{d_2} M \otimes_R T \otimes_R T \xrightarrow{d_2} M \otimes_R T^{\otimes 3} \dots$$

is exact for each R–module M [25, III.1] where $d_n(m \otimes t_1 \otimes \cdots \otimes t_n) = \sum\limits_{i=0,\dots,n} (-1)^i m \otimes t_1 \otimes \cdots \otimes t_i \otimes 1 \otimes t_{i+1} \otimes \cdots \otimes t_n$. This implies in particular that for any affine R-scheme X, we have an identification

$$X(R) = \big\{ x \in X(T) \mid p_1^*(x) = p_2^*(x) \in X(T \otimes_R T) \big\}$$

which holds actually for any R-scheme. Given a T–module N we consider the $T \otimes_R T$–modules $p_1^*(N) = T \otimes_R M$ and $p_2^*(N) = M \otimes_R T$.

A descent data on N is an isomorphism $\varphi : p_1^*(N) \xrightarrow{\sim} p_2^*(N)$ of $T^{\otimes 2}$–modules such that the diagram

$$\begin{array}{ccccc} T \otimes_R T \otimes_R N & & \xrightarrow{\quad \varphi_3 \quad} & & N \otimes_R T \otimes_R T \\ & \searrow^{\varphi_2} & & \nearrow_{\varphi_1} & \\ & & T \otimes_R N \otimes_R T & & \end{array}$$

is commutative where

- $\varphi_3(t_1 \otimes t_2 \otimes n) = \varphi(t_1 \otimes n) \otimes t_2$;
- $\varphi_2(t_1 \otimes t_2 \otimes n) = t_2 \otimes \varphi(t_1 \otimes n)$;
- $\varphi_1(t_1 \otimes n \otimes t_3) = t_1 \otimes \varphi(n \otimes t_3)$

There is a clear notion of morphisms for T–modules equipped with descent data from T to R. If M is an R–module, the identity of M gives rises to a canonical isomorphism $can_M : p_1^*(M \otimes_R T) \xrightarrow{\sim} p_2^*(M \otimes_R T)$, this is a descent data.

Theorem 4.4. *(Faithfully flat descent, see* [25, III, th. 2.1.2] *)*
(1) The functor $M \to (M \otimes_R T, can_M)$ *is an equivalence of categories between the category of* R*–modules and that of* T*–modules with descent data. An inverse functor (the descent functor) is* $(N, \varphi) \mapsto \{n \in N \mid n \otimes 1 = \varphi(1 \otimes n)\}$.
(2) The functor above induces an equivalence of categories between the category of R*–algebras (commutative, unital) and that of* T*–algebras (commutative, unital) with descent data.*

For an exhaustive view, we recommend [39, Tag 023F]. We shall see later examples of descent beyond the case of Zariski covers (e.g. 4.15).

4.4. **The linear case.** An important example is the extension of Swan-Serre's correspondence. A consequence of the faithfully flat descent theorem (and of the fact that the property to be locally free of rank r is local for the flat topology [39, Tag 05B2]) is the following.

Theorem 4.5. *Let* $r \geq 0$ *be an integer.*
(1) Let M *be a locally free* R*–module of rank* r*. Then the* R*–functor* $S \mapsto \mathrm{Isom}_{S-mod}(S^r, M \otimes_R S)$ *is representable by a* GL_r*–torsor* X^M *over* $\mathrm{Spec}(R)$.
(2) The functor $M \mapsto X^M$ *induces an equivalence of categories between the groupoid of locally free* R*–modules of rank* r *and the category of* GL_r*–torsors over* $\mathrm{Spec}(R)$.

Proof. See [6, 2.4.3.1]. □

This implies that the GL_r–torsors are the same with flat topology or with Zariski topology.

Corollary 4.6. *(Hilbert-Grothendieck 90) We have* $H^1_{Zar}(R, \mathrm{GL}_r) = H^1_{fppf}(R, \mathrm{GL}_r)$. *In particular, if* R *is a local (or semilocal ring), we have* $H^1_{fppf}(R, \mathrm{GL}_r) = 1$.

