# Octonions algebras over rings and their norm forms


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# Witt's theorem for quaternion algebras

Let $k$ be a field. For convenience in this talk, 2 will be invertible in the examples.

- Let $Q$ be a quaternion algebra over $k$. It is equipped with a canonical involution $\sigma : Q \to Q$.
- The map $q \to q\sigma(q)$ defines a regular quadratic form $N_Q$ on the $k$-vector space $Q$. It is called the norm form of $Q$.
- Witt has shown that $Q$ is determined by $N_Q$. It means that if a quaternion algebra $Q'$ is such that the quadratic space $(Q', N_{Q'})$ is isometric to $(Q, N_Q)$, then the algebras $Q$ and $Q'$ are isomorphic.

- In concrete terms, assume that $Q = (a, b) = k \oplus k.i \oplus k.j \oplus k.ij$ is the standard quaternion algebra of presentation $i^2 = a$, $j^2 = b$, $ij + ji = 0$. Then
$$\sigma(x + y\, i + z\, j + w\, ij) = x - y\, i - z\, j - w\, ij$$
and
$$N_Q(x + y\, i + z\, j + w\, ij) = x^2 - a\, y^2 - b\, z^2 + ab\, w^2.$$

- $N_Q$ is isometric to the diagonal quadratic form $\langle 1, -a, -b, ab\rangle = \langle 1, -a\rangle \otimes \langle 1, -b\rangle$ which is also denoted by $\langle\langle a, b\rangle\rangle$.

- Then $Q = (a, b)$ is isomorphic to $Q' = (a', b')$ if and only if the quadratic forms $\langle\langle a, b\rangle\rangle$ and $\langle\langle a', b'\rangle\rangle$ are isometric.

- For $k = \mathbb{R}$, it follows that $M_2(\mathbb{R})$ and the Hamilton quaternion algebra $\mathbb{H}$ are the only quaternions algebras.

# A classical result for octonion algebras

- Now consider an octonion algebra $C$ (we'll come back to the definition).
- It is of dimension 8 over $k$ and is equipped with a regular quadratic form $N_C$.
- Springer-Veldkamp showed that $C$ is determined by $N_C$.
- To summarize, there is a one to one correspondence between isomorphism classes of octonion (resp. quaternion) algebras over $k$ and 3-Pfister (resp. 2-Pfister) quadratic forms of $k$.

# Quaternion algebras over rings

- A quaternion algebra $Q/R$ is a an Azumaya algebra over $R$ of dimension four. There are several equivalent definitions.
- One is to require that there exists a faithfully flat cover $S/R$ such that $Q \otimes_R S$ is isomorphic to $M_2(S)$.
- One other is to require that $Q$ is a finitely generated projective $R$-module of rank four equipped with an algebra structure and that $Q \otimes_R R/\mathfrak{m}$ is a $R/\mathfrak{m}$-quaternion algebra for each maximal ideal $\mathfrak{m}$ of $R$.
- Since the automorphism group of $M_2$ is $\mathrm{PGL}_2$, a quaternion algebra over $R$ can be seen as the twist of $M_2(S)$ by the $\mathrm{PGL}_{2,R}$–torsor $\underline{\mathrm{Isom}}(M_2, Q)$ whose functor of points is

  $$\underline{\mathrm{Isom}}(M_2, Q)(R') = R' - \text{isomorphisms of algebras } \; M_2(R') \overset{\sim}{\to} Q \otimes_R R'$$

  for each $R$–ring $R'$.

# Knus-Ojanguren-Sridharan's theorem

- Since inner automorphisms of $M_2$ commute with the determinant and the trace, we get by faithfully flat descent the reduced trace $\mathrm{Tr}_Q$ and the reduced norm map $N_Q$ which is a regular quadratic space over $R$. Note that the involution descends as well within the formula $\sigma(q) = \mathrm{Tr}_Q(q) - q$ and satisfies $N_Q(q) = q\,\sigma(q)$.
- We are given a quaternion algebra $Q$ over a commutative ring $R$.
- Knus et al's theorem states that $N_Q$ determines $Q$.
- Loos-Neher-Petersson defined octonions algebras over rings (2008, see later). A such $R$-algebra is in particular a projective module of rank 8 equipped with a regular 8-dimensional quadratic form.

