LECTURES ON R-EQUIVALENCE ON LINEAR ALGEBRAIC GROUPS

P. GILLE

1. Introduction

As usual[1], the ground field is assumed for simplicity to be of characteristic zero. Given a k-variety X, Y. Manin defined the R-equivalence on the set of k–points $X(k)$ as the equivalence relation generated by the following elementary relation. Denote by $\mathcal{O}$ the semi-local ring of $\mathbf{A}^1_k$ at 0 and 1.

1.1. **Definition.** Two points x_0, $x_1 \in X(k)$ are elementary R-equivalent is there exists $x(t) \in X(\mathcal{O})$, such that $x(0) = x_0$ and $x(1) = x_1$.

We denote then by $X(k)/R$ the set of R-equivalence classes. This invariant measures somehow the defect for parametrizing rationally the k-points of X. The following properties follow readily from the definition.

(**1**) additivity : $(X \times_k Y)(k)/R \cong X(k)/R \times Y(k)/R$;

(**2**) "homotopy invariance" : $X(k)/R \xrightarrow{\sim} X(k(v))/R$.

The plan is to investigate R-equivalence for linear algebraic groups. We focus on the case of tori worked out Colliot-Thélène-Sansuc [CTS1] [CTS2], on the case of isotropic simply connected groups [G5] and of the case of number fields [G1] [C2] and two dimensional geometric fields [CGP] [Pa].

Let G/k be a connected linear algebraic group. First the R-equivalence on $G(k)$ is compatible with the group structure. More precisely, denote by $R(k,G) \subset G(k)$ the R-equivalence class of e. Then $R(k,G)$ is a normal subgroup and $G(k)/R(k,G) \cong G(k)/R$. Therefore $G(k)/R$ has a natural group structure. We can already ask the following optimistic open question based on known examples.

1.2. **Question.** Is $G(k)/R$ an abelian group ?

Notice first the following fact.

1.3. **Lemma.** [G1, II.1.1] *Two points of $G(k)$ which are R-equivalent are elementary equivalent.*

Thus the elementary relation is an equivalence relation.

1.4. **Proposition.** *Let $U \subset G$ be an open subset. Then $U(k)/R \xrightarrow{\sim} G(k)/R$.*

[1] `Version of June 21, 2010.`

Proof. By Grothendieck's theorem, G is an unirational k-variety. It means that there exists a (non-empty) subset V of an affine space and a dominant map $h : V \to G$. We can assume that $e \in h(V(k))$. Then $h(V(k))$ is Zariski dense in G and consists of elements R-equivalent to e. In particular $R(k,G)$ is Zariski dense in G, so $R(k,G).U = G$. Hence $U(k)/R \to G(k)/R$ is surjective. In the way around, we are given two elements u, $u' \in U(k)$ which are R-equivalent in G. By Lemma 1.3, there exists $g \in G(\mathcal{O})$ such that $g(0) = u$ and $g(1) = u'$. But we see that g belongs actually to $U(\mathcal{O})$, so we conclude that u and u' are R-equivalent in U. □

Recall that X is k-rational if X is birationally isomorphic to an affine space.

1.5. **Corollary.** *Let G_1 and G_2 be linear algebraic groups which are rationally equivalent. Then there is a bijection $G_1(k)/R \cong G_2(k)/R$. In particular, if G is k-rational, then $G(k)/R = 1$.*

We say that X a k-variety is stably k-rational if there exists $n \geq 0$ such that $X \times_k \mathbf{A}^n_k$ is k-rational. By the additivity property, we have

1.6. **Corollary.** *If G is stably k-rational, then $G(k)/R = 1$.*

2. Examples

2.1. **Normic torus.** Let L/k be a finite Galois algebra of group Γ. We consider the Weil restriction torus (called also induced or quasitrivial)

$$R_{L/k}(\mathbb{G}_m)$$

which is defined by $R_{L/k}(\mathbb{G}_m)(A) = (A \otimes_k L)^\times$ for each k–algebra A. Each element of Γ induces a homomorphism $\sigma_* : R_{L/k}(\mathbb{G}_m) \to R_{L/k}(\mathbb{G}_m)$. The product $\prod \sigma_*$ gives rise to a norm map

$$N_{L/k} : R_{L/k}(\mathbb{G}_m) \to \mathbb{G}_m$$

whose kernel $T = R^1_{L/k}(\mathbb{G}_m)$ is called the normic torus attached to K/k. The group $T(k)$ contains the image of of $(\sigma - 1).L^\times$ for each $\sigma \in \Gamma$. It generates the subgroup $I_\Gamma.L^\times$ where $I_\Gamma \subset \mathbb{Z}[\Gamma]$ stands for augmentation ideal. We have (see §3.3.2 below)

$$T(k)/R \cong T(k)/I_\Gamma.L^\times.$$

If L/k is a field, the following are equivalent :

(1) T is R–trivial, i.e. $T(F)/R = 1$ for any extension F/k;

(2) Γ is metacyclic, i.e. his Sylow subgroups are cyclic.