This is a special case of a more general statement which holds for $\mathrm{GL}_1(\mathcal{B})$ where $\mathcal{B}$ is a separable R–algebra (for example Azumaya or finite étale) which is a locally free R–module of finite rank, see [17, §4.2].

4.5. **Torsors and cocycles.**

Lemma 4.7. *The map* $\gamma : H^1_{fppf}(S/R, G) \to \check{H}^1_{fppf}(S/R, G)$ *is injective.*

Proof. Once again we limit ourselves to the kernel for simplicity (for the general argument, see [16, §2.2]). If (X, ϕ) gives rise to a cocycle which is cohomologous to the trivial cocycle, it means that there exists a trivialization $\phi' : G \times_R S \xrightarrow{\sim} X \times_R S$ such that the associated cocycle is trivial. We put $x = \phi'(1) \in X(S)$. Then $p_1^*(x) = p_2^*(x) = 1$. Since $X(R)$ identifies with $\{x \in X(S) \mid p_1^*(x) = p_2^*(x)\}$, we conclude that $X(R)$ is non-empty. □

Theorem 4.8. *If* G *is affine, the class map* $H^1_{fppf}(S/R, G) \to \check{H}^1_{fppf}(S/R, G)$ *is an isomorphism.*

Note that by passing to the limit on the flat covers, we get a bijection $H^1_{fppf}(R,G) \to \check{H}^1_{fppf}(R,G)$.

The fact that we can descend torsors under an affine scheme is a consequence of the faithfully flat descent theorem. The sketch is as follows. We are given a cocycle $g \in G(S \otimes_R S)$. We consider the map $L_g^* : (S \otimes_R S)[G] \xrightarrow{\sim} (S \otimes_R S)[G]$ and define φ_g by the diagram

$$\begin{array}{ccc} S \otimes_R S[G] & \xrightarrow[\sim]{\varphi_g} & S[G] \otimes_R S \\ \cong \downarrow \alpha & & \cong \downarrow \beta \\ (S \otimes_R S)[G] & \xrightarrow[\sim]{L_g^*} & (S \otimes_R S)[G] \end{array}$$

where $\alpha(s_1 \otimes f) = (s_1 \otimes 1) p_2^*(f)$ and $\beta(f \otimes s_2) = p_1^*(f)(1 \otimes s_2)$. The cocycle condition implies that ϕ_g is a descent data for the S–algebra $S[G]$. Theorem 4.4 defines an R–algebra $R[X]$ and X is actually a G–torsor denoted by E_g.

This construction is a special case of *Twisting*. More generally, if Y is an affine R–scheme equipped with a left action of G, then the action map $g : Y \times_R (S \otimes_R S) \xrightarrow{\sim} Y \times_R (S \otimes_R S)$ defines a descent data. This gives rises to the twist of Y_g of Y by the one cocycle g. It is affine over R.

A special case is the action of G on itself by inner automorphisms, G_g is called the twisted R–group scheme; it acts on Y_g for Y as above.

Remarks 4.9. (a) The above construction do not depend of choices of cocycles or of trivializations. We can define for a G–torsor E the twist ${}^E Y$ and ${}^E G$.

(b) In practice, the affiness assumption is too strong. More generally we can twist G–schemes equipped with an ample invertible G-linearized bundle, see [5, §6, th. 7 and §10, lemma 6] for details).

4.6. **Examples.** *(a) Vector group schemes.* Let M be a finite locally free R–module of finite rank, we claim that $\check{H}^1(R, \mathbf{W}(M)) = 0$ so that each $\mathbf{W}(M)$–torsor is trivial. We are given a flat cover S/R. Since the complex $M \otimes_R S \xrightarrow{p_1^* - p_2^*} M \otimes_R S \otimes_R S \to M \otimes_R S \otimes_R S \otimes_R S$ is exact, each cocycle $g \in \mathbf{W}(M)(S \otimes_R S) = M \otimes_R S \otimes_R S$ is a coboundary. Thus $\check{H}^1(S/R, \mathbf{W}(M)) = 0$ and $\check{H}^1(R, \mathbf{W}(M)) = 0$.