# The question

- Loos-Neher-Petersson defined octonions algebras over rings (2008, see later). A such $R$-algebra is in particular a projective module of rank 8 equipped with a regular 8-dimensional quadratic form.
- In his Lens lecture on Jordan algebras over rings (May 2012), Holger Petersson asked the following question. Is it true that Springer-Veldkamp's theorem over fields holds for arbitrary rings ?
- In other words, are octonion algebras over rings determined by their norm forms ?

# Summarizing

- | | Quaternions | Octonions |
  |---|---|---|
  | / fields | Witt | Springer-Veldkamp |
  | / rings | Knus-Ojanguren-Sridharan | ? |

- The plan : to discuss slightly more quaternion algebras before embarking to octonions. More precisely, we are interested in understanding the original proof by the geometry of torsors.

# The original proof

- To a regular quadratic space $(V, q)$ over $R$, one can attach its Clifford algebra $C(q)$ and its even part. The Clifford algebra is an Azumaya algebra defined over the discriminant algebra of $q$. For a quaternion algebra $Q$ over $R$, we have $C_0(Q, N_Q) \cong Q \times Q$.

- Clifford algebras provide then the way to pass from rank four quadratic forms of trivial discriminant to quaternion algebras.

- There is no need of Clifford algebras since the semisimple group scheme $\mathrm{SO}(Q, N)$ is of type $A_1 \times A_1$. There is an exact sequence of group schemes

$$1 \to \mu_2 \to \mathrm{SL}_1(Q) \times \mathrm{SL}_1(Q) \xrightarrow{f} \mathrm{SO}(Q, N_Q) \to 1$$

  where $f(x, y).q = xq\,y^{-1}$ for every $q \in Q$.

- It follows that $\mathrm{PGL}_1(Q) \times \mathrm{PGL}_1(Q)$ is the adjoint group of $\mathrm{SO}(Q, N_Q)$. Since $\mathrm{PGL}_1(Q)$ encodes $Q$, we conclude that $N_Q$ determines $Q$.

## The torsor viewpoint

- If we want to think in terms of torsors, we are interested in the kernel of the mapping $H^1(R, \mathrm{PGL}_1(Q)) \to H^1(R, \mathrm{O}(Q, N_Q))$ arising from the embedding $\mathrm{PGL}_1(Q) \to \mathrm{SO}(Q, N_Q) \to O(q, N_Q)$, $x \mapsto h(x).q = x\, q\, x^{-1}$.
- It classifies the isomorphism classes of quaternion $R$-algebras $Q$' such that $N_{Q'}$ is isometric to $N_Q$.
- In term of flat sheaves, we have the isomorphism

$$\big(\mathrm{SL}_1(Q) \times \mathrm{SL}_1(Q)\big)/\mathrm{SL}_1(Q) \ \stackrel{\sim}{\to}\ \mathrm{SO}(Q, N_Q)/\mathrm{PGL}_1(Q)$$

where $\mathrm{SL}_1(Q)$ acts diagonally by right translations. It follows that $\mathrm{SO}(Q, N_Q)/\mathrm{PGL}_1(Q)$ is representable and that the map $\mathrm{SO}(Q, N_Q) \to \mathrm{SO}(Q, N_Q)/\mathrm{PGL}_1(Q)$ admits a section.

# The torsor viewpoint, II

- For extending to the orthogonal group $\mathrm{O}(Q, N_Q)$, the ingredient is to know that $\mathrm{O}(Q, N_Q) = \mathrm{SO}(Q, N_Q) \rtimes \mathbb{Z}/2\mathbb{Z}$ in this case, the splitting being given by the isometry $q \mapsto \sigma(q)$. Note that this element commutes with the subgroup $\mathrm{PGL}_1(Q)$.