If Γ is cyclic, note that (1) is nothing but the theorem 90 of Hilbert.

2.2. **Common norm torus.** Let l be a prime number and let $k_1, ..., k_n$ be étale extensions of k of degree l. We consider the common norm torus T

$$N_{k_1/k}(y_1) = \cdots N_{k_n/k}(y_n)$$

inside the quasitrivial torus $R_{k_1/k}(\mathbb{G}_m)\times \cdots R_{k_n/k}(\mathbb{G}_m)$. Put $M = k_1 \otimes_k \cdots \otimes_k k_n$. We have a natural map

$$\mathbb{G}_m \times N_{M/k}(\mathbb{G}_m) \to R_{k_1/k}(\mathbb{G}_m)\times \cdots R_{k_n/k}(\mathbb{G}_m), \quad (x,y) \mapsto (x\, N_{M/k_1}(y), \cdots, x\, N_{M/k_1}(y))$$

whose image for k-rational points consists of R-trivial elements. We have indeed [G2, §3]

$$T(k)/R = \bigcap_i N_{k_i/k}(k_i^\times) \ / \ (k^\times)^l . N_{M/k}(M^\times).$$

If $l = 2$ and $n = 2$, this group is trivial since T is a quadric which is a k–rational variety.

If l is 2 (resp. odd) and M is a field, Merkurjev has shown that T is R-trivial if and only if $n \leq 2$ (resp. $n \leq 3$) [Me3].

2.3. **Special linear groups.** Let A/k be a central simple algebra and put $G = \mathrm{SL}_n(A)$ for $n \geq 1$. This is the kernel of the reduced norm map

$$\mathrm{GL}_n(A) \to \mathbb{G}_m$$

which is the twisted version of the determinant. The commutator subgroup $[A^\times, A^\times]$ consists of R–trivial elements of $G(k)$ and we have indeed

$$G(k)/R = G(k)/[A^\times, A^\times] = SK_1(A)$$

i.e. this group is independent of $n \geq 1$.

By Wedderburn's theorem $A \cong M_r(D)$ where D is a division algebra and the degree of A is by definition the square root of $\dim_k(D)$. Wang has shown that if $\deg(A)$ is squarefree, then G is R–trivial. Suslin conjectured the converse is true [Su]. The main evidence for Suslin's conjecture is the degree 4 case proven by Merkurjev [Me2][Me7]. If $A/\mathbb{Q}$ is a cyclic division algebra of degree 4, we know that $SK_1(A) = 0$ by a result of Wang but Merkujev showed that the generic point of G does not belong to $[A^\times_{k(G)}, A^\times_{k(G)}]$.

Suslin's conjecture is an explanation to Platonov's examples [P] of divisions algebras D of index l^2 with non-trivial SK_1. Using those examples, Wouters showed recently that Suslin's conjecture is true for generic central simple algebras of index l^2 [W].

2.4. **Projective special linear groups.** Let q be a regular quadratic form over a finite even dimensional k-vector space V. By Cayley parametrisation, the special orthogonal group is a k–rational variety, so $\mathrm{SO}(q)$ is R-trivial. The center of $\mathrm{SO}(q)$ is μ_2 and its adjoint quotient $\mathrm{PSO}(q) = \mathrm{SO}(q)/\mu_2$ occurs as a quotient of $\mathrm{GO}^+(q)$ [KMRT], that is the neutral component of the similarity group of q where

$$\mathrm{GO}^+(q)(R) = \Big\{ (f,a) \in \mathrm{GL}(V)(R) \times R^\times \ \mid \ q \circ f = q \ \ and \ \ \det(f) = a^{\frac{\dim(V)}{2}} \Big\}.$$

We have a commutative exact diagram of reductive groups

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & & \\
1 \arrow[r] & \mu_2 \arrow[r] \arrow[d] & \mathrm{SO}(q) \arrow[r] \arrow[d] & \mathrm{PSO}(q) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, "\times 2"'] & \mathrm{GO}^+(q) \arrow[r] \arrow[d, "\mu"', "=p_1"] & \mathrm{PSO}(q) \arrow[r] & 1 \\
 & \mathbb{G}_m \arrow[r, equal] \arrow[d] & \mathbb{G}_m \arrow[d] & & \\
 & 1 & 1 & &
\end{tikzcd}$$

By the theorem 90 of Hilbert 90, the fibration $\mathrm{GO}^+(q) \to \mathrm{PSO}(q)$ is generically trivial, hence Corallary 1.5 yields a bijection $\mathrm{GO}^+(q)(k)/R \xrightarrow{\sim} \mathrm{PSO}(q)(k)/R$. The multiplier induces an isomorphism [Me5]

$$\mathrm{PSO}(q)(k)/R \xrightarrow{\sim} G(q)/N_X(k).(k^\times)^2$$

where $G(q)$ stands for the image of μ_k namely the similarity factors of the quadratic form q and $N_X(k)$ is the norm group of the projective quadric $X = \{q = 0\}$. More precisely, $N_X(k)$ is the subgroup of $k^\times$ generated by the $N_{L/k}(L^\times)$ for L/k running over the finite field extensions of k such that q_L is isotropic.