(b) An important case is when $G = \Gamma_R$, that is, the *finite constant group scheme* attached to an abstract finite group Γ. We mean that $G(S)$ is the group of locally constant functions $\operatorname{Spec}(S) \to \Gamma$. In other words, $G = \sqcup_{\gamma \in \Gamma} \operatorname{Spec}(R)_\gamma$ so that its coordinate ring identifies with $R^{(\Gamma)}$.

In this case a Γ_R–torsor $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is the same thing than a Galois Γ–algebra S and is called often a Galois cover. A special case is that of a finite Galois extension L/k of fields of group Γ.

(c) As for GL_r, a special nice case is the case of *forms*, that is when G is the automorphism group of some algebraic structure, see [6, §2.2.3] for an exhaustive discussion.

For example, the orthogonal group scheme O_{2n} is the automorphism group of the hyperbolic quadratic form attached to R^n. As regular quadratic forms of rank $2n$ are locally isomorphic to the hyperbolic form for the flat topology, descent theory provides an equivalence of categories between the groupoid of regular quadratic forms of rank $2r$ and $H^1_{fppf}(R, O_{2n})$. This is what we wanted in §3, that is, $H^1(R, O_{2n})$ classifies the isomorphism classes of regular quadratic R–forms of rank $2n$ [10, III.5.2].

(d) Another important example is that of the symmetric group S_n. For any R–algebra S, the group $S_n(S)$ is the automorphism group of the S–algebra $S^n = S \times \cdots \times S$ (n–times). Since finite étale algebras of degree n are locally isomorphic to R^d for the étale topology, the same yoga shows that there is an equivalence of categories between the category of S_n–torsors and that of finite étale R–algebras of rank n.

The inverse functor is defined by descent but can be described explicitely. This is the Galois closure construction done by Serre in [33, §1.5], see also [2].

4.7. **Functoriality issues.** Let $G \to H$ be a monomorphism of R–group schemes. We say that an R-scheme X equipped with a map $f : H \to X$ is a *flat quotient* of H by G if for each R–algebra S the map $H(S) \to X(S)$ induces an injective map $H(S)/G(S) \hookrightarrow X(S)$ and if for each $x \in X(S)$, there exists a flat cover S' of S such that $x_{S'}$ belongs to the image of $H(S') \to X(S')$ (we say that f is "couvrant" in French). If it exists, a flat quotient is unique (up to unique isomorphism); furthermore, if G is normal in H, then X carries a natural structure of R–group schemes, we say in this case that $1 \to G \to H \to X \to 1$ is an exact sequence of R–group schemes (for the flat topology).

Lemma 4.10. *Assume that X is the flat quotient of H by G.*

(1) The map $H \to X$ is a G–torsor.

(2) There is an exact sequence of pointed sets

$$1 \to G(R) \to H(R) \to X(R) \xrightarrow{\varphi} H^1_{fppf}(R, G) \to H^1_{fppf}(R, H)$$

where $\varphi(x) = [f^{-1}(x)]$.

For the proof, see [10, III.4.2, cor. 1.8 and III.4.4].

Remark 4.11. (a) Assume that X is affine (or is equipped with an ample G–linearized invertible sheaf, see [5, §6, th. 7 and §10, lemma 6] for details). Then the category of G–torsors over $\mathrm{Spec}(R)$ is equivalent to the category of couples (F, x) where F is a H–torsor and $x \in ({}^F X)(R)$.

(b) If G is normal in H, then X has natural structure of R–group scheme. In this case (a) rephrases by saying that the category of G–torsors over $\mathrm{Spec}(R)$ is equivalent to the category of couples (F,ϕ) where F is a H–torsor and ϕ a trivialization of the X–torsor ${}^F X$.

(c) Using the extended Swan-Serre correspondence 4.5, an example is that category of SL_r-torsors is equivalent to the category of pairs (M,θ) where M is a locally free R–module of rank r and $\theta : R \xrightarrow{\sim} \Lambda^r(M)$ is a trivialization of the determinant of M.

(d) For an integer d, we have the Kummer exact sequence $1 \to \mu_d \to \mathbb{G}_m \xrightarrow{\times d} \mathbb{G}_m \to 1$. Similarly the category of μ_d–torsors is equivalent to the category of pairs (M,θ) where M is an invertible R–module and $\theta : R \xrightarrow{\sim} M^{\otimes r}$ a trivialization. This is related with §3.2.