- It follows that $\mathrm{O}(Q, N_Q)/\Big(\mathrm{PGL}_1(Q) \times \mathbb{Z}/2\mathbb{Z}\Big)$ is representable and that the map $\mathrm{O}(Q, N_Q) \to O(Q, N_Q)/\Big(\mathrm{PGL}_1(Q) \times \mathbb{Z}/2\mathbb{Z}\Big)$ admits a section.

- Therefore the mapping $H^1(R, \mathrm{PGL}_1(Q)) \to H^1(R, O(Q, N_Q))$ has trivial kernel.

- It ends the quaternions story.

# Octonion algebras

- The present definition is that of Loos-Neher-Petersson and the properties come from the same paper.
- An octonion algebra over $R$ is a non-associative algebra $C$ over $R$ is a f.g. projective $R$–module of rank 8 ; it contains an identity element $1_C$ and admits a norm $n_C : C \to R$ which is uniquely determined by the two following conditions :

  (I) $n_C$ is a regular quadratic form ;

  (II) $n_C(xy) = n_C(x)\, n_C(y)$ for all $x, y \in C$.

- This notion is stable under base extension and descends under faithfully flat base change of rings [LNP].

## The split octonion algebra

- The basic example of an octonion algebra is the split octonion algebra (*ibid*, 4.2) denoted $C_0$ and called the algebra of Zorn vector matrices, which is defined over $\mathbb{Z}$. It can be given by "doubling process". We take

$$C_0 = M_2(\mathbb{Z}) \oplus M_2(\mathbb{Z}), \quad 1_{C_0} = (1, 0)$$

  and use the canonical involution $\sigma = \mathrm{Tr} - id$ of $M_2(\mathbb{Z})$ for defining the multiplication law.

- For $x, y, u, v \in C_0$, we have

$$(x, y).(u, v) = (x\, u + \sigma(v)\, y, v\, x + y\, \sigma(u))$$

  and define the norm by

$$n_{C_0}(x, y) = \det(x) - \det(y).$$

- If $C$ is a octonion algebra over $R$, there exists a flat cover $S/R$ such that $C \otimes_R S \cong C_0 \otimes_R S$ [LNP].

# Descent

- If $C$ is a octonion algebra over $R$, there exists a flat cover $S/R$ such that $C \otimes_R S \cong C_0 \otimes_R S$.
- The automorphism group sheaf $\underline{\mathrm{Aut}}(C_0)$ is representable by the Chevalley group of type $G_2$, denoted by $G_2/\mathbb{Z}$.
- According to the yoga of forms, the étale (or flat) cohomology set $H^1(R, G_2)$ classifies the isomorphism of octonion algebras over $R$.
- Since $G_2 = \mathrm{Aut}(G_2)$, it classifies as well the semisimple group $R$-schemes of type $G_2$.

# Quotients

- We are given an octonion $R$-algebra. We have a natural map $j : \mathrm{Aut}(C) \to \mathrm{O}(C, n_C)$. It is a closed immersion.
- When we take the cohomology, we get a map
$$j_* : H^1(R, \mathrm{Aut}(C)) \to H^1(R, \mathrm{O}(C, n_C))$$
This map attaches to a class $[C']$ of octonion algebras the class of the quadratic space $[(C', n_{C'})]$.
- We are then interested in the kernel of $j_*$. By a general result of Colliot-Thélène and Sansuc, we know that the quotient sheaf $O(C, n_C)/\mathrm{Aut}(C)$ is representable by an affine scheme $X = \mathrm{Spec}(A)$ of finite type over $\mathrm{Spec}(R)$ so that $O(C, n_C) \to X$ is a $\mathrm{Aut}(C)$-torsor.
- If the answer of Petersson's question is yes, then the kernel of $H^1(A, \mathrm{Aut}(C)) \to H^1(A, \mathrm{O}(C, n_C))$ is trivial. In particular, the $\mathrm{Aut}(C)$-torsor $O(C, n_C) \to X$ must be trivial.