Another very interesting example is the case of $\mathrm{Spin}(q)$ worked out by Chernousov, Merkurjev and Rost [CM], see also [G5]. The vanishing of $\mathrm{Spin}(q)(k)/R$ for certain q is a key ingredient in Voevodsky's proof of the Milnor conjecture [Vo].

2.5. **Specialization methods.** The examples are not independent of each other. Let l be a prime and assume that the base field k admits a primitive l–root of unity ζ_l.

For a field $k((x))((y))$ of iterated Laurent serie power, one can show that the special linear group of the tensor product of symbol algebras

$$A/k((x))((y)) := (a, x)_{\zeta_l} \otimes (b, y)_{\zeta_l}$$

degenerates to the normic torus T attached to the bicyclic field extension $k(\sqrt[l]{a}, \sqrt[l]{b})$; it gives rise to a surjective induced map $SK_1(A) \to T(k)/R$. Platonov's examples are constructed in such a way [P].

Similarly, there is a relation between R–equivalence for certain quadratic forms over iterated Laurent serie fields and common norm tori of quadratic extensions [G2].

3. R-equivalence on tori

We shall use that the category of k–tori is anti-equivalent to the category of Γ_k-lattices, i.e. the category of lattices equipped with a continuous action of Γ_k. One way is to associate to a k-torus T its Galois module of characters defined by $\widehat{T} = \mathrm{Hom}_{k_s-gp}(T \times_k k_s, \mathbb{G}_{m,k_s})$.

3.1. **Coflasque modules.** Let Γ be a finite group. We denote by $C(\Gamma)$ the following semigroup:

Generators : [M], M Γ-lattice;

Relations : $[P] = 0$, P permutation Γ-lattice.

In other words, two Γ–lattices M, N have same class in $C(\Gamma)$ if $M \oplus P \cong N \oplus Q$ with P, Q permutation Γ-lattice.

3.1. **Definition.** Let M be a Γ-lattice. We say that M is invertible if there exists a Γ-lattice N such that its class is invertible in $C(\Gamma)$.

In other words, invertible Γ-modules are direct summands of permutation modules.

3.2. **Definition.** Let M be a Γ-lattice. We say that M is coflasque if $H^1(\Gamma', M) = 0$ for all subgroups $\Gamma' \subset \Gamma$.

We say that M is flasque if the dual module M^0 is coflasque. By Shapiro's lemma, it follows that permutation lattices are flasque and coflasque. More generally, invertible Γ-lattices are flasque and coflasque.

3.3. **Remark.** This notion is stable by change of groups $f : \widetilde{\Gamma} \to \Gamma$: if M is a coflasque Γ-lattice, then it is a coflasque $\widetilde{\Gamma}$-lattice as well. If f is surjective, then the converse is true. Therefore this notion makes sense for profinite groups.

3.4. **Lemma.** [CTS2, 0.6] *Let M be a Γ-lattice.*

(1) *M admits a coflasque resolution, that is an exact sequence of Γ-modules*
$$0 \to C \to P \to M \to 0$$
such that P is permutation and C is coflasque.

(2) *M admits a flasque resolution, that is an exact sequence of Γ-modules*
$$0 \to M \to P \to F \to 0$$
such that P is permutation and F is flasque.

(3) *The class of F in $C(\Gamma)$ depends only of M.*

We get then an additive map
$$\begin{array}{rcl} p : \{\ \Gamma\text{-lattices}\} & \longrightarrow & C(\Gamma) \\ M & \mapsto & [F]. \end{array}$$

3.2. **Flasque resolution of tori.** We are given a k-torus T, its character group $\widehat{T}$ is a Γ_k–lattice. The kernel of the action $\Gamma_k \to \mathrm{Aut}(\widehat{T})$ is of finite index, this is the Galois group of the minimal splitting field k_T/k. We denote by $\Gamma(T)$ its Galois group.

We say that T is coflasque (resp. flasque) if $\widehat{T}$ a flasque (resp. coflasque) Γ_k-lattice. Equivalently, $\widehat{T}$ a coflasque (resp. flasque) $\Gamma(T)$-lattice. By dualizing Proposition 3.4, we get a flasque resolution of the torus T, namely

$$1 \to S \xrightarrow{i} E \xrightarrow{f} T \to 1,$$

where E is an induced torus and S is a flasque torus.