Examples 4.7.1. $\mathbb{G}_m$ is the flat quotient of GL_r by SL_r and $\mathbb{G}_m$ is the flat quotient of $\mathbb{G}_m$ by μ_d.

There are of course many more functorial properties for example when G is commutative normal.

4.8. **Étale covers.** We remind to the reader that an étale morphism of rings $R \to S$ is a smooth morphism of relative dimension zero [27, §I.3]. There are several alternative definitions, for example, S is a flat R–module such that for each R–field F, then $S \otimes_R F$ is an étale F–algebra (i.e. a finite geometrically reduced F–algebra).

Examples 4.12. *(a) A localization morphism $R \to R_f$ is étale.*

(b) If d is invertible in R, the Kummer morphism $\mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^d$ is étale.

(c) More generally, if d is invertible in R and $r \in R^\times$, then $S = R[x]/(x^d - r)$ is a finite étale R–algebra.

For an R–group scheme G, we define the subset $H^1_{ét}(R,G)$ of $\check{H}^1_{fppf}(R,G)$ of classes ot torsors which are trivialized by an étale cover. We define similarly $\check{H}^1_{ét}(R,G)$

Proposition 4.13. *If G is affine smooth, then we have $H^1_{ét}(R,G) = H^1_{fppf}(R,G)$.*

Sketch. Smoothness is a local property with respect to flat topology so that any G–torsor E is smooth affine over R. According to the existence of quasi-sections [12, 17.16.3], E admits locally sections with respect of the étale topology.

4.9. **Isotrivial torsors and Galois cohomology.** We are given a Galois R–algebra S of group Γ. The action isomorphism $\mathrm{Spec}(S) \times_R \Gamma_S \xrightarrow{\sim} \mathrm{Spec}(S) \times_R \mathrm{Spec}(S)$ reads as the isomorphism $S \otimes_R S \xrightarrow{\sim} S \otimes_R R^{(\Gamma)} = S^{(\Gamma)}$. A 1-cocycle is then an element $z = (z_\gamma)_{\gamma \in \Gamma} \in G(S \otimes_R S) = G(S)^{(\Gamma)}$ satisfying a certain relation.

Since Γ acts on the left on S, it acts as well on the left on $G(S)$.

Lemma 4.14. *(see* [16, lemme 2.2.3]*) A Γ-uple $z = (z_\sigma)_{\sigma \in \Gamma} \in G(S^{(\Gamma)}) = G(S)^{(\Gamma)}$ is a 1–cocycle for S/R if and only if*

$$z_{\sigma\tau} = z_\sigma \, \sigma(z_\tau)$$

for all $\sigma, \tau \in \Gamma$.

We find that $Z^1(S/R,G)$ is the set of Galois cocycles $Z^1(\Gamma, G(S))$ and that $\check{H}^1(S/R,G)$ is the set of non-abelian Galois cohomology $H^1(\Gamma, G(S)) = Z^1(\Gamma, G(S))/\sim$ where two cocycles z, z' are cohomologous if $z_\gamma = g^{-1}\, z'_\gamma\, \sigma(g)$ for some $g \in G(S)$.

An interesting case is when G is the constant group scheme associated to an abstract group Θ. In this case, we have $Z^1(S/R,G) = \mathrm{Hom}_{R-gp}(\Gamma_S, \Theta_S)$ and $\check{H}^1(S/R,G) = \mathrm{Hom}_{S-gp}(\Gamma_S, \Theta_S)/\Theta_R(S)$. In particular, if S is connected, we have $Z^1(S/R,G) = \mathrm{Hom}_{R-gp}(\Gamma, \Theta)$ and $\check{H}^1(S/R,G) = \mathrm{Hom}_{gp}(\Gamma, \Theta)/\Theta$.

Remark 4.15. Galois descent is then a special case of faithfully flat descent. The reader can check that the category of R-modules is equivalent to the category of couples (N, ρ) where N is a S-module equipped with a semilinear action of Γ (i.e. $\rho(\sigma)(\lambda\,.\,n) = \sigma(\lambda)\,.\,\rho(\sigma)(n)$).

