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Forms of Lie algebras over Laurent polynomial rings

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(joint work with Arturo Pianzola)

1. Introduction

Let S be a scheme. In algebraic geometry, the term "form" or S-form of an object over S is used to describe another object over S that "locally look the same" to the given one, in the sense that the two objects become isomorphic after applying a suitable change to the base S.

This leads to Galois cohomology and more generally to étale cohomology. In this talk we shall discuss mainly the so-called isotrivial situation, namely that of objects V over an affine scheme $S = \mathrm{Spec}(R)$ such that there exists a finite étale covering $S' = \mathrm{Spec}(R')$ which makes the objects isomorphic after base change S'/S.

A perfect example for us is that of the punctured affine line $\mathbb{G}_m = \mathrm{Spec}(\mathbb{C}[t^{\pm 1}])$ which affords standard Kummer coverings of degree d, namely $\mathbb{G}_m \to \mathbb{G}_m$, $t = t^d$. We deal also with the analogous n-variables version of this example, namely $R_n = \mathbb{C}[t_1^{\pm 1}, .., t_n^{\pm 1}]$ and its covering $R_{n,m} = \mathbb{C}[t_1^{\frac{\pm 1}{m}}, .., t_n^{\frac{\pm 1}{m}}]$.

We are interested in classifying semisimple and reductive group schemes over $\mathrm{Spec}(R_n)$ and also their Lie algebras. There is a strong motivation for doing this coming from the theory of extended affine Lie algebras (EALAs for short. See [AABFP]). These are infinite dimensional complex Lie algebras defined by a set of axioms. In nullity one (i.e. when $n = 1$) EALAs are nothing but the affine Kac-Moody Lie algebras. Neher has shown how to construct EALAs out of the centreless cores. The most interesting class of EALAs has the remarkable property their centreless cores are known to be (multi)loop Lie algebras over R_n.

2. Loop algebras

2.1. **Definition.** Let $\mathfrak{g}$ be a semisimple complex Lie algebra. If $\sigma = (\sigma_1, ..., \sigma_n)$ is a family of finite order commuting automorphisms of $\mathfrak{g}$ whose orders divide an integer m, then we can define the Lie algebra

$$\mathcal{L}(\mathfrak{g},\sigma) = \bigoplus_{(i_1,\dots,i_n)\in\mathbb{Z}^n} t_1^{\frac{i_1}{m}} \cdots t_n^{\frac{i_n}{m}} \, \mathfrak{g}_{i_1,\dots,i_n} \subset \mathfrak{g} \otimes R_{n,m}$$

where

$$\mathfrak{g}_{i_1,\dots,i_n} = \Big\{ X \in \mathfrak{g} \mid \sigma_j(X) = \zeta_m^{i_j} X \quad \forall j = 1, \dots, n \Big\}$$

stands for the eigenspace attached to the common diagonalization of the σ_j, where $\zeta_m = e^{\frac{2i\pi}{m}}$.

Since the eigenspaces are m-periodic in each coordinate, $\mathcal{L}(\mathfrak{g},\sigma)$ is a R_n-module. The relations $[\mathfrak{g}_{i_1,\dots,i_n}, \mathfrak{g}_{i'_1,\dots,i'_n}] \subset \mathfrak{g}_{i_1+i'_1,\dots,i_n+i'_n}$ provides $\mathcal{L}(\mathfrak{g},\sigma)$ with an R_n-Lie algebra structure. This Lie algebra is called the (multi)loop algebra of the pair $(\mathfrak{g},\sigma)$. Note that it is independent of the choice of It is easy to see that

$$\mathcal{L}(\sigma,\mathfrak{g}) \otimes_{R_n} R_{n,m} \xrightarrow{\sim} \mathfrak{g} \otimes_{R_n} R_{n,m},$$

as $R_{n,m}$-Lie algebras. Thus $\mathcal{L}(\sigma,\mathfrak{g})$ is an R_n-form of $\mathfrak{g} \otimes R_n$ (or simply of $\mathfrak{g}$, for simplicity of terminology).

A natural question is to classify all $R-n$-forms of $\mathfrak{g}$, and in particular classify and characterize multiloop algebras, among all forms. We should note that Lie theorists are interested in classifying these objects over $\mathbb{C}$. However there is a "rigidity" result (called the centroid trick) which shows that two R_n-forms $\mathcal{L}$ and $\mathcal{L}'$ of $\mathfrak{g}$ are $\mathbb{C}$-isomorphic if and only if there exists $f \in \mathrm{GL}_n(\mathbb{Z}) = \mathrm{Aut}_{\mathbb{C}}(R_n)$ such that $f^*\mathcal{L}$ is R_n-isomorphic to $\mathcal{L}'$ [GP1]. We should concentrate then on the classification/characterization question over R_n.

In nullity one this program was carried out in [P]. The cohomological approach yields a new proof of the classification of the affine Kac-Moody algebras. In this case, all forms are loop algebras.

In higher nullity $n \geq 2$ the authors tried hard to show that it is also the case, but it is not (see the Margaux algebra [GP1]) ! A possible way to construct exotic objects over R_n would be by relaxing the splitting condition, namely to look at R_n-forms $\mathcal{L}$ of $\mathfrak{g}$ which are trivialized by a general faithfully flat base change R'/R_n. But we have shown (Isotriviality Theorem, [GP2]) that this approach is futile: all relevant objects are trivialized by some generalized Kummer covering $R_{n,m}/R_n$.