3.5. **Theorem.** *The characteristic map $T(k) \to H^1(k,S)$ induces an isomorphism*

$$T(k)/R \xrightarrow{\sim} H^1(k,S).$$

If the theorem is true, we should have $H^1(k,S) \xrightarrow{\sim} H^1(k(t),S)$ by property (2) of the introduction. The proof goes by proving that fact before.

3.6. **Lemma.** *Let S/k be a flasque torus as above. Then*

$$H^1(k,S) \xrightarrow{\sim} H^1\big(\Gamma_k, S(\mathcal{O}_{k_s})\big) \xrightarrow{\sim} H^1\big(\Gamma_k, S(k_s(t))\big) \xrightarrow{\sim} H^1(k(t),S).$$

Proof. Tensorising the split sequence of Galois modules

$$1 \to k_s^\times \to k_s(t)^\times \to \bigoplus_{x \in \mathbf{A}^1(k_s)} \mathbb{Z} \to 0$$

by $\widehat{S}^0$ provides the split exact sequence of Γ_k- modules

$$1 \to S(k_s) \to S(k_s(t)) \to \bigoplus_{M \in (\mathbf{A}^1)_0} \mathrm{Coind}_k^{k(M)}(\widehat{S}^0) \to 0$$

Since S/k is flasque, $H^1\big(k, \mathrm{Coind}_k^{k(M)}(\widehat{S}^0)\big) = H^1(k(M), \widehat{S}^0) = 0$, so the long exact sequence of cohomology yields an isomorphism $H^1(k,S) \xrightarrow{\sim} H^1\big(\Gamma_k, S(k_s(t))\big)$. The last isomorphism is true for an arbitrary torus and the midle one follows of the fact that $S(\mathcal{O}_{k_s})$ is a direct summand of $S(k_s(t))$. □

We can now proceed to the proof of Theorem 3.5.

Proof. We have the exact sequence

$$E(k) \xrightarrow{f} T(k) \xrightarrow{\delta} H^1(k,S) \to H^1(k,E) = 1,$$

whose last term vanishes by Hilbert 90. We want to show that $f(E(k)) = R(k,T)$. One way is obvious: since E is a k-rational variety, we have $f(E(k)) \subset R(k,T)$. In the other hand, we have the exact sequence of Γ_k-modules

$$1 \to S(\mathcal{O}_{k_s}) \to E(\mathcal{O}_{k_s}) \to T(\mathcal{O}_{k_s}) \to 1.$$

We have then the following commutative diagram

$$\begin{array}{ccccccc}
E(k) & \xrightarrow{f} & T(k) & \xrightarrow{\delta} & H^1(k,S) & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow\wr & & \\
E(\mathcal{O}) & \xrightarrow{f} & T(\mathcal{O}) & \xrightarrow{\delta} & H^1(\Gamma_k, S(\mathcal{O}_{k_s})) & \longrightarrow & H^1(\Gamma_k, E(\mathcal{O}_{k_s})) = 1,
\end{array}$$

where the last term vanishes by the Lemma (note that the evaluation at 0 provides a splitting of all vertical maps). We are given $x \in R(k,T)$. It exists $x(t) \in T(\mathcal{O})$ such that $x(0) = e$ and $x(1) = x$. From the Lemma we get that $\delta(x(t)) = 0 \in H^1\big(\Gamma_k, S(\mathcal{O}_{k_s})\big)$. So by diagram chase, there exists $y \in E(\mathcal{O})$ lifting x. By specializing at 1, we get that $x = x(1) = f(y(1)) \in f(E(k))$ as desired. □

3.3. **Examples, local fields.**

3.3.1. *Case of a cyclic group.* The proof of the following fact is based on cyclotomic polynomials,

3.7. **Proposition.** *(Endo-Miyata* [EM]*,* [CTS1, prop. 2]*) Assume that* Γ *is a cyclic group. Let* M *be* Γ*-lattice. Then the following are equivalent:*

(1) *M is flasque;*
(2) *M is coflasque;*
(3) *M is invertible in $C(\Gamma)$.*

3.8. **Corollary.** *Let T/k be a torus split by a cyclic extension L/k. Then $T(k)/R = 1$.*

3.9. **Corollary.** *Let $T/\mathbb{Q}$ be a torus. Then $T(\mathbb{Q})$ is dense in $T(\mathbb{R})$.*

3.3.2. *Normic tori.* Let L/k be a finite Galois extension of group Γ. The normic torus $R^1_{L/k}(\mathbb{G}_m)$ is the kernel of the norm map $R_{L/k}(\mathbb{G}_m) \to \mathbb{G}_m$. We have an exact sequence

$$1 \to R^1_{L/k}(\mathbb{G}_m) \to R_{L/k}(\mathbb{G}_m) \xrightarrow{N_{L/k}} \mathbb{G}_m \to 1.$$

3.10. **Proposition.** *For the norm torus $T = R^1_{L/k}(\mathbb{G}_m)$, we have*

$$T(k)/R \xrightarrow{\sim} H^{-1}(\Gamma, L^\times) = \mathrm{Ker}(L^\times \to k^\times)/I_\Gamma \,.\, L^\times.$$

In particular, it vanishes in the cyclic case; this is a version of Hilbert 90.