# An evidence

- The case of local rings. From Demazure-Grothendieck, we know that local rings behave as fields in the theory of reductive group schemes : Bruhat decomposition, Borel-Tits's theory, classification,..
- **Theorem** (Bix, 1981) *Springer-Veldkamp's theorem holds for local rings.*
- In the case of a local ring of a smooth variety over a field, we can arrive to the same conclusion by using a huge machinery : Balmer-Walter for purity of Witt groups, Chernousov-Panin's results for purity of $G_2$–torsors.
- This shows the $\mathrm{Aut}(C)$-torsor above $O(C, n_C) \to O(C, n_C)/\mathrm{Aut}(C)$ is locally trivial for the Zariski topology.

# A first counterexample

- **Proposition**. *Let $G/\mathbb{R}$ be the anisotropic form of $G_2$ and consider its embedding in the anisotropic real semisimple algebraic groups $\mathrm{SO}_8$. Then the $G$–torsors $\mathrm{SO}_8 \to \mathrm{SO}_8/G$ and $O_8 \to O_8/G$ are not split.*

- The group $G$ is the automorphism group of the Cayley real octonions. Its norm form is $x_1^1 + x_2^2 + \cdots + x_8^2$.

- Proof : We reason by sake of contradiction. Assume that $p : \mathrm{SO}_8 \to \mathrm{SO}_8/G = B$ is a split $G$–torsor. Then the surjective map $\mathrm{SO}_8(\mathbb{R}) \to (\mathrm{SO}_8/G)(\mathbb{R})$ is a split topological $G(\mathbb{R})$–fibration.

- Proof : We reason by sake of contradiction. Assume that $p : \mathrm{SO}_8 \to \mathrm{SO}_8/G = B$ is a split $G$–torsor. It means there is a $G$-isomorphism $\mathrm{SO}_8 \cong G \times_{\mathbb{R}} B$. By taking the real points, we get a decomposition $\mathrm{SO}_8(\mathbb{R}) \cong G(\mathbb{R}) \times B(\mathbb{R})$. It follows that the homotopy group $\pi_n(G(\mathbb{R}))$ is a direct summand of $\pi_n(\mathrm{SO}_8(\mathbb{R}))$ for all $n \geq 1$. From the tables, we have $\pi_6(G(\mathbb{R})) \cong \mathbb{Z}/3\mathbb{Z}$ and $\pi_6(\mathrm{SO}_8(\mathbb{R})) = 0$, hence a contradiction.
- For $O_8$, the argument is similar since $\pi_6(O_8(\mathbb{R})) = 0$ as well.

- The countexample still works after extension over $\mathbb{C}$. Indeed $G_2(\mathbb{C})$ (resp. $\mathrm{SO}_8(\mathbb{C})$) has the the same homotopy type than the compact group $G(\mathbb{R})$ (resp. $\mathrm{SO}_8(\mathbb{R})$).
- It works then for the split case over $\mathbb{C}$ and enables us to strengthen the statement.
- **Proposition**. *Let $R$ be a $\mathbb{Q}$–ring and let $C$ be a octonion algebra over $R$. Then the $\mathrm{Aut}(C)$–torsors $\mathrm{SO}(C, n_C) \to \mathrm{SO}(C, n_C)/\mathrm{Aut}(C)$ and $\mathrm{O}(C, n_C) \to \mathrm{O}(C, n_C)/\mathrm{Aut}(C)$ are not split.*

## Concluding remarks and questions

- It is a quite high invariant which provides the counterexample.
- Is there a mod 3 invariant whose vanishing detects whether a $R$-form $C'$ of $C$ with same norm is isomorphic to $C$ ?
- In small enough dimension, does Springer-Veldkamp's theorem holds ?
- In particular, what is the situation for rings of integers of number fields ?
- For $\mathbb{Z}$, there are only two classes of octonions and the answer is yes.

MERCI !