In practice the construction of counterexamples lead to technical complications because one needs to isolate the class of loop algebras. In the two dimensional case, we have conjectured that the only counterexamples are Margaux-like, so that they are can be described by an invertible module over a 2-loop Azumaya algebra which is rationally a division algebra.

For classical types A, B, C and D, Alexander Steinmetz has shown that the conjecture is true with the possible exception of small dimension cases [SZ]. This uses work of Parimala and also cancellation theorems (Bass, Suslin, Knus, Bertuccioni,...).

2.2. **Internal characterization of loop algebras.** The first characterization is given by grading considerations. We have proven in [GP2] that an R_n-form $\mathcal{L}$ of $\mathfrak{g}$ is a loop algebra if and only we there exists a $\mathbb{Z}^n$-grading on $\mathcal{L}$ together with a trivialization $\mathcal{L} \otimes_{R_n} R_{n,m} \cong \mathfrak{g} \otimes R_{n,m}$. which is a graded isomorphism.

This explains somehow why exotic algebras were not considered (gradings are an essential ingredient of EALAs, and the Margaux example is constructed in such a way as to "break" the grading). The previous criterion is of external nature (since it appeals to gradings that are put on the Lie algebras under consideration). We have another internal characterization of loop algebras which is much more useful in practice.

Theorem 2.1. *[GP3] Let $\mathcal{L}$ be a R_n-form of $\mathfrak{g}$. The following are equivalent:*
(i) $\mathcal{L}$ is a loop algebra.
(ii) $\mathcal{L}$ carries a maximal Cartan algebra, that is a subalgebra $\mathcal{C}$ which is locally (for the Zariski topology) a direct summand of $\mathcal{L}$ and whose geometric fibers are Cartan algebras in the usual sense.

According to [SGA3, XIV.4], (ii) is equivalent to the fact that the semisimple adjoint R_n-group scheme $\mathbf{Aut}(\mathcal{L})^0$ is "toral', i.e. it admits a maximal R_n-torus. The hard implication is $(ii) \Longrightarrow (i)$. That $(i) \Longrightarrow (ii)$ is a consequence of Borel-Mostow's theorem [BM]. If $\mathcal{L} = \mathcal{L}(\mathfrak{g}, \sigma)$, then since the σ_i generate an abelian subgroup of $\mathrm{Aut}(\mathfrak{g})$, we know that there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is stable under the σ_i. Then $\mathcal{L}(\mathfrak{h}, \sigma)$ is a Cartan subalbebra of $\mathcal{L}$.

3. **The main results**

We denote by $F_n = \mathbb{C}((t_1))...((t_n))$ the iterated Laurent series field in n-variables. An important fact is that $\pi_1(R_n, \, .\,) \cong \mathcal{G}al(F_n) \xrightarrow{\sim} (\hat{\mathbb{Z}})^n$. This implies that R_n and F_n have the "same" finite étale coverings.

Theorem 3.1. *The tensor product $\otimes_{R_n} F_n$ induces a one to one correspondence: between isomorphisms of loop R_n-forms of $\mathfrak{g}$ and F_n-forms of $\mathfrak{g}$.*

As in Tits classification ([T] over F_n), the problem reduces to that of "anisotropic objects". The proof of the main theorem proceeds by several delicate steps, and by looking closely at the abelian subgroups of $\mathbf{Aut}(\mathfrak{g})(\mathbb{C})$. A crucial fact, based on Bruhat-Tits theory, is the following:

Theorem 3.2. *Let σ be an anisotropic n-tuple of commuting automorphims of* $\mathrm{Aut}(\mathfrak{g})$ *of finite order (which amounts to the common centralizer of all the σ_i in* $\mathrm{Aut}(\mathfrak{g})$ *being finite). Let σ' be another n-uple. Then the following are equivalent :*

(1) σ and σ' are conjugated under $\mathrm{Aut}(\mathfrak{g})(\mathbb{C})$*;*

(2) $\mathcal{L}(\mathfrak{g}, \sigma) \cong \mathcal{L}(\mathfrak{g}, \sigma')$ as R_n-Lie algebras.

Essentially, the classification of finite abelian subgroups of $\mathrm{Aut}(\mathfrak{g})(\mathbb{C})$ provides the classification of loop algebras. But it is not easy to classify these subgroups! The only general result is about p-elementary abelian subgroups due to Griess. This is sufficient to provide many interesting loop algebras, specially for the exceptional groups G_2, F_4, E_8. The remarkable fact is that we can go the other way around; indeed one knows quite well semisimple F_n-Lie algebras and groups.

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Irreducible representations of Lie algebra of vector fields on a torus and chiral de Rham complex

Yuly Billig

(joint work with Vyacheslav Futorny)

In this talk we discuss representation theory of a classical infinite-dimensional Lie algebra – the Lie algebra $\mathrm{Vect}(\mathbb{T}^N)$ of vector fields on a torus,

$$\mathrm{Vect}(\mathbb{T}^N) = \mathrm{Der}\mathbb{C}[t_1^{\pm1}, \ldots, t_N^{\pm1}] = \mathop{\oplus}_{p=1}^{N} \mathbb{C}[t_1^{\pm1}, \ldots, t_N^{\pm1}] \frac{\partial}{\partial t_p}. \tag{1.1}$$

This algebra has a class of representations of a geometric nature – tensor modules, since vector fields act on tensor fields of any given type via Lie derivative. Tensor modules are parametrized by finite-dimensional representations of gl_N, with the fiber of a tensor bundle being a gl_N-module W:

$$T = \mathbb{C}[q_1^{\pm1}, \ldots, q_N^{\pm1}] \otimes W \tag{1.2}$$