3.11. **Sketch of proof.** Define the map

$$\begin{array}{ccc}
R_{L/k}(\mathbb{G}_m)^\Gamma & \xrightarrow{f} & R^1_{L/k}(\mathbb{G}_m) \\
(y_\sigma) & \mapsto & \prod_{\sigma\in\Gamma} \sigma(y_\sigma)/y_\sigma.
\end{array}$$

One shows that this map is surjective and its kernel is a flasque k-torus. Theorem 3.5 yields that $T(k)/R = H^{-1}(\Gamma, L^\times)$.

3.3.3. *Local fields.* Assume here that we deal with a p-adic field K. Tate's duality for tori [**?**, II.5.8] states that the natural pairing

$$H^1(K,T) \times H^1(K,\widehat{T}) \to H^2(K,\mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

is a perfect duality of finite groups.

3.12. **Corollary.** *Let T/K be a K-torus and let $1 \to S \to E \to T \to 1$ be a flasque resolution. Then*

$$T(K)/R \xrightarrow{\sim} H^1(\Gamma,\widehat{S})^D.$$

In the case of norm tori, we have a nice formula.

3.13. **Example.** Let $T = R^1_{L/K}(\mathbb{G}_m)$ be the norm torus of a Galois extension L/K of group Γ. Then we have

$$T(K)/R \xrightarrow{\sim} H^3(\Gamma,\mathbb{Z})^D$$

Proof. We use the flasque resolution which arises in the proof of Proposition 3.10. Then we have an exact sequence of Γ-modules

$$0 \to \widehat{T} \to \widehat{E} \to \widehat{S} \to 0.$$

By Shapiro's lemma, we get an isomorphism

$$H^1(\Gamma,\widehat{S}) \xrightarrow{\sim} H^2(\Gamma,\widehat{T}).$$

In the other hand, from the sequence $0 \to \mathbb{Z} \to \mathbb{Z}[\Gamma] \to \widehat{T} \to 0$, we get an isomorphism $H^2(\Gamma,\widehat{T}) \xrightarrow{\sim} H^3(\Gamma,\mathbb{Z})$. $\square$

We know that for bicyclic groups $\Gamma = \mathbb{Z}/n\mathbb{Z}\times\mathbb{Z}/n\mathbb{Z}$, we have $H^2(\Gamma,\mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} H^3(\Gamma,\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$. This provides an example of torus T/K such that $T(K)/R \neq 1$.

3.4. R-trivial tori, Voskresenskiĭ's conjecture.

3.4.1. *R-trivial tori.* Let T/k be a k-torus.

3.14. **Theorem.** *The following are equivalent:*

(i) T is R-trivial, i.e. $T(F)/R = 1$ for all extensions F/k;

(ii) There exists a k–torus T' such that $T \times_k T'$ is a k-rational variety;

(iii) $p(\widehat{T})$ is invertible in $C(\Gamma)$.

The proof of $(ii) \Longrightarrow (iii)$ requires some geometry, see for example [C1, §5].

Proof. $(i) \Longrightarrow (ii)$: Denote by $\xi : \mathrm{Spec}(k(T)) \to T$ the generic point of the torus T. Then

$$\xi \in \mathrm{Im}\Big(E(k(T)) \xrightarrow{f} T(k(T))\Big).$$

This means that there exists an open subset U of T such that $f^{-1}(U) \cong U \times_k S$. Hence $T \times_k S$ is k-birational to the k-rational variety E. Thus $T \times_k S$ is k-rational.

$(iii) \Longrightarrow (i)$: We assume that there exists a k-torus T' such that $T \times_k T'$ is a k-rational variety. Then $T(k)/R \times T'(k)/R = 1$, so $T(k)/R = 1$. The same holds for any extension F/k. $\square$

3.4.2. *Stably k-rational tori.* Given a k-torus T, we have the following characterisation of stably k-rational tori.

3.15. **Theorem.** *The following are equivalent:*

(i) *T is the quotient of two induced tori;*

(ii) *T is a stably rational k-variety;*

(iii) $p(\widehat{T}) = 0 \in C(\Gamma)$.

The proof of $(ii) \Longrightarrow (iii)$ is the same than for Theorem 3.14, so it requires as well some geometry.