We say that torsor E under an R–group scheme G is isotrivial if it is split by a Galois finite étale cover. This is subclass of torsors which can be explicited by Galois cohomology computations and this is a preliminary question is it is the case. For example, for the ring of Laurent polynomials in characteristic zero and a reductive group scheme, this is the case [17].

5. Exercices

R is a commutative (unital) ring.

Exercice 1. Let M be a locally free R–module of rank $2n \geq 2$ equipped with a regular quadratic form q. Show that locally for the flat topology that (M, q) is hyperbolic. [Hint: one can deal first with the case of a local ring where 2 is invertible].

Exercice 2. Prove Lemma 3.1.

Exercice 3. Let R' be a finite locally free R–algebra. Let $r \geq 0$ be an integer.

(1) Show that the R–functor $S \mapsto \mathrm{End}_{S \otimes_R R'}\Big(S \otimes_R R')^r\Big)^\times$ is representable by an affine R–group scheme. We denoted it by $\widetilde{G} = R_{R'/R}(\mathrm{GL}_r)$ (the Weil restriction).
(2) Show that the category of $\widetilde{G}$–torsors is equivalent to the category of locally free R'-modules of rank r.
(3) Give an interpretation of the map $H^1(R, \mathrm{GL}_r) \to H^1(R, \widetilde{G})$ and show that this map is not in general injective nor surjective.
(4) We denote by $M = (R')^r$ the underlying R–module. Construct a map $\widetilde{G} \to \mathrm{GL}(M)$ and discuss the cohomological consequences.

Exercice 3. (Special case of Grothendieck-Margaux [26]) Let G be an affine R–group scheme of finite presentation and let $R_\infty = \varinjlim_\lambda R_\lambda$ be a colimit of R–algebras. For each λ, we put $G_\lambda = G \times_R R_\lambda$ and similarly $G_\infty = G \times_R R_\infty$.

Define a map $\varinjlim H^1_{fppf}(R_\lambda, G_\lambda) \to H^1_{fppf}(R_\infty, G_\infty)$ and show that it is bijective.

Exercice 4. Let $d \geq 1$ be an integer and let R' be a $\mathbb{Z}/d\mathbb{Z}$–Galois extension. We denote by σ the canonical generator of $\mathbb{Z}/d\mathbb{Z}$.

(1) Show that the formula $N(y) = y\, \sigma(y) \dots \sigma^{r-1}(y)$ defines a group scheme homomorphism $N : R_{R'/R}(\mathbb{G}_m) \to \mathbb{G}_m$.
(2) Show that $1 \to \ker(N) \to R_{R'/R}(\mathbb{G}_m) \to \mathbb{G}_m \to 1$ is an exact sequence of R–group schemes.
(3) Deduce an exact sequence involving $H^1(R, \ker(N))$.
(4) Show that the flat quotient of $R_{R'/R}(\mathbb{G}_m)$ by $\mathbb{G}_m$ exists in the category of schemes and is isomorphic to $\ker(N)$.
(5) Construct an exact sequence

$$R^\times \to (R')^\times \xrightarrow{\sigma - 1} \ker(N)(R) \to \ker\big(\mathrm{Pic}(R) \to \mathrm{Pic}(R')\big).$$

(6) Discuss the case of the coordinate ring $A = R[\ker(N)]$ of $\ker(N)$.
(7) For $R = \mathbb{R}$ and $S = \mathbb{C}$, is the $\mathbb{G}_m$-torsor $R_{S/R}(\mathbb{G}_m) \to R_{S/R}(\mathbb{G}_m)/\mathbb{G}_m$ trivial?

Exercice 5. Let B be standard Borel R–subgroup of upper triangular matrices of $\mathrm{GL}_{2,R}$.

(1) Show that the flat quotient of $\mathrm{GL}_{2,R}$ by B exists in the category of R–schemes and is isomorphic to the projective line.

(2) Deduce an exact sequence of pointed sets

$$1 \to B(R) \to \mathrm{GL}_2(R) \to \mathbb{P}^1(R) \to H^1_{fppf}(R, B) \to H^1_{fppf}(R, \mathrm{GL}_2).$$

(3) For R local, show that $H^1_{fppf}(R, B) = 1$ and that $H^1_{fppf}(R, \mathbb{G}_a) = 1$.

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