3.16. **Sketch of proof.** $(i) \Longrightarrow (ii)$: Assume that there is an exact sequence $1 \to E_1 \to E_2 \to T \to 1$ where E_1, E_2 are quasi-trivial tori. By Hilbert 90, T is R-trivial and the same argument as in the proof of Theorem 3.14 shows that $T \times_k E_1$ is birationally k-isomorphic to E_2. Since induced tori are k-rational varieties, we conclude that T is stably k-rational.
$(iii) \Longrightarrow (i)$: Let $1 \to S \to E \to T \to 1$ be a flasque resolution of T. Our hypothesis is that there exist quasi-trivial tori E_1, E_2 such that $E_2 = S \times E_1$. Replacing S by $S \times E_1$ and E by $E \times E_1$, we conclude that T is the quotient of induced tori.

We can now state Voskresenskiĭ's conjecture.

3.17. **Conjecture.** If T is stably k-rational, it is k-rational.

There are few evidences for this conjecture. Even the case of tori split by cyclic extensions is not known [V].

4. Case of isotropic simply connected groups

For simplicity, we deal with a semisimple simply connected group G/k which is assumed to be absolutely k–simple, i.e. $G \times_k k_s \cong \mathrm{SL}_{n,k_s}, \mathrm{Spin}_{2n+1,k_s}, \mathrm{Sp}_{2n;k_s}$, etc... We assume G to be isotropic, that is, G carries a proper k-parabolic subgroup P. We denote by $R_u(P)$ its unipotent radical.

We denote by $G(k)^+ \subset G(k)$ the normal subgroup generated by the conjugates of $R_u(P)(k)$; this group does not depend of the choice of P. We denote by Z the center of $G(k)$. Tits simplicity theorem states that a proper normal subgroup of $G(k)^+$ is a subgroup of $Z(k)$ [T1]. So simplicity statements for the abstract group $G(k)$ boils down to the vanishing of the Whitehead group $W(k, G) = G(k)/G(k)^+$. For $G = \mathrm{SL}_n(D)$ with $n \geq 2$, we have $W(k, G) \cong SK_1(D)$ which is nothing but $G(k)/R$. It is surprizing since $G(k)^+$ consists of elements which can be linked to 1 within a mapping $\mathbf{A}^1_k \to G$. This is actually a general fact.

4.1. **Theorem.** [G5, 7.2] $W(k,G) \xrightarrow{\sim} G(k)/R$.

Our interpretation is that the R-equivalence is the right extension to arbitrary reductive groups to the Whitehead groups. The key step to show the homotopy invariance property.

4.2. **Theorem.** [G5, 5.8] $W(k,G) \xrightarrow{\sim} W(k(t),G) \xrightarrow{\sim} W(k((t)),G)$.

4.3. **Sketch of proof.** The fact that the map $G(k[[t]]) \to W(k((t)),G)$ is onto is a quite easy application of Bruhat-Tits theory. We shall use that fact for each closed point M of the affine line $\mathbf{A}^1_k$ be used, namely

$$G(\widehat{O}_M) \twoheadrightarrow W(\widehat{F}_M, G)$$

where $\widehat{O}_M \cong k(M)[[\pi_M]]$ stands for the completion of $k[t]$ at the point M and $\widehat{K}_M = Frac(\widehat{O}_M)$. We want to show that $G(k(t)) = G(k(t))^+ \, G(k)$. We are given $g \in G(k(t))$ which can be written $g = h_M \, g_M$ with $h_M \in G(\widehat{F}_M)^+$ and $g_M \in G(\widehat{O}_M)$. Put $U = R_u(P)$ and consider the unipotent radical U^- of a k–parabolic subgroup P^- which is opposite to P. We know [BoT3, §6] that $G(E)^+$ is generated by $U(E)$ and $U^-(E)$ for an arbitrary field extension E/k. We can then approximate strongly the h_M by an element $h \in G(k(t))^+$, that is

$$h^{-1} \, h_M \in G(\widehat{O}_M)$$

for all $M \in \mathbf{A}^1$. Up to replace g by $h^{-1} \, g$, we can then assume that

$$g \in G(k[t]) = G(k(t)) \cap \prod_M G(\widehat{O}_M).$$

Margaux-Soulé's theorem states that $G(k[t])$ is generated by $G(k)$ and $U(k[t])$ [Ma], so $g \in G(k(t))^+ \, G(k)$ as desired.

We have shown that a torus T is R-trivial if and only if G is a direct summand of a k-rational variety so a fortiori iff T is a retract rational variety (*ibid*, 5.9), this is a "retraction of a k–rational variety", a notion due to Saltman [Sa]. So it is natural to ask the following

4.4. **Question.** Let H be a reductive k–group. If H is R-trivial, is H a retract k–rational variety ?

By an important characterisation of retract rational varieties, this is to ask whether the map $H(A) \to H(A/\mathfrak{m}_A)$ is onto for an arbitrary local algebra A.

5. Reductive groups

Our purpose is to compute concretly the group $G(k)/R$ for reductive groups over nice fields.

5.1. **Flasque resolution of reductive groups.** Recall that a linear algebraic group G/k is reductive if it is connected and has trivial unipotent radical. We say that G is quasi-trivial if DG is simply connected and if its coradical torus $E := G/DG$ is quasi-trivial. A flasque resolution of G is an exact sequence of k–groups

$$1 \to S \to \widetilde{G} \to G \to 1$$

such that S is a flasque k–torus and $\widetilde{G}/k$ is a quasi-trivial reductive group.

Flasque resolutions exist and are essentially unique. One way to show the existence is by using geometry after Voskresenskiĭ [V], Borovoi-Kunyavskiĭ [BK] and Colliot-Thélène [C2].

5.1. **Theorem.** *Let X be a smooth compactification of G. Let S be the Néron-Severi torus of X, i.e. of character module* $\mathrm{Pic}(X\times_k k_s)$. *Let $(\mathcal{T}, t_0)/X$ be the universal S-torsor of (X, e) and denote by $\widetilde{G} = G \times_X \mathcal{T}$. Then $(\widetilde{G}, t_0)$ admits a unique structure of algebraic groups such that the mapping $\widetilde{G} \to G$ defines a flasque resolution of G.*

Let us explain what means here "universal torsor" [Sk]. The point $e \in G(k) \subset X(k)$ gives rise to a decomposition

$$H^1_{\acute{e}t}(X, S) \;=\; H^1(k, S) \oplus H^1(X\times_k k_s, S)^{\Gamma_k} \;=\; H^1(k, S) \oplus \mathrm{Hom}_{\Gamma_k}\big(\widehat{S}, \mathrm{Pic}(X\times_k k_s)\big).$$

This is to say that the class of the S-torsor $\mathcal{T}/X$ maps to $(0, id)$.

5.2. **Sketch of proof.** *S is flasque.* Let Y be the variety of Borel subgroups of G. Since Y is a geometrically connected variety, Borovoi and Kunyavskiĭ noticed that S is flasque iff $S \times_k k(Y)$ is flasque [BK]. This trick permits to assume that G is quasi-split, that is G carries a Borel subgroup B. Let T be a maximal k-torus of B, then G is k-birational to $G/B \times B$. Since G/B is a k-rational variety (Borel-Tits) and $B = R_u(B) \rtimes T$, it follows that T and G are stably k-birationnally equivalent.

The point is that the class of $\widehat{S}$ in the semigroup $C(\Gamma_k)$ does not depend of the choice of the compactification, and depends only of the stably birational class of the variety [Vo, §4.4], G in our case. So we are reduced to the case of a smooth compactification of the torus T which is Voskresenskiĭ [V, §4.6].

$\widetilde{G}$ is a k-group. The point here is the fact that the S-torsor $\widetilde{G} \to G$ is "multiplicative", namely

$$p_1^*([\widetilde{G}]) + p_2^*([\widetilde{G}]) = m^*([\widetilde{G}]) \in H^1_{\acute{e}t}(G, S).$$

The choice of an isomorphism determines then a k-group structure on $\widetilde{G}$ [C2, §5].

The k-group $\widetilde{G}$ is quasi-trivial. The derived group $\widetilde{G}$ is semisimple and is simply connected iff $\mathrm{Pic}(D\widetilde{G} \times_k k_s) = 0$. Since the map $D\widetilde{G} \times_k k_s \to \widetilde{G} \times_k k_s$

is split, it is enough to check that $\mathrm{Pic}(\widetilde{G} \times_k k_s) = 0$. We consider the exact sequence

$$0 \longrightarrow k_s[G]^\times / k_s^\times \longrightarrow \mathrm{Div}_{\mathcal{T}_{k_s} \backslash \widetilde{G}_{k_s}} \longrightarrow \mathrm{Pic}(\mathcal{T}_{k_s}) \longrightarrow \mathrm{Pic}(\widetilde{G}_{k_s}) \to 0.$$

But $\mathrm{Pic}(\mathcal{T} \times_k k_s) = 0$, hence $\mathrm{Pic}(\widetilde{G} \times_k k_s) = 0$. Therefore $k_s[\widetilde{G}/D\widetilde{G}]^\times / k_s^\times ==$ $k_s[\widetilde{G}]^\times / k_s^\times$ is a permutation Galois module, so the coradical torus of $\widetilde{G}$ is quasi-trivial.

As for tori, it is interesting for R-equivalence.

5.3. **Lemma.** *Let $1 \to S \to \widetilde{G} \to G \to 1$ be a flasque resolution. Then the characteristic map $\varphi_k ; G(k) \to H^1(k,S)$ gives rise to an exact sequence*

$$\widetilde{G}(k)/R \to G(k)/R \to H^1(k,S) \to H^1(k,\widetilde{G}) \to H^1(k,G)$$

Note that we have an exact sequence for the centers $1 \to S \to Z(\widetilde{G}) \to Z(G) \to 1$. Technically speaking, it is important since it shows that the map $H^1(k,S) \to H^1(k,\widetilde{G})$ factorises by $H^1(k, Z(\widetilde{G}))$.

The computation of $G(k)/R$ essentially decomposes to the quasi-trivial case and to the control on the image of the characteristic map.

5.2. **Norm priciple and norm groups.** We are given the exact sequence $1 \to D\widetilde{G} \to \widetilde{G} \xrightarrow{f} E \to 1$ and would like to control the image of $R(k,\widetilde{G})$ inside $E(k)$. The key ingredient is the norm principle of Gille-Merkurjev [G1] [Me4] which reads as follows

$$N_{L/k}\Big(f_L(R(L,\widetilde{G})\Big) \subset f_k(R(k,\widetilde{G}))$$

for field extension L/k. If $\widetilde{G}_L$ is quasi-split (i.e. admits a Borel L–subgroup), $\widetilde{G}$ is L–rational, and $R(L,\widetilde{G}) = \widetilde{G}(L)$ surjects onto $E(L)$. We have then the inclusion

$$N_{L/k}(E(L)) \subset f_k(R(k,\widetilde{G})) \subset E(k).$$

By taking all norm groups for finite fields extensions which quasi-split $\widetilde{G}$, we get then the inclusion

$$N_X(k,E) = f_k(R(k,\widetilde{G})) \subset E(k)$$

where X stands for the variety of Borel subgroups of $\widetilde{G}$.

5.3. **Fields of cohomological dimension ≤ 2.** We shall discuss here the case of fields of cohomological dimension ≤ 2, e.g. function fields of surfaces or totally imaginary number fields.

Norm groups: We have shown that $N_X(k) = k^\times$ [G3, th. 6]. By decomposing the quasi-trivial torus E, it is easy to see that the previous inclusion yields that $f_k(R(k,\widetilde{G})) = E(k)$ [BK, appendix]. It follows that

$$D\widetilde{G}(k)/E \to \widetilde{G}(k)/R$$

is onto.

Surjectivity of the characteristic map: Using the theorem 90 of Hilbert, the exact sequence $1 \to D\widetilde{G} \to \widetilde{G} \to E \to 1$ yields that the map $H^1(k, D\widetilde{G}) \to H^1(k, \widetilde{G})$ is onto. If Serre's conjecture II holds[2] for $D\widetilde{G}$, we have $H^1(k, D\widetilde{G}) = 1$ and can conclude that the characteristic map $G(k) \to H^1(k, S)$ is trivial.

We know that $H^1(k, Z(\widetilde{G})) \to H^1(k, D\widetilde{G})$ is trivial [G3, th. 6] which is enough to conclude. We have then proven the following

5.4. **Theorem.** *Let $u : \widetilde{G} \to G$ be a flasque resolution of the reductive group G/k defined over a field of cohomological dimension ≤ 2. Put $S = \ker(u)$. Then we have an exact sequence*

$$D\widetilde{G}(k)/R \to G(k)/R \to H^1(k, S) \to 1.$$

In several cases, in particular by the rationality results of Chernousov-Platonov [CP], we know that $\widetilde{G}$ is a k-rational variety, which enables us to conclude of the vanishing of $\widetilde{G}(k)/R$.

5.4. **p-adic fields and totally imaginary number fields.** If k is a p-adic field, Voskresenskiĭ has proven that $D\widetilde{G}(k)/R = 1$ (utside ot type A, the job is done by the previous general statement). We have

$$G(k)/R \xrightarrow{\sim} H^1(k, S) \cong H^1(k, \widehat{S})^D$$

which generalizes the case of tori.

Similarly, if k is a totally imaginary number field, we know that $D\widetilde{G}(k)/R = 1$ by Platonov et al ([PR], see also [G1, III.1.1]), then

$$G(k)/R \xrightarrow{\sim} H^1(k, S)$$

which generalizes as well the case of tori. In particular, $G(k)/R$ is a finite abelian group which depends only of the center of G.

5.5. **Geometric fields.** If k is the function field of a complex surface, we have shown that the groups are very isotropic exactly as in preceding case [CGP]. This permits to conclude that

$$G(k)/R \xrightarrow{\sim} H^1(k, S).$$

Furthermore, this is a finite group (*loc. cit*, §3.2).

5.6. **Open question.** Let k be a finitely generated field over $\mathbb{Q}$ or $\mathbb{C}$. Let G/k be reductive group. Is the group $G(k)/R$ finite ?

[2]Serre's vanishing conjecture II is known in several cases, see [G7] for a survey.

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UMR 8553 du CNRS, École Normale Supérieure, 45 rue d'Ulm, 75005 Paris, France.

E-mail address: `gille@ens.fr`